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HANDBOOK OF DIFFERENTIAL EQUATIONS

Stationary Partial Differential Equations

VOLUME 2

Edited by

Michel Chipot

Pavol Quittner

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VOLUME II

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Edited by

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Preface

This handbook is Volume II in a series devoted to stationary partial differential equations. Similarly as Volume I, it is a collection of self-contained, state-of-the-art surveys written by well-known experts in the field.

The topics covered by this handbook include existence and multiplicity of solutions of superlinear elliptic equations, bifurcation phenomena, problems with nonlinear boundary conditions, nonconvex problems of the calculus of variations and Schrödinger operators with singular potentials. We hope that these surveys will be useful for both beginners and experts and speed up the progress of corresponding (rapidly developing and fascinating) areas of mathematics.

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M. Chipot and P. Quittner

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CHAPTER 1

The Dirichlet Problem for Superlinear Elliptic Equations

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Contents

HANDBOOK OF DIFFERENTIAL EQUATIONS
Stationary Partial Differential Equations, volume 2
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0. Introduction

Boundary value problems for nonlinear elliptic partial differential equations have been a major focus of research in nonlinear analysis for decades. In this survey we discuss semilinear equations like

$$\begin{cases} -\Delta u + a(x)u = f(x,u), & x \in \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

where Ω is a domain in $\mathbb{R}^N$, $N \geqslant 2$, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is superlinear, that is, $f(x,t)/t \to \infty$ as $|t| \to \infty$. The model nonlinearity is the homogeneous function

$$f(x,t) = |t|^{p-2}t \quad \text{with } p > 2. \tag{0.2}$$

The continuation method or other classical methods based on the Leray–Schauder degree do not apply easily to (0.1) because there are no a priori bounds for the solutions. This is inherent to superlinear nonlinearities. In fact, for the model nonlinearity (0.2) with $p > 2$, $p < 2N/(N-2)$ if $N \geqslant 3$, $a \in L^\infty(\Omega)$, and bounded Ω, there exist infinitely many solutions which are unbounded in the H^1 norm. On the other hand, if $\Omega \neq \mathbb{R}^N$ is star-shaped, $a(x) \equiv \text{const} \geqslant 0$, f is as in (0.2) with $p \geqslant 2N/(N-2)$, $N \geqslant 3$, then (0.1) has no solution except the trivial one $u \equiv 0$. Here we know a posteriori that the solutions are bounded but the Leray–Schauder methods do not apply due to a lack of compactness.

After some initial work during the 1960s and early 1970s, Ambrosetti and Rabinowitz established in the seminal paper [5] several variational methods to obtain solutions of (0.1) on bounded domains, most notably the mountain pass theorem and variations thereof. These methods have been refined and extended to deal, for instance, with unbounded domains or the critical exponent case $f(x,t) = b(x)|t|^{4/(N-2)}t$ which is closely related to the Yamabe problem from differential geometry. With these methods more complicated partial differential equations with variational structure can now be investigated. Moreover, qualitative properties of the solutions have been discovered in recent years, in particular, on the nodal structure and the symmetry of the solutions.

The goal of this chapter is to present some basic ideas in a simple setting and to survey selected results on the Dirichlet problem (0.1) with superlinear nonlinearity. The chapter consists of three sections. In Section 1 we deal with positive solutions of (0.1), in Section 2 with sign-changing solutions on bounded domains and in Section 3 we treat the unbounded domain $\Omega = \mathbb{R}^N$. No effort is being made to be as general as possible. Neither did we try to write a comprehensive survey on (0.1). For example, we do not present results on the bifurcation of solutions nor for the p-Laplace operator, nor do we treat singularly perturbed equations in detail. These topics require separate surveys. Fortunately, there are a number of well written monographs about (0.1) where the reader can find additional information and further references. We would like to mention the books by Rabinowitz [79], Struwe [87], Chang [41], Ghoussoub [52], Kavian [56], Schechter [81], Willem [95], Chabrowski [39,40], Kielhöfer [57]. We concentrate on topics not being covered in these monographs, although a certain overlap cannot be avoided for natural reasons. Of course, the choice of topics is also influenced by our own research interests.

0.1. *Conditions on the nonlinearity*

For the convenience of the reader we list here conditions on f which we use at various places in the chapter. The critical exponent is defined by

$$2^* = \begin{cases} +\infty & \text{if } N = 2, \\ \frac{2N}{N-2} & \text{if } N \geqslant 3. \end{cases}$$

Given $f : \Omega \times \mathbb{R} \to \mathbb{R}$ we let $F : \Omega \times \mathbb{R} \to \mathbb{R}$, $F(x,t) := \int_0^t f(x,s)\,\mathrm{d}s$, be the primitive of f.

- (f_0) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with $f(x,t) = \mathrm{o}(t)$ as $t \to 0$. There exist $C > 0$ and $p \leqslant 2^*$ such that $|f(x,t)| \leqslant C(|t| + |t|^{p-1})$ for all $x \in \Omega$, $t \in \mathbb{R}$.
- (f_0') $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is differentiable in $t \in \mathbb{R}$, and the derivative f_t is a Carathéodory function with $f_t(x,0) = 0$. There exist $C > 0$ and $p \leqslant 2^*$ such that $|f_t(x,t)| \leqslant C(1 + |t|^{p-2})$ for all $x \in \Omega$, $t \in \mathbb{R}$.
- (f_1) For every $x \in \Omega$ the function $\mathbb{R} \setminus \{0\} \to \mathbb{R}$, $t \mapsto f(x,t)/|t|$, is strictly increasing.
- (f_1') $f_t(x,t) > f(x,t)/t$ for every $x \in \Omega$ and every $t \neq 0$.
- (f_2) There exist $R \geqslant 0$ and $\theta > 2$ such that $0 < \theta F(x,t) \leqslant f(x,t)t$ for all $x \in \Omega$, $|t| > R$. If $\Omega = \mathbb{R}^N$ it is required that $R = 0$.
- (f_3) $\lim_{|t| \to \infty} F(x,t)/t^2 = +\infty$, uniformly in x.
- (f_4) There exists $m > 0$ so that $t \mapsto f(x,t) + mt$ is strictly increasing for all $x \in \Omega$.
- (f_5) $\inf_{t \neq 0} f(x,t)/t > -\infty$.

1. Positive solutions

1.1. *Existence of positive solutions*

We consider first the problem

$$\begin{cases} -\Delta u + a(x)u = u^{p-1}, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth domain in $\mathbb{R}^N$, $a \in L^\infty(\Omega)$ and $2 < p \leqslant 2^*$. In order to obtain a solution of (1.1) we assume that $-\Delta + a$ is positive, i.e., there exists $c > 0$ such that, for every $u \in H_0^1(\Omega)$,

$$\int_\Omega |\nabla u|^2 + a(x)u^2\,\mathrm{d}x \geqslant c \int_\Omega u^2\,\mathrm{d}x. \tag{1.2}$$

Our main tool is the Rellich compactness theorem.

THEOREM 1.1. *Let Ω be bounded and let $1 \leqslant p < 2^*$. Then the injection $H_0^1(\Omega) \subset L^p(\Omega)$ is compact.*

THEOREM 1.2. *Assume that Ω is bounded*, $2 < p < 2^*$ *and* (1.2) *is satisfied*. *Then there is a solution of* (1.1).

PROOF. By Theorem 1.1 and lower semicontinuity, it is easy to verify that

$$\mu = \inf_{\substack{v \in H_0^1(\Omega) \\ \|v\|_{L^p}=1}} \int_\Omega |\nabla v|^2 + a(x)v^2 \,\mathrm{d}x \tag{1.3}$$

is achieved by some $\bar{v}$. After replacing $\bar{v}$ by $|\bar{v}|$, we may assume that $\bar{v} \geqslant 0$. It follows from the Lagrange multiplier rule that

$$-\Delta\bar{v} + a(x)\bar{v} = \mu\bar{v}^{p-1}.$$

A solution of (1.1) is then given by $\bar{u} = \mu^{1/(p-2)}\bar{v}$. Indeed, $\bar{u} > 0$ on Ω by the strong maximum principle. □

REMARK 1.3. There are other ways to prove Theorem 1.2. Instead of minimizing as in (1.3) one can minimize the functional

$$\Phi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + a(x)u^2 \,\mathrm{d}x - \frac{1}{p}\int_\Omega |u|^p \,\mathrm{d}x$$

on the Nehari manifold

$$\begin{aligned}\mathcal{N} &= \bigl\{u \in H_0^1(\Omega)\colon\ u \neq 0,\ \Phi'(u)u = 0\bigr\} \\ &= \left\{u \in H_0^1(\Omega)\colon\ u \neq 0,\ \int_\Omega |\nabla u|^2 + a(x)u^2 \,\mathrm{d}x = \int_\Omega |u|^p \,\mathrm{d}x\right\}.\end{aligned}$$

A critical point of the constrained functional $\Phi|_{\mathcal{N}}$ is a critical point of Φ, hence a solution of (1.1). The map,

$$\bigl\{v \in H_0^1(\Omega)\colon\ \|v\|_{L^p} = 1\bigr\} \to \mathcal{N}, \qquad v \mapsto \left(\int_\Omega |\nabla v|^2 + a(x)v^2 \,\mathrm{d}x\right)^{1/(p-2)} v,$$

is a diffeomorphism with inverse

$$\mathcal{N} \to \bigl\{v \in H_0^1(\Omega)\colon\ \|v\|_{L^p} = 1\bigr\}, \qquad u \mapsto \frac{u}{\|u\|_{L^p}}.$$

It maps the minimizer $\bar{v}$ of (1.3) to the minimizer $\bar{u}$ of Φ on $\mathcal{N}$. The solution can also be obtained via the mountain pass theorem from [5]. In fact,

$$\Phi(\bar{u}) = \inf_{u \neq 0}\max_{t \geqslant 0}\Phi(tu) = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\Phi\bigl(\gamma(t)\bigr),$$

where Γ consists of all continuous paths $\gamma:[0,1]\to H_0^1(\Omega)$ with $\gamma(0)=0$ and $\Phi(\gamma(1))<0$. These three different approaches are equally valid for (1.1). They allow different generalizations. The mountain pass approach leads to the most general existence results for positive solutions of $-\Delta u+a(x)u=f(x,u)$ with Dirichlet boundary conditions. This approach is most widely used in the literature. Minimizing over the Nehari manifold requires more conditions on the nonlinearity f. When these are satisfied one can find nodal solutions on the Nehari manifold and obtain useful additional information, in particular on the nodal structure and the symmetry. Minimizing as in (1.3) only makes sense for homogeneous nonlinearities.

The critical case $p=2^*$ and the supercritical case $p>2^*$ are more delicate.

THEOREM 1.4. *Suppose that* $N\geqslant 3$, $p\geqslant 2^*$, $\Omega\neq\mathbb{R}^N$ *is star-shaped with smooth boundary, and* $a(x)\equiv\lambda\geqslant 0$. *Then there is no solution of* (1.1).

PROOF. By the Pohozaev identity [78], if

$$\begin{cases}-\Delta u=f(u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases}$$

then

$$N\int_\Omega F(u)\,\mathrm{d}x-\frac{N-2}{2}\int_\Omega|\nabla u|^2\,\mathrm{d}x=\frac{1}{2}\int_{\partial\Omega}|\nabla u|^2\sigma\cdot\nu\,\mathrm{d}\sigma,$$

where $F(u)=\int_0^u f(s)\,\mathrm{d}s$ and ν denotes the unit outward normal to $\partial\Omega$. For a solution u of (1.1) we therefore obtain

$$-\lambda\int_\Omega u^2\,\mathrm{d}x+\left(\frac{N}{p}-\frac{N-2}{2}\right)\int_\Omega|u|^p\,\mathrm{d}x=\int_{\partial\Omega}\frac{|\nabla u|^2}{2}\sigma\cdot\nu\,\mathrm{d}\sigma\geqslant 0.$$

It follows that $u=0$ or $\lambda\leqslant 0$, hence $\lambda=0$. If $\lambda=0$, then $p=2^*$ and $\nabla u=0$ on $\partial\Omega$, so by (1.1)

$$0=-\int_\Omega\Delta u\,\mathrm{d}x=\int_\Omega u^{p-1}\,\mathrm{d}x,$$

and therefore, $u=0$. □

REMARK 1.5. In the situation of Theorem 1.4, there is, in fact, no positive nor sign-changing solution $u\neq 0$ of

$$\begin{cases}-\Delta u+\lambda u=|u|^{p-2}u, & x\in\Omega,\\ u=0 & \text{on } \partial\Omega.\end{cases}\tag{1.4}$$

As in the proof of Theorem 1.4, for a solution $u\neq 0$ to exist we must have $\lambda=0$, $p=2^*$, and then $\nabla u=0$ on $\partial\Omega$. Now $u=0$ follows from the unique continuation principle.

We assume now that Ω is bounded and $a(x) \equiv -\lambda$, where $0 < \lambda < \lambda_1(\Omega)$ and

$$\lambda_1(\Omega) = \min_{\substack{v \in H_0^1(\Omega) \\ \|v\|_{L^2}=1}} \int_\Omega |\nabla v|^2 \, \mathrm{d}x.$$

We consider the minimization problem

$$S_\lambda = \inf_{\substack{v \in H_0^1(\Omega) \\ \|v\|_{L^{2^*}}=1}} \int_\Omega |\nabla v|^2 - \lambda v^2 \, \mathrm{d}x. \tag{1.5}$$

In order to solve this problem, we need two tools: the following Brezis–Lieb lemma from [31] and the strict inequality $S_\lambda < S$, where

$$S = \inf_{\substack{v \in H^1(\mathbb{R}^N) \\ \|v\|_{L^{2^*}}=1}} \int_{\mathbb{R}^N} |\nabla v|^2 \, \mathrm{d}x.$$

LEMMA 1.6. *Let (u_n) be a bounded sequence in $L^p(\Omega)$, $1 \leqslant p < \infty$, such that $u_n \to u$ a.e. on Ω. Then*

$$\lim_{n\to\infty} \left(\|u_n\|_{L^p}^p - \|u_n - u\|_{L^p}^p \right) = \|u\|_{L^p}^p.$$

LEMMA 1.7. *Let $N \geqslant 4$ and $\lambda > 0$. Then $S_\lambda < S$.*

PROOF. The instanton

$$U(x) = \frac{[N(N-2)]^{(N-2)/4}}{[1 + |x|^2]^{(N-2)/2}} \tag{1.6}$$

is a minimizer for S. Since $U|_\Omega \notin H_0^1(\Omega)$, we have to use a truncation ψ. We can assume that $B_\rho(0) \subset \Omega$. Let $\psi \in \mathcal{D}(\Omega)$, $\psi \geqslant 0$, be such that $\psi \equiv 1$ on $B_\rho(0)$. Using

$$U_\varepsilon(x) = \psi(x)\varepsilon^{(2-N)/2} U\left(\frac{x}{\varepsilon}\right)$$

an asymptotic analysis shows that $S_\lambda < S$. □

REMARK 1.8. When $N = 3$, the situation is more delicate. Consider the unit ball $\Omega = B_1(0) \subset \mathbb{R}^3$. Then we have

$$0 < \lambda \leqslant \frac{\lambda_1(\Omega)}{4} \quad \Longrightarrow \quad S_\lambda = S$$

and

$$\frac{\lambda_1(\Omega)}{4} < \lambda < \lambda_1(\Omega) \quad \Longrightarrow \quad S_\lambda < S.$$

The next result is due to Brezis and Nirenberg [32].

THEOREM 1.9. *Let $N \geqslant 4$, $p = 2^*$, $0 < \lambda < \lambda_1(\Omega)$ and $a(x) \equiv -\lambda$. Then S_λ is achieved and there exists a solution of* (1.1).

PROOF. Let $(v_n) \subset H_0^1(\Omega)$ be a minimizing sequence for S_λ: $\|v_n\|_{L^{2^*}} = 1$,

$$\int_\Omega |\nabla v_n|^2 - \lambda v_n^2 \, \mathrm{d}x \to S_\lambda.$$

By Rellich's theorem, we can assume, going if necessary to a subsequence:

$$\begin{aligned} &v_n \rightharpoonup v \quad \text{in } H_0^1(\Omega), \\ &v_n \to v \quad \text{in } L^2(\Omega), \\ &v_n \to v \quad \text{a.e. on } \Omega. \end{aligned}$$

With $p = 2^*$ and $w_n = v_n - v$, we obtain from Lemma 1.6,

$$1 - \lim_{n\to\infty} \|w_n\|_{L^p}^p = \|v\|_{L^p}^p.$$

It follows that

$$\begin{aligned} S_\lambda &= \int |\nabla v|^2 - \lambda v^2 \, \mathrm{d}x + \lim_{n\to\infty} \int |\nabla w_n|^2 \, \mathrm{d}x \\ &\geqslant S_\lambda \|v\|_{L^p}^2 + S \lim_{n\to\infty} \|w_n\|_{L^p}^2 \\ &= S_\lambda \|v\|_{L^p}^2 + S\big(1 - \|v\|_{L^p}^p\big)^{2/p}. \end{aligned}$$

Since, by Lemma 1.7, $S_\lambda < S$, necessarily $\|v\|_{L^p}^p = 1$ and v is a minimizer for S_λ. As before, (1.1) is solvable. □

The case $\lambda = 0$ and $p = 2^*$ is more complicated. Using the instantons $U_{\varepsilon,y}(x) = \varepsilon^{(2-N)/2} U((x-y)/\varepsilon)$ with U from (1.6) one shows that the infimum S_0 in (1.5) is not achieved on a domain $\Omega \neq \mathbb{R}^N$ (see, e.g., [95], p. 32). Bahri and Coron [8] showed that the topology of the domain plays an important role.

THEOREM 1.10. *Suppose $\Omega \subset \mathbb{R}^N$ is a bounded domain with nontrivial topology in the sense that it has a nontrivial homology group $H_k(\Omega; \mathbb{Z}_2) \neq 0$ for some $k \geqslant 1$. Then* (1.1) *with $a(x) \equiv 0$ and $p = 2^*$ has a solution.*

We now consider the case $\Omega = \mathbb{R}^N$, $a(x) \equiv 1$, $2 < p < 2^*$ and the boundary value problem

$$\begin{cases} -\Delta u + u = u^{p-1} & \text{on } \mathbb{R}^N, \\ u > 0 & \text{on } \mathbb{R}^N, \\ u(x) \to 0, & |x| \to \infty. \end{cases} \tag{1.7}$$

The corresponding minimization problem is

$$S_p = \inf_{\substack{v \in H^1(\mathbb{R}^N) \\ \|v\|_{L^p} = 1}} \int_{\mathbb{R}^N} |\nabla v|^2 + v^2 \, \mathrm{d}x.$$

Let us recall that $H^1(\mathbb{R}^N) = H_0^1(\mathbb{R}^N)$. We shall use the Schwarz symmetrization and radial Sobolev inequalities.

Let $V_N = m(B_1(0))$ be the Lebesgue measure of the unit ball in $\mathbb{R}^N$. The Schwarz symmetrization of a measurable subset A of $\mathbb{R}^N$ is defined by

$$A^* = B_R(0), \quad \text{where } R \text{ is given by } m\bigl(B_R(0)\bigr) = V_N R^N = m(A).$$

The Schwarz symmetrization of a measurable function $u : \mathbb{R}^N \to [0, \infty[$ is defined by

$$u^*(x) = \sup\bigl\{t > 0\colon\ x \in \{u > t\}^*\bigr\}.$$

THEOREM 1.11. *Let $u : \mathbb{R}^N \to [0, \infty[$ be measurable and let $f : [0, \infty[\to \mathbb{R}$ be continuous. Then*

$$\int_{\mathbb{R}^N} f\bigl(u^*\bigr) \, \mathrm{d}x = \int_{\mathbb{R}^N} f(u) \, \mathrm{d}x.$$

Let $1 \leqslant p < \infty$ and $u \in W^{1,p}(\mathbb{R}^N)$, $u \geqslant 0$. Then the Pólya–Szegö inequality holds:

$$\bigl\|\nabla u^*\bigr\|_{L^p} \leqslant \|\nabla u\|_{L^p}.$$

See [34] or [96] for a simple proof.

We denote by $H_r^1(\mathbb{R}^N)$ the space of radial functions of $H^1(\mathbb{R}^N)$. Let us recall that a function u is radial if $u = u(|x|)$. The Schwarz symmetrization of a measurable function is radial.

The following results are due to Strauss [85].

LEMMA 1.12. *Let $N \geqslant 2$. There exists $c(N) > 0$ such that, for every $u \in H_r^1(\mathbb{R}^N)$,*

$$\bigl|u(x)\bigr| \leqslant c(N) \|u\|_{L^2}^{1/2} \|\nabla u\|_{L^2}^{1/2} |x|^{(1-N)/2} \quad \text{a.e. on } \mathbb{R}^N.$$

THEOREM 1.13. *Let $2 < p < 2^*$. Then there is a radial solution of problem* (1.7).

PROOF. Let $(v_n) \subset H^1(\mathbb{R}^N)$ be a minimizing sequence for S_p: $\|v_n\|_{L^p} = 1$,

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 + v_n^2 \, \mathrm{d}x \to S_p.$$

By Theorem 1.11, we can replace (v_n) by (v_n^*). By Theorem 1.1 and Lemma 1.12, we can assume, going if necessary to a subsequence:

$$\begin{aligned} v_n^* &\rightharpoonup v \quad \text{in } H^1(\mathbb{R}^N), \\ v_n^* &\to v \quad \text{in } L^p(\mathbb{R}^N), \end{aligned}$$

where v is a radial function. Clearly, v is a minimizer for S_p and (1.7) is solvable. □

The existence of nonradial entire solutions $u \in H^1(\mathbb{R}^N)$ of (1.1) and more general equations will be discussed in Section 3.

1.2. *Uniqueness of positive solutions*

The problem of uniqueness of positive solutions is mostly solved for symmetric domains and is closely related to the symmetry of solutions. Let us first recall a celebrated result proved in 1979 by Gidas, Ni and Nirenberg [53].

THEOREM 1.14. *Let Ω be the unit ball in $\mathbb{R}^N$. Assume that f is $\mathcal{C}^1$ and $u \in \mathcal{C}^2(\overline{\Omega})$ satisfies*

$$\begin{cases} -\Delta u = f(u), & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then u is a radial function and $u'(r)$ is negative.

Consider the problem

$$\begin{cases} -\Delta u + \lambda u = u^{p-1}, & x \in \Omega = B_1(0), \\ u > 0, & x \in \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

where $\Omega = B_1(0)$ is the unit ball in $\mathbb{R}^N$. Uniqueness is proved when

$\lambda = 0, 2 < p < 2^*$, Gidas, Ni and Nirenberg [53], 1979,

$\lambda > 0, 2 < p < 2^*$, Kwong [59], 1989,

$\lambda < 0, 2 < p \leqslant 2^*$, Srikanth [84], 1993.

Concerning problem (1.7), uniqueness for $2 < p < 2^*$ is proved in [59] after the pioneering work of Coffman [42] in 1972.

The situation differs when Ω is an annulus, $\Omega = \{x \in \mathbb{R}^N\colon\ r < \|x\| < R\}$ for some $R > r > 0$. Let us recall a particular case of the principle of symmetric criticality proved in 1979 by Palais.

DEFINITION 1.15. The action of a topological group G on a normed space X is a continuous map

$$G \times X \to X : (g, u) \to gu$$

such that

$$1 \cdot u = u,$$
$$(gh)u = g(hu),$$
$$u \mapsto gu \quad \text{is linear.}$$

The action is isometric if $\|gu\| \equiv \|u\|$. The orbit of an element $u \in X$ is the set $Gu := \{gu\colon\ g \in G\}$, and the space of fixed points is defined by

$$\mathrm{Fix}(G) = \{u \in X\colon\ gu = u \text{ for all } g \in G\} = \bigl\{u \in X\colon\ Gu = \{u\}\bigr\}.$$

A function $\varphi : X \to \mathbb{R}$ is invariant if $\varphi \circ g = \varphi$ for every $g \in G$.

EXAMPLE 1.16. Assume that Ω is invariant by rotations: for every $g \in SO(N)$, $g\Omega = \Omega$. The action of $SO(N)$ on $H_0^1(\Omega)$ is defined by

$$gu(x) = u\bigl(g^{-1}x\bigr).$$

The space $\mathrm{Fix}(SO(N))$ is the space $H_{0,r}^1(\Omega)$ of radial functions in $H_0^1(\Omega)$. From Theorem 1.1 and Lemma 1.12, it follows that the injection $H_{0,r}^1(\Omega) \subset L^p(\Omega)$ is compact for $2 < p < 2^*$. Moreover, if Ω is an annulus

$$\Omega = \bigl\{x \in \mathbb{R}^N\colon\ \rho < |x| < R\bigr\}$$

or an exterior domain

$$\Omega = \bigl\{x \in \mathbb{R}^N\colon\ \rho < |x|\bigr\},$$

the injection $H_{0,r}^1 \subset L^p(\Omega)$ is compact for $2 < p \leqslant \infty$.

Let us recall that, for every open subset Ω of $\mathbb{R}^N$,

$$S(\Omega) := \inf_{\substack{u \in H_0^1(\Omega)\\ \|u\|_{2^*}=1}} \|\nabla u\|_2^2 = S$$

and $S(\Omega)$ is never achieved except when $\Omega = \mathbb{R}^N$.

Since the “radial infimum” is achieved when Ω is an annulus, we have

$$S = \inf_{\substack{u \in H_0^1(\Omega) \\ \|u\|_{L^{2^*}}=1}} \|\nabla u\|_{L^2}^2 < \mu_{2^*,r} = \min_{\substack{u \in H_{0,r}^1(\Omega) \\ \|u\|_{L^{2^*}}=1}} \|\nabla u\|_{L^2}^2.$$

We deduce

$$\mu_p = \min_{\substack{u \in H_0^1(\Omega) \\ \|u\|_{L^p}=1}} \|\nabla u\|_{L^2}^2 < \mu_{p,r} = \min_{\substack{u \in H_{0,r}^1(\Omega) \\ \|u\|_{L^p}=1}} \|\nabla u\|_{L^2}^2 \quad \text{for } 2^* - \varepsilon < p < 2^*. \tag{1.9}$$

Using the symmetric criticality principle, we construct, for $2 < p < \infty$, a radial solution of the problem

$$\begin{cases} -\Delta u = u^{p-1}, & x \in \Omega = \{x \in \mathbb{R}^N \colon r < \|x\| < R\}, \\ u > 0, & x \in \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.10}$$

For $2^* - \varepsilon < p < 2^*$ inequality (1.9) yields a nonradial solution of problem (1.10). Thus we have proved the following theorem.

THEOREM 1.17. *For* $2^* - \varepsilon < p < 2^*$, *problem* (1.10) *has at least two solutions, one radial and one nonradial. Moreover, the least energy solution is nonradial.*

The above result is due to Brezis and Nirenberg [32]. Following their work, there are related results for multiple positive nonradial solutions of semilinear elliptic equations on expanding annular domains, by Coffman [43] for $N = 2$, by Li [62] for $N \geqslant 4$, and by Byeon [35] (and Catrina and Wang [38] independently) for $N = 3$. We set $\Omega_a = \{x \in \mathbb{R}^N \colon a < |x| < a + 1\}$ and consider

$$\begin{cases} -\Delta u + u = u^{p-1}, & x \in \Omega_a, \\ u > 0, & x \in \Omega_a, \\ u = 0, & x \in \partial\Omega_a, \end{cases} \tag{1.11}$$

where $N \geqslant 2$, $2 < p < 2^*$.

THEOREM 1.18. *The number of not rotationally equivalent, nonradial solutions of* (1.11) *tends to infinity as* $a \to \infty$.

In [38] the exact symmetry of these nonradial positive solutions was examined and positive solutions having a prescribed symmetry can be constructed from a local minimization procedure. The article [93] has results for the critical exponent problems. More results on multiple positive solutions will be mentioned in Section 1.6.

1.3. *The Nehari manifold*

Let $\Omega \subset \mathbb{R}^N$ be a (not necessarily bounded) smooth domain. We consider the problem

$$\begin{cases} -\Delta u + a(x)u = f(x,u), & x \in \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.12}$$

where $a \in L^\infty(\Omega)$. Concerning the nonlinearity we assume:

(f_0) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with $f(x,t) = o(t)$ as $t \to 0$, uniformly in x. There exist $C > 0$ and $p \leqslant 2^*$ such that $|f(x,t)| \leqslant C(|t| + |t|^{p-1})$ for all $x \in \Omega$, $t \in \mathbb{R}$.

(f_1) For every $x \in \Omega$ the function $\mathbb{R} \setminus \{0\} \to \mathbb{R}$, $t \mapsto f(x,t)/|t|$, is strictly increasing.

By (f_0) the functional

$$\Phi : E = H_0^1(\Omega) \to \mathbb{R}, \qquad \Phi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + a(x)u^2 \,\mathrm{d}x - \int_\Omega F(x,u)\,\mathrm{d}x,$$

where $F(x,t) = \int_0^t f(x,s)\,\mathrm{d}s$, is well defined and $\mathcal{C}^1$. Critical points of Φ are weak solutions of (1.12). The Nehari manifold is defined by

$$\mathcal{N} = \{u \in E \setminus \{0\}\colon\ \Phi'(u)u = 0\}.$$

We define also

$$\mathcal{N}_\pm = \{u \in \mathcal{N}\colon\ \pm u \geqslant 0\} \quad \text{and} \quad \beta_\pm := \inf\{\Phi(u)\colon\ u \in \mathcal{N}_\pm\}.$$

It is clear that $\mathcal{N}$ contains all nontrivial critical points of Φ. Moreover, in order to find the least energy positive (resp. negative) solution of (1.12), it suffices to minimize Φ on $\mathcal{N}_+$ (resp. $\mathcal{N}_-$).

We begin with a geometric description of $\mathcal{N}$. We assume that f satisfies (f_0) and (f_1), or the following differentiable versions:

(f_0') $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is differentiable in $t \in \mathbb{R}$ with $f(x,t) = o(t)$ as $t \to 0$, uniformly in x. The derivative f_t is a Carathéodory function. There exist $C > 0$ and $p \leqslant 2^*$ such that $|f_t(x,t)| \leqslant C(1 + |t|^{p-2})$ for all $x \in \Omega$, $t \in \mathbb{R}$.

(f_1') $f_t(x,t) > f(x,t)/t$ for every $x \in \Omega$ and every $t \neq 0$.

Clearly, (f_0') and (f_1') imply (f_0) and (f_1). For $u \in SE := \{u \in E\colon\ \|u\| = 1\}$, the map,

$$\begin{aligned} &\psi_u : \mathbb{R}^+ \to \mathbb{R}, \\ &\lambda \mapsto \frac{1}{\lambda}\Phi'(\lambda u)u = \int_\Omega |\nabla u|^2 + a(x)u^2\,\mathrm{d}x - \int_\Omega \frac{f(x,\lambda u)}{\lambda} u\,\mathrm{d}x, \end{aligned} \tag{1.13}$$

is strictly decreasing by (f_1). The set

$$U := \{u \in SE\colon\ \psi_u(\lambda) < 0 \text{ for some } \lambda > 0\}$$

is an open subset of SE. For $u \in U$ there exists a unique $\lambda_u > 0$ such that $\psi_u(\lambda_u u) = 0$, that is, $\lambda_u u \in \mathcal{N}$.

PROPOSITION 1.19. (a) *If f satisfies* (f_0) *and* (f_1) *then the map*

$$h : U \to \mathcal{N}, \qquad u \mapsto \lambda_u u$$

is a homeomorphism with inverse $\mathcal{N} \to U$, $v \mapsto v/\|v\|$.

(b) *If f satisfies* (f_0') *and* (f_1') *then $\mathcal{N}$ is a $\mathcal{C}^1$-manifold with tangent space $T_u\mathcal{N} = \{v \in E\colon\ \Phi''(u)[u, v] + \Phi'(u)v = 0\}$. The map h is a $\mathcal{C}^1$-diffeomorphism. If $u \in \mathcal{N}$ is a critical point of the constrained functional $\Phi|_{\mathcal{N}}$, then u is a critical point of Φ.*

PROOF. (a) is a simple consequence of the fact that ψ_u is strictly decreasing and that $\psi_u(\lambda)$ is continuous in (u, λ).

(b) Here $\Phi \in \mathcal{C}^2(E)$ and the implicit function theorem applied to the map

$$U \times \mathbb{R}^+, \qquad (u, \lambda) \mapsto \psi_u(\lambda)$$

yields that $u \mapsto \lambda_u$ is $\mathcal{C}^1$. The claims follow because $T_u\mathcal{N}$ is transversal to $\mathbb{R}^+u$. □

We can describe the set U if f is superlinear:

(f_2) There exist $R > 0$ and $\theta > 2$ such that $0 < \theta F(x,t) \leqslant f(x,t)t$ for all $x \in \Omega$, $|t| > R$.

LEMMA 1.20. *Suppose* (f_0), (f_1) *and* (f_2) *hold. Then $U = \{u \in SE\colon\ Q(u) > 0\}$, where $Q(u) := \int_\Omega (|\nabla u|^2 + a(x)u^2)\,\mathrm{d}x$. If $-\Delta + a$ is positive then $U = SE$ and $\mathcal{N}$ is bounded away from* 0.

The easy proof is left to the reader.

THEOREM 1.21. *If* (f_0) *and* (f_1) *hold then $\beta_\pm = \inf_{u\in\mathcal{N}} \Phi(u) \geqslant 0$ and every minimizer of Φ on $\mathcal{N}_+$* (*resp.* $\mathcal{N}_-$) *is a positive* (*resp. negative*) *solution of* (1.12).

We do not claim that $\beta_\pm$ is achieved. This requires additional conditions on f and a; see Theorem 1.23. Since $\mathcal{N}$ is not, in general, a differentiable manifold, the Lagrange multiplier rule is not applicable. We shall use a general deformation lemma. By definition $\Phi^c = \Phi^{-1}(]-\infty, c])$ and $S_\delta = \{u \in X\colon\ \mathrm{dist}(u, S) \leqslant \delta\}$.

LEMMA 1.22. *Let X be a Banach space, $\Phi \in \mathcal{C}^1(X, \mathbb{R})$, $S \subset X$, $c \in \mathbb{R}$, $\varepsilon > 0$, $\delta > 0$, such that*

$$\forall u \in \Phi^{-1}\big([c - 2\varepsilon, c + 2\varepsilon]\big) \cap S_{2\delta}\colon \qquad \big\|\Phi'(u)\big\| \geqslant \frac{8\varepsilon}{\delta}.$$

Then there exists $\eta \in \mathcal{C}([0,1] \times X, X)$ such that

(i) *if* $t = 0$ *or if* $u \notin \Phi^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap S_{2\delta}$ *then* $\eta(t, u) = u$;
(ii) $\eta(1, \Phi^{c+\varepsilon} \cap S) \subset \Phi^{c-\varepsilon}$;
(iii) $\eta(t, \cdot)$ *is a homeomorphism of* X $\forall t \in [0, 1]$;
(iv) $\|\eta(t, u) - u\| \leqslant \delta$ $\forall u \in X$, $\forall t \in [0, 1]$;
(v) $\Phi(\eta(\cdot, u))$ *is nonincreasing* $\forall u \in X$;
(vi) $\Phi(\eta(t, u)) < c$ $\forall u \in \Phi^c \cap S_\delta$, $\forall t \in [0, 1]$.

A proof can be found in [95], Lemma 1.4.

PROOF OF THEOREM 1.21. The inequality $\beta_\pm \geqslant 0$ follows from the fact that the map ψ_u from (1.13) is strictly decreasing. Now let $u \in \mathcal{N}^+$ be such that $\Phi(u) = \beta_+$. We shall prove that $\Phi'(u) = 0$.

It follows from assumption (f_1) that

$$\Phi(su) < \Phi(u) = \beta_+ \quad \text{for } 0 < s \neq 1.$$

If $\Phi'(u) \neq 0$, then there exists $\delta > 0$ and $\lambda > 0$ such that

$$\|v - u\| \leqslant 3\delta \quad \Longrightarrow \quad \Phi'(v) \geqslant \lambda.$$

Clearly $\beta_0 = \max\{\Phi(u/2), \Phi(3u/2)\} < \beta_+$. For $\varepsilon = \min\{(\beta_+ - \beta_0)/2, (\lambda\delta)/8\}$ and $S = B_\delta(u)$, Lemma 1.22 yields a deformation η such that

- $\eta(1, v) = v$ if $v \notin \Phi^{-1}([\beta_+ - 2\varepsilon, \beta_+ + 2\varepsilon])$,
- $\eta(1, \Phi^{\beta_+ + \varepsilon} \cap B(u, \delta)) \subset \Phi^{\beta_+ - \varepsilon}$,
- $\Phi(\eta(1, v)) \leqslant \Phi(v)$ for all $v \in H_0^1(\Omega)$.

Let us define, for $1/2 < s < 3/2$,

$$h(s) = \max\bigl\{\eta(1, su), 0\bigr\} \in H_0^1(\Omega).$$

It is clear that

$$\max_{1/2 \leqslant s \leqslant 3/2} \Phi\bigl(h(s)\bigr) < \beta_+.$$

We shall prove that $h([1/2, 3/2]) \cap \mathcal{N}^+ \neq \emptyset$, contradicting the definition of β_+. Since

$$\Phi'\left(h\left(\frac{1}{2}\right)\right)h\left(\frac{1}{2}\right) = \frac{1}{2}\Phi'\left(\frac{u}{2}\right)u > 0,$$
$$\Phi'\left(h\left(\frac{3}{2}\right)\right)h\left(\frac{3}{2}\right) = \frac{3}{2}\Phi'\left(\frac{3u}{2}\right)u < 0,$$

the existence of $s \in (1/2, 3/2)$ with $\Phi'(h(s))h(s) = 0$, i.e., $h(s) \in \mathcal{N}^+$, follows from the intermediate value theorem. □

1.4. *Existence of ground states*

In order to prove the existence of a minimizer of Φ on $\mathcal{N}_+$, we assume

(f$_3$) $\lim_{|t|\to\infty} F(x,t)/t^2 = +\infty$, uniformly in x.

THEOREM 1.23. *Suppose that Ω is bounded and that f satisfies* (f$_0$) *with $p < 2^*$,* (f$_1$) *and* (f$_3$). *Moreover, suppose that $-\Delta + a$ is positive. Then there exists a minimizer of Φ on $\mathcal{N}_+$ and, hence, a positive solution of* (1.12).

PROOF. Let $(u_n) \subset \mathcal{N}_+$ be a minimizing sequence: $\Phi(u_n) \to \beta_+$. Let us define $t_n = \|u_n\|$ and $v_n = u_n/t_n$. We can assume that $v_n \rightharpoonup v$ in $H_0^1(\Omega)$. Since, for every $R > 0$,

$$\frac{R^2}{2}\left(1 + \int_\Omega a v_n^2 \,\mathrm{d}x\right) - \int_\Omega F(x, Rv_n)\,\mathrm{d}x = \Phi(Rv_n) \leqslant \Phi(u_n),$$

we have

$$\frac{R^2}{2}\left(1 + \int_\Omega a(x) v^2 \,\mathrm{d}x\right) - \int_\Omega F(x, Rv)\,\mathrm{d}x \leqslant \beta_+,$$

so that $v \neq 0$. If (t_n) is unbounded, we can assume that $t_n \to +\infty$. We obtain, from (f$_3$) and Fatou's lemma, the contradiction

$$\begin{aligned} +\infty &= \int_\Omega \liminf\left(-\frac{a(x)v_n^2}{2} + \frac{F(x, t_n v_n)}{t_n^2}\right)\mathrm{d}x \\ &\leqslant \lim_{n\to\infty} \int_\Omega \left(-\frac{a(x)v_n^2}{2} + \frac{F(x, t_n v_n)}{t_n^2}\right)\mathrm{d}x = \frac{1}{2}. \end{aligned}$$

It follows that (u_n) is bounded in $H_0^1(\Omega)$. Going if necessary to a subsequence, we can assume that $u_n \rightharpoonup u$ in $H_0^1(\Omega)$. By (f$_0$), since $(u_n) \subset \mathcal{N}_+$,

$$\begin{aligned} \int_\Omega f(u)u\,\mathrm{d}x &= \lim_{n\to\infty} \int_\Omega f(u_n)u_n\,\mathrm{d}x \\ &= \lim_{n\to\infty} \left(\int_\Omega a(x)u_n^2\,\mathrm{d}x + |\nabla u_n|^2\right) > 0. \end{aligned}$$

In particular, $u \neq 0$. Since $u \geqslant 0$, there exists $t \geqslant 0$ such that $tu \in \mathcal{N}^+$ and

$$\Phi(tu) \leqslant \liminf_{n\to\infty} \Phi(tu_n) \leqslant \liminf_{n\to\infty} \Phi(u_n) = \beta_+.$$

Hence tu is a minimizer of Φ on $\mathcal{N}_+$. □

Theorem 1.23 is due to Liu and Wang [67]. Note that (f$_3$) is weaker than (f$_2$). Efforts in weakening (f$_2$) have been made in [46,82] (see also the references therein).

1.5. *Symmetry of the ground state solution*

In this section we assume that Ω is invariant by rotations: $g\Omega = \Omega$ for every $g \in SO(N)$. Theorem 1.17 shows that, even when f is independent of x, the ground state is, in general, not radial. However when $f = f(|x|, u)$, a partial symmetry is always preserved by the ground state.

We arbitrarily choose a fixed direction P in $\mathbb{R}^N$ which we will refer to as the north pole direction. Let $R > 0$ and $\mathrm{d}\sigma$ denote the standard measure on $\partial B_R(0)$. The symmetrization A^* of a measurable set $A \subset \partial B_R(0)$ is defined as the closed geodesic ball in $\partial B_R(0)$ centered at the north pole and whose $\mathrm{d}\sigma$-measure equals that of A. The foliated Schwarz symmetrization B^* of a Borel set $B \subset \mathbb{R}^N$ is defined on any sphere $\partial B_R(0)$ by

$$B^* \cap \partial B_R(0) = \big(B \cap \partial B_R(0)\big)^*$$

and the foliated Schwarz symmetrization of a Borel function $u : \mathbb{R}^N \to [0, \infty[$ is defined by

$$u^*(x) = \sup\big\{t > 0\colon\ x \in \big(\partial B_R(0) \cap \{u > t\}\big)^*\big\}.$$

See [83] for the Pólya–Szegö inequality corresponding to the foliated Schwarz symmetrization.

The following result is proved in [19] by combining an elementary symmetrization, the polarization (see [34]) and the maximum principle.

THEOREM 1.24. *We assume* (f_0), (f_1) *and*

(A_1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ *is Hölder continuous on* $\Omega \times [-R, R]$ *for every* $R > 0$,

(A_2) Ω *is radially symmetric,*

(A_3) a *and* $f(\cdot, t)$ *are radial functions for every* $t \in \mathbb{R}$.

Then every minimizer u *of* Φ *on* $\mathcal{N}_+$ *or* $\mathcal{N}_-$ *is a foliated Schwarz symmetric solution of* (1.12).

A related result can be found in [76], Theorem 3.1.

1.6. *Multiple positive solutions*

Using a more topological argument Benci, Cerami and Passaseo [21,23] were able to establish the following result about the impact of the domain topology on the solution structure. Consider

$$\begin{cases} -\Delta u + \lambda u = u^{p-1}, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.14}$$

on a bounded domain $\Omega \subset \mathbb{R}^N$.

THEOREM 1.25. *Suppose the Lusternik–Schnirelmann category of the domain satisfies* $\operatorname{cat}(\Omega) \geqslant 2$.

(a) *If* $2 < p < 2^*$ *then for* λ *sufficiently large* (1.14) *has at least* $\operatorname{cat}(\Omega) + 1$ *solutions*.

(b) *If* $\lambda \geqslant 0$ *then for* $p < 2^*$ *sufficiently close to* 2^* (1.14) *has at least* $\operatorname{cat}(\Omega) + 1$ *solutions*.

The case $p = 2^*$ is of special interest. It is closely related to the Yamabe problem from differential geometry. This critical case is analytically more difficult because the embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is not compact. As discussed in Section 1.1, Brezis and Nirenberg [32] obtained one solution for $0 < \lambda < \lambda_1$ if $N \geqslant 4$, and, in the case $N = 3$, showed that there exists $\lambda^* \in [0, \lambda_1)$ so that (1.14) has a solution for $\lambda^* < \lambda < \lambda_1$. In the case $\lambda = 0$ and $p = 2^*$, Bahri and Coron [8] obtained one solution if the domain has nontrivial homology: $H_k(\Omega; \mathbb{Z}_2) \neq 0$ for some $k \geqslant 1$. The existence of multiple positive solutions of (1.14) for $p = 2^*$ is not known.

REMARK 1.26. More results about the effect of the topology and geometry of the domains on the solutions structure have been given for the singularly perturbed nonlinear elliptic equation

$$\begin{cases} -\varepsilon^2 \Delta u + a(x)u = u^{p-1}, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.15}$$

and variations thereof. In fact, Theorem 1.25 can easily be reformulated for (1.15). Typical results are concerned with the existence of multiple positive solutions and their limiting shape as $\varepsilon \to 0$. This was first done for the least energy solutions in [75]. Singularly perturbed equations like (1.15) have been a very active area of research during the last fifteen years and the number of papers abound. A discussion of this topic goes beyond the scope of our survey.

1.7. *The method of moving planes*

This section is related to Section 1.5. We consider the problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.16}$$

where the domain Ω has some symmetry. We shall prove, by the method of moving planes, that, under some assumptions, all the solutions of (1.16) inherit the symmetry of Ω.

The method of moving planes was introduced by Alexandrov in 1962 in the context of minimal surfaces and was used by Serrin in 1971 and Gidas, Ni and Nirenberg [53] in the study of semilinear elliptic equations. The method was extended and simplified by Berestycki and Nirenberg [26]. We describe a result of [26], following [29].

We shall need the maximum principle for small domains. Let w be a solution of

$$\begin{cases} -\Delta w + c(x)w \geqslant 0 & \text{in } \Omega, \\ w \geqslant 0 & \text{on } \partial\Omega. \end{cases} \tag{1.17}$$

The standard form of the maximum principle asserts that, if $c(x) \geqslant 0$ in Ω then $w(x) \geqslant 0$ in Ω. In Stampacchia's form we use a weaker assumption. Let us recall that

$$S = \inf_{\substack{u \in H_0^1(\Omega) \\ \|u\|_{2^*}=1}} \|\nabla u\|_2^2$$

is independent of Ω.

LEMMA 1.27. *Assume that $w \in H^1(\Omega)$ satisfies* (1.17) *with*

$$\|c_-\|_{N/2} < S. \tag{1.18}$$

Then $w \geqslant 0$ in Ω.

PROOF. It suffices to multiply (1.17) by w_-, to integrate by parts and to use (1.18). □

Assumption (1.18) is always satisfied if $\|c_-\|_\infty < \infty$ and $|\Omega|$ is sufficiently small.

THEOREM 1.28. *Let f be locally Lipschitz and let Ω be bounded, convex in some direction, say x_1, and symmetric with respect to the plane $x_1 = 0$. Then any solution $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ of* (1.16) *is symmetric with respect to x_1 and $\partial u/\partial x_1 < 0$ for $x_1 > 0$ in Ω.*

PROOF. Let us define

$$a = \sup\{x_1 \colon (x_1, y) \in \Omega\},$$
$$\Omega_\lambda = \{(x_1, y) \in \Omega \colon x_1 > \lambda\}, \quad 0 < \lambda < a,$$
$$w_\lambda(x) = u(2\lambda - x_1, y) - u(x_1, y), \quad x \in \Omega_\lambda.$$

The function w_λ is well defined on Ω_λ since Ω is convex in the direction x_1 and symmetric with respect to the plane $x_1 = 0$. We shall prove that

$$w_\lambda(x) \geqslant 0 \quad \text{for } 0 < \lambda < a,\ x \in \Omega_\lambda.$$

In order to see this, we define

$$c_\lambda(x) = \begin{cases} \frac{f(u(x_1,y)) - f(u(2\lambda - x_1, y))}{w_\lambda(x_1,y)} & \text{if } w_\lambda(x) \neq 0, \\ 0 & \text{if } w_\lambda(x) = 0. \end{cases}$$

The function w_λ satisfies

$$\begin{cases} -\Delta w_\lambda + c_\lambda w_\lambda = 0 & \text{in } \Omega_\lambda, \\ w_\lambda \geqslant 0 & \text{on } \partial\Omega_\lambda. \end{cases} \tag{1.19}$$

Moreover, since u is bounded and f is locally Lipschitz,

$$\sup_{0<\lambda<a} \sup_{x\in\Omega_\lambda} \left|c_\lambda(x)\right| < \infty. \tag{1.20}$$

Let us also define

$$\Lambda = \{0 < \lambda < a\colon\ w_\lambda \geqslant 0 \text{ in } \Omega_\lambda\}.$$

By (1.19), (1.20) and Lemma 1.27, Λ is not empty. Clearly Λ is closed in $(0, a)$. Let us prove that Λ is open. Let $\mu \in \Lambda$ and let K be a smooth compact subset of Ω_μ such that $|\Omega_\lambda \setminus K|$ is sufficiently small for λ near μ. From

$$w_\mu \geqslant c > 0 \quad \text{in } K,$$

it follows that

$$w_\lambda \geqslant 0 \quad \text{in } K \text{ for } \lambda \text{ near } \mu.$$

Lemma 1.27 implies that

$$w_\lambda \geqslant 0 \quad \text{in } \Omega_\lambda \setminus K.$$

Thus $w_\lambda \geqslant 0$ in Ω_λ for λ near μ. Hence Λ is open in $(0, a)$ and $\Lambda =]0, a[$. It follows immediately that, on Ω,

$$u(-x_1, y) \geqslant u(x_1, y).$$

Since $\tilde{u}(x_1, y) = u(-x_1, y)$ is also a solution of (1.16), one finds that $u(x_1, y) = u(-x_1, y)$. It is easy to conclude that $\partial u/\partial x_1 < 0$ for $x_1 > 0$ using Hopf's lemma. □

REMARK 1.29. (a) Theorem 1.14 follows directly from Theorem 1.28. It is interesting to note that Theorem 1.28 is applicable to domains like cubes.

(b) The method of moving planes is very flexible and has been adapted to a large variety of problems. It is not possible to give a bibliography within this survey. The surveys by Berestycki [24] and by Brezis [29] contain many references.

(c) With respect to Section 1.5 the assumptions on Ω and on f are somewhat stronger, but the results are applicable to any positive solution.

1.8. *A priori bounds for positive solutions*

Topological methods require the existence of a priori bounds for the set of all positive solutions. In this section we briefly describe particular cases of three classical results. We denote by λ_1 the first eigenvalue of $-\Delta$ on $H_0^1(\Omega)$ and by $e_1 > 0$ the corresponding eigenfunction. Throughout this section we assume that Ω is a smooth bounded domain in $\mathbb{R}^N$ and that $f : \overline{\Omega} \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous.

The first result is due to Brezis and Turner [33]. The proof uses Hardy's inequality.

THEOREM 1.30. *Assume that*

$$\liminf_{u\to\infty} \frac{f(x,u)}{u} > \lambda_1 \quad \textit{uniformly for } x \in \overline{\Omega},$$

and

$$\lim_{u\to\infty} \frac{f(x,u)}{u^\alpha} = 0 \quad \textit{uniformly for } x \in \overline{\Omega}, \textit{ where } \alpha = \frac{N+1}{N-1}.$$

Then there exists $c > 0$ such that if $u \in H_0^1(\Omega)$ satisfies

$$\begin{cases} -\Delta u = f(x,u) + te_1 & \textit{in } \Omega, \\ u > 0 & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega, \end{cases}$$

for some $t \geqslant 0$, we have $\|u\|_\infty \leqslant c$.

The second result is due to de Figueiredo, Lions and Nussbaum [51]. The proof uses the Pohozaev identity.

THEOREM 1.31. *Assume that Ω is convex, $f : \mathbb{R}^+ \to \mathbb{R}^+$ is locally Lipschitz and*

$$\limsup_{u\to\infty} \frac{f(u)}{u} > \lambda_1 \quad \textit{and} \quad \lim_{u\to\infty} \frac{f(x,u)}{u^\alpha} = 0, \quad \textit{where } \alpha = \frac{N}{N-2}.$$

Then there exists $c > 0$ such that, if u satisfies

$$\begin{cases} -\Delta u = f(u) & \textit{in } \Omega, \\ u > 0 & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega, \end{cases}$$

we have $\|u\|_{L^\infty} \leqslant c$.

The third result is due to Gidas and Spruck [54]. The proof uses a blow-up argument and Liouville theorems for the space $\mathbb{R}^N$ or the half-space $\mathbb{R}^N_+$.

THEOREM 1.32. *Assume that there exists a continuous function* $h:\overline{\Omega}\to(0,\infty)$ *such that*

$$\lim_{u\to\infty}\frac{f(x,u)}{u^\alpha}=h(x)\quad \textit{uniformly for } x\in\overline{\Omega},\ \textit{where } 1<\alpha<\frac{N+2}{N-2}.$$

Then there exists $c>0$ *such that, if* $u\in\mathcal{C}^2(\Omega)\cap\mathcal{C}(\overline{\Omega})$ *satisfies*

$$\begin{cases}-\Delta u=f(x,u) & \textit{in } \Omega,\\ u>0 & \textit{in } \Omega,\\ u=0 & \textit{on } \partial\Omega,\end{cases}$$

we have $\|u\|_\infty\leqslant c$.

2. Nodal solutions on bounded domains

In this section we report on recent results concerning nodal solutions of

$$\begin{cases}-\Delta u+a(x)u=f(x,u), & x\in\Omega,\\ u=0 & \text{on } \partial\Omega.\end{cases}\tag{2.1}$$

Here Ω is a bounded smooth domain in $\mathbb{R}^N$, $N\geqslant 2$, though some general results apply to unbounded domains. In the first two subsections we present two ideas how to localize critical points of the associated functional outside of the set of positive or negative functions. In the third subsection we give a nonlinear version of Courant's nodal domain theorem for eigenfunctions of the Laplace operator. This gives an upper bound on the number of nodal domains of a solutions of (2.1) related to the min–max description of the critical value. In the Sections 2.4 and 2.5 we prove the existence and some properties of least energy nodal solutions. Finally, in Section 2.6 we study the existence of multiple nodal solutions.

2.1. *A natural constraint*

In this subsection Ω may be unbounded. We consider the problem (2.1) with f satisfying (f_0) and (f_1) from Section 1.3. Recall the functional $\Phi:E=H_0^1(\Omega)\to\mathbb{R}$ and the Nehari manifold

$$\mathcal{N}=\{u\in E\setminus\{0\}:\ \Phi'(u)u=0\}$$

from Section 1.3. In Theorem 1.21 we showed that a minimizer of Φ on $\mathcal{N}^+=\{u\in\mathcal{N}:\ u\geqslant 0\}$ is a positive solution of (2.1). In order to obtain nodal solutions, we consider the nodal Nehari set

$$\begin{aligned}\mathcal{S}&=\{u\in E:\ u^+\in\mathcal{N},\ u^-\in\mathcal{N}\}\\ &=\{u\in E:\ u^+\neq 0\neq u^-,\Phi'(u)u^+=0=\Phi'(u)u^-\}.\end{aligned}$$

Clearly, $\mathcal{S} \subset \mathcal{N}$ contains the set of all nodal solutions.

THEOREM 2.1. *Suppose* (f_0) *and* (f_1) *hold. Then* $\beta := \inf \Phi(\mathcal{S}) \geqslant 0$, *and every minimizer of* Φ *on* $\mathcal{S}$ *is a nodal solution of* (2.1).

In Section 2.4 we shall prove the existence of a minimizer of Φ on $\mathcal{S}$. Since the maps $E \to E$, $u \mapsto u^{\pm}$ are continuous but not differentiable, $\mathcal{S}$ is not a differentiable manifold even if $\mathcal{N}$ is one (as in Proposition 1.19).

PROOF OF THEOREM 2.1. Clearly $\beta = \inf \Phi(\mathcal{S}) \geqslant \inf \Phi(\mathcal{N}) \geqslant 0$ by Theorem 1.21. Let $u \in \mathcal{S}$ be a minimizer and suppose $\Phi'(u) \neq 0$. As a consequence of (f_1), for any $v \in \mathcal{N}$, the function $\mathbb{R}_+ \ni t \mapsto \Phi(tv) \in \mathbb{R}$ achieves its unique maximum at $t = 1$. Therefore

$$\Phi\big(su^+ + tu^-\big) = \Phi\big(su^+\big) + \Phi(tu^-) < \Phi\big(u^+\big) + \Phi(u^-) = \Phi(u) \tag{2.2}$$

for $(s,t) \in \mathbb{R}_+^2 \setminus \{(1,1)\}$. By the continuity of Φ' there exist $\alpha, \delta > 0$ such that $\|\Phi'(v)\| \geqslant \alpha$ for $v \in U_{3\delta}(u)$. Setting

$$g : D = \left(\frac{1}{2}, \frac{3}{2}\right) \times \left(\frac{1}{2}, \frac{3}{2}\right) \to E, \qquad g(s,t) = su^+ + tu^-,$$

(2.2) implies $\beta_0 := \max \Phi \circ g(\partial D) < \beta$. For $\varepsilon := \min\{\frac{\beta-\beta_0}{2}, \frac{\alpha\delta}{8}\}$ and $S = U_\delta(u)$, Lemma 1.22 yields a deformation η such that $\eta_1 := \eta(1, \cdot) : E \to E$ satisfies:

- $\eta_1(v) = v$ if $\Phi(v) \leqslant \beta - 2\varepsilon$,
- $\eta_1(\Phi^{\beta+\varepsilon} \cap S) \subset \Phi^{\beta-\varepsilon}$,
- $\Phi(\eta_1(v)) \leqslant \Phi(v)$ for all $v \in E$.

It follows that $\max \Phi \circ \eta_1 \circ g(D) < \beta$ and that $h := \eta_1 \circ g = g$ on ∂D. We shall show that $h(D) \cap \mathcal{S} \neq \emptyset$, which gives the contradiction $\inf \Phi|_{\mathcal{S}} < \beta$. Consider the map

$$\psi : D \to \mathbb{R}^2, \qquad \psi(s,t) := \left(\frac{1}{s}\Phi'\big(h(s,t)^+\big)h(s,t)^+, \frac{1}{t}\Phi'\big(h(s,t)^-\big)h(s,t)^-\right)$$

and observe that $\psi(s,t) = (0,0)$ is equivalent to $h(s,t) \in \mathcal{S}$. For $(s,t) \in \partial D$ we have

$$\psi(s,t) = \big(\Phi'\big(su^+\big)u^+, \Phi'(tu^-)u^-\big)$$

because $h = g$ on ∂D. This implies $\deg(\psi, D, (0,0)) = 1$, hence $\psi(s,t) = (0,0)$ for some $(s,t) \in D$. $\square$

REMARK 2.2. (a) The proof of Theorem 2.1 is a two-dimensional version of the proof of Theorem 1.21. A minimizer of Φ on $\mathcal{N}^+$ or $\mathcal{N}^-$ is a local minimum of Φ on $\mathcal{N}$, hence a critical point of mountain pass type. Though not being a manifold, $\mathcal{S}$ is a kind of co-dimension 1 subset of $\mathcal{N}$, a co-dimension 2 subset of E. Hence a minimizer of Φ on $\mathcal{S}$ is a critical point of Morse index 2 (if nondegenerate).

(b) In [17], Lemma 3.2, Bartsch and Weth prove that $\mathcal{S} \cap H^2(\Omega)$ is a co-dimension 2 submanifold of $E \cap H^2(\Omega)$ if f satisfies the conditions (f_0') and (f_1').

(c) In [67] Liu and Wang give a slightly different proof of the above result without using the deformation lemma and therefore requiring less smoothness of the functional.

2.2. *Localizing critical points*

A basic idea for localizing critical points can be formulated in a very general setting. Let X be a topological space and $\varphi : \mathcal{G} \subset [0,\infty) \times X \to X$ be a continuous semiflow on X. Here $\mathcal{G} = \{(t,u) \in [0,\infty) \times X\colon\ 0 \leqslant t < T(u)\}$ is an open subset of $[0,\infty) \times X$, where $T(u) \in (0,\infty]$ is the maximal existence time of the trajectory $t \mapsto \varphi(t,u)$. We often write $\varphi^t(u) = \varphi(t,u)$.

Given $B \subset A \subset X$ we call B positively invariant in A if, for $u \in B$ and $T > 0$ with $\varphi^t(u) \in A$, $0 \leqslant t \leqslant T$, it follows that $\varphi^T(u) \in B$. If even $\varphi^T(u) \in \operatorname{int} B$ then B is said to be strictly positively invariant. In the case $A = X$, we simply call B (strictly) positively invariant. The notion of invariant sets has also been exploited in [65].

Recall that a continuous map $\gamma : (C,D) \to (A,B)$ between pairs $D \subset C$, $B \subset A$ of topological spaces is nullhomotopic if there exists a homotopy $H : (C \times [0,1], D \times [0,1]) \to (A,B)$ with $H(x,0) = \gamma(x)$ and $H(x,1) \in B$ for all $x \in C$.

LEMMA 2.3. *Let $A \subset X$ be positively invariant, $B \subset A$ strictly positively invariant. Let $f : (C,D) \to (A,B)$ be not nullhomotopic, C a metric (paracompact) space. Then*

$$\mathcal{A}(B) := \bigl\{x \in C \bigm| \exists t \geqslant 0\colon\ \varphi^t\bigl(f(x)\bigr) \in B\bigr\} \neq C.$$

PROOF. We argue by contradiction. If $\mathcal{A}(B) = C$ then for each $x \in C$ there exists $\tau(x) \geqslant 0$ with $\varphi^{\tau(x)}(f(x)) \in \operatorname{int}_A B$. Choose a neighborhood V_x of x in C with

$$\varphi^{\tau(x)}\bigl(f(y)\bigr) \in \operatorname{int}_A B \quad \text{for all } y \in V_x.$$

Let $(\pi_j)_{j\in J}$ be a partition of unity subordinated to V_x: $\operatorname{supp} \pi_j \subset V_{x_j}$. Now we define $\sigma : C \to [0,\infty)$, $\sigma(x) := \sum_{j\in J} \pi_j(x)\tau(x_j)$, and $H : C \times I \to A$, $H(x,f) := \varphi^{t\sigma(x)}(f(x))$. This homotopy shows that f is nullhomotopic. $\square$

There also exists an equivariant version of Lemma 2.3 when a group G acts on A and C. The extension is straightforward and therefore omitted.

In a typical application, φ is the negative gradient flow of a functional $\Phi : X \to \mathbb{R}$, and B contains a sublevel set $\Phi^b = \{u \in X\colon\ J(u) \leqslant b\}$. A and B are closed and one wants to find a critical point in $A \setminus B$. For $x \in C \setminus \mathcal{A}(B)$ and $u := f(x)$, one then has

$$\varphi^t(u) \notin B, \quad \text{hence } \Phi\bigl(\varphi^t(u)\bigr) > b \text{ for all } t.$$

Then $c := \lim_{t\to\infty} \Phi(\varphi^t(u)) \geqslant b$ and there exists a sequence $t_n \to \infty$ with

$$\bigl\|\nabla\Phi\bigl(\varphi^{t_n}(u)\bigr)\bigr\|^2 = \frac{\mathrm{d}}{\mathrm{d}t} J\bigl(\varphi^t(u)\bigr)\Bigr|_{t=t_n} \to 0.$$

Thus $u_n := \varphi^{t_n}(u)$, $n \in \mathbb{N}$, is a $(\mathrm{PS})_c$-sequence in $A \setminus B$. If $u_n \to \bar{u}$ then $\bar{u} \notin B$ because otherwise $\bar{u} = \varphi^t(\bar{u}) \in \mathrm{int}_A B$, hence $\phi^{t_n}(u) \in \mathrm{int}_A B$ for some t_n, a contradiction. Thus $\bar{u} \in \overline{A} \setminus B$.

We shall now present examples of strictly positively invariant sets which can be used to find nodal solutions of (2.1). Thus we are interested in finding critical points of the associated functional Φ outside of

$$P^{\pm} = \{u \in E\colon \pm u \geqslant 0 \text{ a.e.}\}. \tag{2.3}$$

Since P^+ and P^- have empty interior one cannot use $B = P^+ \cup P^-$.

EXAMPLE 2.4. Suppose Ω is bounded, and (f_0') and

(f_4) there exists $m > 0$ so that $t \mapsto f(x,t) + mt$ is strictly increasing for all $x \in \Omega$ hold. We take

$$\langle u, v\rangle_m := \langle \nabla u, \nabla v\rangle_{L^2} + m\langle u, v\rangle_{L^2}$$

as scalar product in E and write $\|\cdot\|_m$ for the corresponding norm. Setting $g(x,u) = f(x,u) + mu$ and $G(x,t) = \int_0^t g(x,s)\,\mathrm{d}s$ we can write Φ as

$$\Phi(u) = \frac{1}{2}\|u\|_m^2 - \int_\Omega G(x,u)\,\mathrm{d}x.$$

Thus the gradient of Φ with respect to the above scalar product has the form

$$\nabla\Phi = \mathrm{Id} - K \quad \text{with } K(u) = (-\Delta + m)^{-1}\big(g(\cdot, u)\big).$$

By (f_0'), K leaves $X = \mathcal{C}^1(\overline{\Omega}) \cap E$ invariant. The cones $P_X^{\pm} := X \cap P^{\pm}$ have nonempty interior in X because Ω is bounded. We write $u \leqslant v$ if $v - u \in P_X^+$, and $u \ll v$ if $v - u \in \mathrm{int}(P_X^+)$. As a consequence of the strong maximum principle, K is strictly order preserving, that is,

$$u < v \quad \Longrightarrow \quad K(u) \ll K(v) \quad \text{for } u, v \in X.$$

We consider the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi^t(u) = -\nabla\Phi\big(\varphi^t(u)\big) = -\varphi^t(u) + K\big(\varphi^t(u)\big)$$

which induces a flow on X (and E). Since K is strictly order preserving the set $P_X^{\pm} \setminus \{0\}$ is strictly positively invariant. More generally, if $u \in X$ is a subsolution, that is, $u \leqslant K(u)$, then $u + (P_X^{\pm} \setminus \{0\})$ is strictly positively invariant. This follows from the fact that for $v > 0$ the vector field $-\nabla\Phi$ points at $u + v$ into $u + \mathrm{int}\, P_X^+$,

$$u + v - \nabla\Phi(u + v) = K(u + v) \gg K(u).$$

Similarly, if u is a supersolution then $u + (P_X^{\pm} \setminus \{0\})$ is strictly positively invariant. In order to find nodal solutions above some level $\alpha > 0$ one can work with $B := \Phi^\alpha \cup P_X^+ \cup P_X^-$ which is strictly positively invariant for φ if there are no nodal solutions at the level α.

EXAMPLE 2.5. Suppose Ω is bounded, and (f_0') and

(f_5) $\inf_{t\neq 0} f(x,t)/t > -\infty$

hold. Then we choose $m > 0$ so that $f(x,t) + mt > 0$ for all $t > 0$, $f(x,t) + mt < 0$ for all $t < 0$, and define $\langle\cdot,\cdot\rangle_m$, X and the order relation $\ll$ as in Example 2.4. The gradient of Φ with respect to this metric is not necessarily order preserving but it does satisfy

$$u > 0 \implies K(u) \gg 0 \quad \text{and} \quad u < 0 \implies K(u) \ll 0.$$

It follows as in Example 2.4 that the cones $P_X^{\pm} \setminus \{0\}$ are strictly positively invariant.

If Ω is unbounded or if f is only a Carathéodory function, the approach presented in Example 2.4 does not work because then one cannot work in $X = \mathcal{C}^1(\overline{\Omega}) \cap E$. Either the cones $P_X^{\pm}$ have empty interior or there is no flow on X due to a lack of regularity. Here one can often replace $P^{\pm}$ by their open neighborhoods in E.

DEFINITION 2.6. Let $K : E \to E$ be a continuous operator on a Banach space E. A set $C \subset E$ is said to be K-attractive if there exists $\varepsilon_0 > 0$ so that $K(\operatorname{clos}(U_\varepsilon(C))) \subset U_\varepsilon(C) = \{u \in E \colon \operatorname{dist}(u, C) < \varepsilon\}$ for $0 < \varepsilon < \varepsilon_0$.

LEMMA 2.7. *Let E be a Banach space, $\Phi \in \mathcal{C}^1(E)$ and $C = C_1 \cup \cdots \cup C_n \subset E$ be a finite union of convex sets. Suppose $\nabla\Phi = \mathrm{Id} - K$, and each C_j is K-attractive. Then Φ has no critical points in* $\operatorname{clos}(U_{\varepsilon_0}(C)) \setminus C$, *where ε_0 is from Definition* 2.6. *Given $\varepsilon \in (0, \varepsilon_0]$ there exists a pseudo-gradient vector field $V : E \setminus \operatorname{Fix}(K) \to E$ for Φ so that* $\operatorname{clos}(U_\varepsilon(C))$ *is strictly positively invariant for the flow associated to $-V$.*

PROOF. For $\varepsilon \in (0, \varepsilon_0]$ and $u \in \partial U_\varepsilon(C_j)$, we have $K(u) \in U_\varepsilon(C_j)$. Since $C_1, \ldots, C_n$ are convex, a standard partition of unity argument yields a locally Lipschitz continuous map $\widetilde{K} \setminus \operatorname{Fix}(K) \to E$ such that

$$\widetilde{K}(u) \in U_\varepsilon(C_j) \quad \text{for } u \in \partial U_\varepsilon(C_j),\ j = 1, \ldots, n,$$

and

$$\|u - \widetilde{K}(u)\| \leqslant 2\|\Phi'(u)\|, \quad \Phi'(u)\big(u - \widetilde{K}(u)\big) \geqslant \frac{1}{2}\|\Phi'(u)\|^2,$$

for $u \in E \setminus \operatorname{Fix}(K)$. □

Here are two examples of K-attractive sets.

EXAMPLE 2.8. Suppose f satisfies (f_0) and (f_5). Thus there exists $m > 0$ as in Example 2.5, and we define $K(u) = (-\Delta + m)^{-1}(f(\cdot, u) + mu)$. If $\limsup_{t\to 0} |f(x,t)|/|t| <$

$\lambda_1(\Omega)$ uniformly in Ω then it has been proved in [12], Lemma 3.1, that $P^{\pm} = \{u \in E\colon \pm u \geqslant 0 \text{ a.e.}\}$ is K-attractive. We sketch the argument in a more complicated parameter dependent situation (see Lemma 3.17).

EXAMPLE 2.9. Suppose f satisfies (f_0) and (f_4). Set $E = H_0^1(\Omega)$ and $K(u) = (-\Delta + m)^{-1}(f(\cdot, u) + mu)$ as in Example 2.4. In [44] it is proved for bounded Ω that, for a strict subsolution $u \in E \cap W^{2,2}$, the cone $u + P^+$ is K-attractive.

EXAMPLE 2.10. When Ω is unbounded, neighborhoods of shifted positive and negative cones in the directions of the first eigenfunctions have been proved to be K-attractive in [66]. The situation is more delicate here and some conditions on the spectrum of the linear operator have be to assumed.

The Examples 2.4, 2.5 and 2.8 together with Lemma 2.7 yield strictly positively invariant sets which can be used to find nodal solutions of (2.1) or (2.2) with the help of Lemma 2.3. We shall do this in Sections 2.4 and 2.6.

2.3. *Upper bounds on the number of nodal domains*

In this subsection $\Omega \subset \mathbb{R}^N$ may be unbounded. A nodal domain of a continuous function $u : \Omega \to \mathbb{R}$ is a connected component of $\Omega \setminus u^{-1}(0)$. We write $\mathrm{nod}(u) \in \mathbb{N}_0 \cup \{\infty\}$ for the number of nodal domains of u and set $\mathrm{nod}(u) = 0$ for $u = 0$.

LEMMA 2.11. *Suppose f satisfies* (f_0). *Then every weak solution $u \in E$ of* (2.1) *is continuous. If $\Omega_0 \subset \Omega$ is a nodal domain of u then $u \cdot \chi_{\Omega_0} \in H_0^1(\Omega)$.*

PROOF. First observe that $f(\cdot, u)/u \in L^{N/2}_{\mathrm{loc}}(\Omega)$ by (f_0) and the Sobolev embedding theorem. The Brezis–Kato theorem [30] implies $u \in L^q_{\mathrm{loc}}(\Omega)$ for every $2 \leqslant q < \infty$ and therefore $f(\cdot, u) \in L^s_{\mathrm{loc}}(\Omega)$ for $s > N/2$. Then u is continuous by elliptic regularity. The last statement has been proved in [73], Lemma 1. □

Now we suppose that (f_0) and (f_1) hold. Let $\mathcal{N}$ be the Nehari manifold and $\mathcal{S} \subset \mathcal{N}$ the nodal Nehari set.

PROPOSITION 2.12. *Suppose* (f_0) *and* (f_1) *hold. Let u be a critical point of Φ and fix $n \in \mathbb{N}$.*

(a) *If $\Phi(u) \leqslant \inf \Phi(\mathcal{S}) + n \cdot \inf \Phi(\mathcal{N})$ then* $\mathrm{nod}(u) \leqslant n + 1$.

(b) *If $\Phi(u) \leqslant \inf_{v_1,\dots,v_n} \sup \Phi(C(v_1, \dots, v_n))$ then* $\mathrm{nod}(u) \leqslant n$. *The infimum extends over all n-tuples of linearly independent elements $v_1, \dots, v_n \in E$, and $C(v_1, \dots, v_n) := \{\sum_{i=1}^n \lambda_i \cdot v_i\colon \lambda_1, \dots, \lambda_n \geqslant 0\}$.*

(c) *If f is odd and $\Phi(u) \leqslant \inf_{V \subset E, \dim(V)=n} \sup \Phi(V)$ then* $\mathrm{nod}(u) \leqslant n$. *Here the infimum extends over all n-dimensional linear subspaces of E.*

PROOF. (a) Let u have k nodal domains $\Omega_1, \dots, \Omega_k$ such that $v_1 := u \cdot \chi_{\Omega_1} > 0$ and $v_2 := u \cdot \chi_{\Omega_2} < 0$. Clearly, $v_j := u \cdot \chi_{\Omega_j} \in \mathcal{N}$ and $v_1 + v_2 \in \mathcal{S}$. Theorem 1.21 implies that v_j is not a minimizer of Φ on $\mathcal{N}$, hence

$$\Phi(u) = \Phi(v_1 + v_2) + \sum_{j=3}^{k} \Phi(v_j) > \inf \Phi(\mathcal{S}) + (k-2) \inf \Phi(\mathcal{N}).$$

This implies $k \leqslant n + 1$.

(b) and (c) Suppose $\mathrm{nod}(u) > n$ and let $\Omega_1, \dots, \Omega_n$ be nodal domains of u. Then $v_i := u \cdot \chi_{\Omega_i} \in \mathcal{N}$ and $\Phi(v_i) = \max_{\lambda \geqslant 0} \Phi(\lambda v_i)$. This implies

$$\Phi(u) > \Phi(v_1 + \cdots + v_n) \geqslant \inf_{v_1, \dots, v_n} \sup \Phi\big(C(v_1, \dots, v_n)\big).$$

If f is odd then $\Phi(v_i) = \max_{\lambda \in \mathbb{R}} \Phi(\lambda v_i)$, hence

$$\Phi(u) > \Phi(v_1 + \cdots + v_n) \geqslant \inf_V \sup \Phi(V).$$ □

If $\Phi \in \mathcal{C}^2(E)$ then the Morse index provides a lower bound for the number of nodal domains.

THEOREM 2.13. *Suppose* (f_0') *and* (f_1') *hold and let* u *be a critical point of* Φ *with Morse index* $\mu(u)$. *Then* $\mathrm{nod}(u) \leqslant \mu(u)$.

PROOF. Let $\Omega_1, \dots, \Omega_n$ be nodal domains of u and set $v_j := u \cdot \chi_{\Omega_j} \in \mathcal{N}$. Then

$$\begin{aligned} \Phi''(u)[v_j, v_j] &= \int_\Omega |\nabla v_j|^2 \, \mathrm{d}x - \int_\Omega f_t(x, u) v_j^2 \, \mathrm{d}x \\ &< \int_\Omega |\nabla v_j|^2 \, \mathrm{d}x - \int_\Omega f(x, v_j) v_j \, \mathrm{d}x \\ &= 0 \end{aligned}$$

and therefore, $\mu(u) \geqslant n$. □

The results of this section should be compared with Courant's nodal domain theorem for eigenfunctions of the Laplacian.

2.4. *The existence of nodal solutions*

In this subsection $\Omega \subset \mathbb{R}^N$ is required to be bounded.

THEOREM 2.14. *Suppose* (f_0') *with* $p < 2^*$ *and* (f_2) *hold. If the second Dirichlet eigenvalue* $\lambda_2(-\Delta + a(x)) > 0$, *then* (2.1) *has a nodal solution*.

PROOF. We first observe that (f_0') and (f_2) imply (f_5), so there exists $m > 0$ with $f(x,t)t + mt^2 > 0$ for $t \neq 0$, $x \in \Omega$. Proceeding as in Example 2.5 we consider the scalar product $\langle u, v\rangle_m := \langle \nabla u, \nabla v\rangle_{L^2} + m\langle u, v\rangle_{L^2}$ on $E = H_0^1(\Omega)$ and let φ^t be the corresponding negative gradient flow of the energy functional associated to (2.1).

Let $e_1 > 0$, e_2 be linearly independent elements of $X = E \cap \mathcal{C}^1(\overline{\Omega})$. By (f_2), there exist $R > 0$ so that $\Phi(u) \leqslant 0$ for $u \in \operatorname{span}\{e_1, e_2\}$, $\|u\| \geqslant R$. We define

$$C := \{u = se_1 + te_2 \colon s \in [-R, R], t \in [0, R]\}$$

and $D := \partial C \subset \operatorname{span}\{e_1, e_2\}$. Let $E_1 \subset E$ be the first eigenspace of $-\Delta + a(x)$. Since $\lambda_2(-\Delta + a(x)) > 0$, there exist $\alpha, \rho > 0$ so that

$$\Phi(u) \geqslant \alpha \quad \text{for } u \in E_1^{\perp}, \|u\| = \rho.$$

Setting $S := \{u \in E_1^{\perp} \colon \|u\| = \rho\}$ the inclusion $(C, D) \hookrightarrow (E, E \setminus S)$ is well defined and not nullhomotopic. We fix $\beta \in (0, \alpha)$ so that (2.1) has no nodal solutions on the level β. If no such β exists the theorem is proved anyway.

As stated in Example 2.4, the flow φ^t leaves X invariant and $B := (X \cap \Phi^{\beta}) \cup P_X^+ \cup P_X^-$ strictly positively invariant. By construction we have $B \cap S = \emptyset$ which implies that the inclusion $(C, D) \hookrightarrow (X, B)$ is not nullhomotopic. Now Lemma 2.3 yields $u \in C$ so that $\varphi^t(u) \notin B$ for every $t > 0$. It follows that there exists a sequence $t_n \to \infty$ with $\|\nabla\Phi(\varphi^{t_n}(u))\|_m^2 = |\frac{\mathrm{d}}{\mathrm{d}t}\Phi(\varphi^{t_n}(u))| \to 0$ as $n \to \infty$.

Thus $u_n := \varphi^{t_n}(u)$, $n \in \mathbb{N}$, is a $(\mathrm{PS})_c$-sequence for some $c \geqslant \beta$, hence $u_n \to \bar{u}$ in E along a subsequence. By the ω-limit lemma from [10] $u_n \to \bar{u}$ in X. This implies $\bar{u} \notin B$ because $\bar{u}$ is a critical point of Φ, $u_n \notin B$, and B is strictly positively invariant. Thus $\bar{u}$ is a nodal solution of (2.1). □

THEOREM 2.15. *Suppose* (f_0') *with* $p < 2^*$ *and* (f_2) *hold. If* $0 \notin \sigma(-\Delta + a(x))$ *then* (2.1) *has a nodal solution.*

This theorem is a simple consequence of basic Morse theory which we recall here.

THEOREM 2.16. *Let X be a Banach space, $\Phi \in C^1(X, \mathbb{R})$, and φ^t a (local) flow on X with the properties*:

(GRAD) *If $\Phi'(u) \neq 0$ then $t \mapsto \Phi(\varphi^t(u))$ is strictly decreasing.*

(COMP) *If $\Phi(\varphi^t(u)) \to c \in \mathbb{R}$ as $t \to \infty$ then $\{\varphi^t(u) \colon t \geqslant 0\}$ is relatively compact.*

Let H_* be any homology theory and let $C_k(\Phi, u) := H_k(\Phi^c, \Phi^c \setminus \{u\})$, $c = \Phi(u)$, be the kth critical group of Φ at u. Finally, let $B \subset X$ be strictly positively invariant such that Φ has only finitely many critical points $u_1, \ldots, u_k \in X \setminus B$. Setting $P_k := \operatorname{rank} H_k(X, B)$ and $\beta_k(u_i) := \operatorname{rank} C_k(\Phi, u_i)$, there exists $Q \in \mathbb{N}_0[[t]]$ such that

$$\sum_{k=0}^{\infty}\left(\sum_{j=1}^{n}\beta_k(u_i)\right)t^k = \sum_{k=0}^{\infty} P_k t^k + (1+t)Q(t).$$

The proof is a variation of standard arguments, cf. [41], Theorem I.4.3.

PROOF OF THEOREM 2.15. We consider Φ, φ^t and X as in the proof of Theorem 2.14 and recall that the conditions (GRAD) and (COMP) hold. By Theorem 2.14, we may assume $\lambda_1(-\Delta + a(x)) < 0$. Then $\alpha e_1 > 0$ is a strict supersolution of (2.1) for $\alpha > 0$ small. Here $e_1 > 0$ is a first Dirichlet eigenfunction of $-\Delta + a(x)$. By Example 2.4, $A := (\alpha e_1 + P_X^+) \cup (-\alpha e_1 + P_X^-)$ is strictly positively invariant. Condition (f_2) implies that $H_k(X, \Phi^{-\beta}) = \{0\}$ for $\beta > 0$ large (see [41], Lemma III.2.3), any $k \in \mathbb{Z}$. (f_2) also implies $\Phi(tu) \to -\infty$ as $t \to \infty$ for $u \neq 0$. It is now easy to show that $B := A \cup \Phi^{-\beta}$ can be deformed into $\Phi^{-\beta}$ by a deformation of the form $(t, u) \mapsto tu$ for $u \in B$, $0 \leqslant t \leqslant T(u)$. As a consequence $H_k(X, B) = \{0\}$, $k \in \mathbb{Z}$. Since B is strictly positively invariant, we can apply Theorem 2.16. By assumption, $0 \in X \setminus B$ is nondegenerate, hence $C_\mu(\Phi, 0) \neq \{0\}$, where μ is the Morse index of 0. It follows that there exists a critical point $\bar{u} \in X \setminus B$, $\bar{u} \neq 0$, with $C_{\mu+1}(\Phi, \bar{u}) \neq \{0\}$ or $C_{\mu-1}(\Phi, \bar{u}) \neq \{0\}$. Clearly $\bar{u}$ is a nodal solution of (2.1). □

Next we assume (f_1), that is, $t \mapsto f(x,t)/|t|$ is strictly increasing on $\mathbb{R}^+$ and on $\mathbb{R}^-$. Recall the nodal Nehari set

$$\mathcal{S} = \left\{ u \in E \colon u^\pm \neq 0, \int_\Omega \nabla u \nabla u^\pm = \int_\Omega f(x,u) u^\pm \right\}$$

from Section 2.1.

THEOREM 2.17. *Suppose* (f_0') *with* $p < 2^*$, (f_1') *and* (f_2) *hold. If* $\lambda_2(-\Delta + a(x)) > 0$ *then* $\inf \Phi(\mathcal{S}) > 0$ *and it is achieved by a nodal solution with precisely two nodal domains and Morse index* 2.

A minimizer of $\Phi|_\mathcal{S}$ is a least energy nodal solution. Geometric properties of these on radial domains will be investigated in Section 2.5.

PROOF OF THEOREM 2.17. Consider a minimizing sequence $e_n \in \mathcal{S}$, $n \in \mathbb{N}$, of $\Phi|_\mathcal{S}$, and define

$$C_n := \left\{ u = s e_n^+ + t e_n^- \colon s \in [-R_n, R_n], t \in [0, R_n] \right\},$$

where $R_n > 0$ is chosen so that $\Phi(u) \leqslant 0$ for $u \in \operatorname{span}\{e_n^+, e_n^-\}$, $\|u\| \geqslant R_n$. Observe that $\Phi(e_n) = \max \Phi(C_n)$ as in (2.2). Now, the same argument as in the proof of Theorem 2.14 yields a nodal solution $u_n \in \mathcal{S}$ with $\Phi(u_n) \leqslant \Phi(e_n)$. Moreover, $\Phi(u_n)$ is bounded away from 0. In fact, $\Phi(u_n) \geqslant \inf \Phi(\{u \in E_1^\perp \colon \|u\| = \rho\}) > 0$, where E_1 is the first eigenspace of $-\Delta + a(x)$ and $\rho > 0$ is small. Thus $(u_n)_n$ is a minimizing sequence of $\Phi|_\mathcal{S}$ consisting of critical points. Thus $\inf \Phi|_\mathcal{S}$ is achieved by the Palais–Smale condition and because $\mathcal{S}$ is closed in E. The equality $\operatorname{nod}(\bar{u}) = 2$ for a minimizer $\bar{u}$ follows from Proposition 2.12. By Remark 2.2(b), $\mathcal{S} \cap H^2(\Omega)$ is a submanifold of $E \cap H^2(\Omega)$ with co-dimension 2. The Hessian $\Phi''(\bar{u})$ of Φ at a minimizer $\bar{u}$ of $\Phi|_\mathcal{S}$ is therefore positive semidefinite on the tangent space $V := T_{\bar{u}}(\mathcal{S} \cap H^2(\Omega)) \subset E \cap H^2(\Omega)$ which has co-dimension 2 in

$E \cap H^2(\Omega)$. It follows that $\Phi''(u)$ is positive semidefinite on $\overline{V}^E \subset E$ which has codimension 2 in E. $\square$

REMARK 2.18. (a) The first existence results for one nodal solution under similar hypotheses as those considered here are due to [15,36,50]. Improved versions and related results can be found in [17,37,45]. In [90] of Wang, besides the positive and negative solutions given in [5] by Ambrosetti and Rabinowitz a third solution was found by two different methods: linking and the Morse theory. No nodal information was given for the third solution in [90], but the methods used there seem to be suggestive for the nodal nature of the solution, motivating some of the results mentioned above. Using symmetries of the domain, the existence of nodal solutions can be proved more easily. Results of this type abound, in particular, for radially symmetric domains.

(b) In the recent paper [67] of Liu and Wang, it is proved that $\inf \Phi(\mathcal{S}) > 0$ is achieved by a nodal solution with precisely two nodal domains provided (f_0), (f_1) and the condition (f_3), that is, $\lim_{|t|\to\infty} F(x,t)/t^2 = \infty$ uniformly in $x \in \Omega$, hold. Observe that this condition is weaker than the classical Ambrosetti–Rabinowitz condition (f_2). Under the same conditions a positive and a negative solutions are given as in Theorem 1.23. The proofs is in the same spirit of that for Theorem 1.23.

2.5. *Geometric properties of least energy nodal solutions on radial domains*

We consider

$$\begin{cases} -\Delta u + a\big(|x|\big)u = f\big(|x|, u\big) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

when Ω is a ball or an annulus and a and f are radial. It is then natural to ask about the symmetry of solutions of (2.4). We shall address this question for the least energy nodal solution when it exists.

THEOREM 2.19. *Suppose Ω is a ball or an annulus, $a \in L^\infty(\Omega)$ and f are radial, and f satisfies* (f_0) *and* (f_1). *Suppose, moreover, that f is Hölder continuous on $\Omega \times [-R, R]$ for every $R > 0$. Then every least energy nodal solution is foliated Schwarz symmetric.*

This result is due to [19]. Its proof is based on an elementary symmetrization, the polarization, and the maximum principle as the one of Theorem 1.24.

If $f = f(u)$ is independent of x the problem whether a least energy nodal solution is radial has been settled recently by Aftalion and Pacella [1]. The remaining part of this section contains several results from [1]. We begin with the following observation relating the number of nodal domains to the Morse index.

THEOREM 2.20. *Suppose Ω is a ball or an annulus, and $f \in C^1(\mathbb{R})$ satisfies $f(0) \geqslant 0$. Let u be a radial solution of*

$$\begin{cases} -\Delta u = f(u) & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega. \end{cases}$$

Then the Morse index of u is at least $(N+1)(\mathrm{nod}(u)-1)$.

PROOF. Let $\Omega = A(r,R) = \mathrm{int}\{x \in \mathbb{R}^N\colon\ r \leqslant |x| \leqslant R\}$ and let $r = r_0 < r_1 < \cdots < r_{k-1} < r_k = R$ be such that $A_i = A(r_{i-1}, r_i)$, $i = 1,\dots,k = \mathrm{nod}(u)$, are the nodal domains of u. We consider the domains $B_i = A(r_{i-1}, r_{i+1})$ and $B_{ij} = \{x \in B_i\colon\ x_j < 0\}$, $i = 1,\dots,k-1$, $j = 1,\dots,N$. Let μ_{ij} be the first Dirichlet eigenvalue of $-\Delta - f'(u)$ in B_{ij} and $\psi_{ij} > 0$, a corresponding eigenfunction. We claim that $\mu_{ij} < 0$. If this has been proved let $v_{ij} \in H_0^1(B_i) \subset H_0^1(\Omega)$ be the extension of ψ_{ij} which is odd in x_j. Then v_{ij} is a Dirichlet eigenfunction of $-\Delta - f'(u)$ in B_i with eigenvalue $\mu_{ij} < 0$. Since v_{ij} changes sign, there exists a positive eigenfunction $v_{i0} \in H_0^1(B_i) \subset H_0^1(\Omega)$ of $-\Delta - f'(u)$ with eigenvalue $\mu_{i0} < \mu_{ij} < 0$. It follows that the quadratic form $v \mapsto \langle(-\Delta - f'(u))v, v\rangle_{L^2}$ is negative on $\mathrm{span}\{v_{ij}\colon\ i = 1,\dots,k-1,\ j = 0,\dots,N\}$. Since the v_{ij}, $i = 1,\dots,k-1$, $j = 0,\dots,N$, are linearly independent by construction, the negative eigenspace of $-\Delta - f'(u)$ in $H_0^1(\Omega)$ has dimension at least $(k-1)(N+1) = (\mathrm{nod}(u)-1)(N+1)$.

It remains to prove $\mu_{ij} < 0$. Observe that $u_j := \partial u/\partial x_j$ solves $-\Delta u_j = f'(u)u_j$ in Ω. Let $\partial_0 B_i = \{x \in \mathbb{R}^N\colon\ |x| = r_{i+1}\}$ be the outer boundary of B_i. Since u is radial, $u_j(x) = 0$ if $x_j = 0$. Suppose $u > 0$ on $A(r_i, r_{i+1}) \subset B_i$ and $u < 0$ on $A(r_{i-1}, r_i) \subset B_i$. Then $u_j(x) > 0$ for $x \in \partial B_i$ with $x_j < 0$, and $C_{ij} := \{x \subset B_{ij}\colon\ u_j(x) < 0\} \subsetneq B_{ij}$ is a nonempty open subset of B_{ij} with $\partial C_{ij} \subset B_{ij} \cup \{x\colon\ x_j = 0\}$.

It follows that $u_j = 0$ on ∂C_{ij}, hence $u_j \in H_0^1(C_{ij}) \subset H_0^1(B_{ij})$. Therefore

$$
\begin{aligned}
\mu_{ij} &= \inf_{v \in H_0^1(B_{ij})} \frac{\langle(-\Delta - f'(u))v, v\rangle_{L^2(B_{ij})}}{\|v\|_{L^2(B_{ij})}} \\
&< \frac{\langle(-\Delta - f'(u))u_j, u_j\rangle_{L^2(C_{ij})}}{\|u_j\|_{L^2(C_{ij})}} = 0.
\end{aligned}
$$
□

COROLLARY 2.21. *Suppose Ω is a ball or an annulus, and $f:\mathbb{R} \to \mathbb{R}$ is independent of $x \in \Omega$ and satisfies* (f_0') *and* (f_1'). *Then a least energy nodal solution of* (2.1) *is not radial.*

PROOF. The proof of Theorem 2.17 shows that a minimizer of Φ on $\mathcal{S}$ has Morse index 2. Now the result follows from Theorem 2.20. □

THEOREM 2.22. *Suppose Ω is a ball, and $f = f(u)$ satisfies* (f_0') *and* (f_1'). *Let u be a least energy nodal solution of* (2.1). *Then the nodal set* $\mathrm{clos}(u^{-1}(0) \cap \Omega)$ *intersects* $\partial\Omega$.

PROOF. Arguing by contradiction we suppose u has constant sign near $\partial\Omega$. By Theorem 2.19, we may assume that u is axially symmetric around the axis $\{0\} \times \mathbb{R} \subset \mathbb{R}^N$. In particular, u is even in $x_1,\dots,x_{N-1}$. Now, as in the proof of Theorem 2.20, one sees that the Morse index of u is at least N. Thus we are done if $N \geqslant 3$ because the Morse index of u is 2. In the case $N=2$, one can use the angle derivative $\partial u/\partial\theta$ and the fact that u is foliated Schwarz symmetric but not radial. Details can be found in [1]. □

REMARK 2.23. It is tempting to conjecture that in the situation of Theorem 2.22 and its proof, the least energy nodal solution u, being even in $x_1,\dots,x_{N-1}$, is odd in x_N and that its nodal surface is the set $\{x\in\Omega\colon\ x_N=0\}$. For the singularly perturbed problem

$$\begin{cases} -\varepsilon^2\Delta u+u=|u|^{p-2}u & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases}$$

this has been proved by Wei and Winter [94] if ε is small.

2.6. *Multiple nodal solutions on a bounded domain*

On an interval $\Omega=(a,b)\subset\mathbb{R}$, the problem,

$$\begin{cases} -u''+a(x)u=f(x,u) & \text{for } a<x<b,\\ u(a)=u(b)=0, \end{cases} \tag{2.5}$$

has infinitely many nodal solutions if f is superlinear in the sense of (f_2). A rather general result in this direction is due to Struwe [87]. It is an open problem whether (2.1) has infinitely many solutions assuming (f_0) with $p<2^*$ and (f_2). The existence of multiple (nodal) solutions has been proved under additional conditions on the domain or on f. We present several such results in this and the next subsection.

THEOREM 2.24. *Suppose f satisfies* (f_0') *with* $p<2^*$, (f_1') *and* (f_2). *If in addition f is odd in u then there exists a sequence $\pm u_k$, $k\geqslant k_0$, of nodal solutions of* (2.1) *with the properties*:

(a) $\|u_k\|_E\to\infty$ *as* $k\to\infty$;
(b) $2\leqslant \operatorname{nod}(u_k)\leqslant k$;
(c) *if $u<u_k$ is a subsolution of* (2.1) *then* $u\leqslant 0$;
(d) *if $u>u_k$ is a supersolution of* (2.1) *then* $u\geqslant 0$.

Here $k_0=\mu(-\Delta+a(x))+\dim\ker(-\Delta+a(x))+1$, where $\mu(-\Delta+a(x))$ denotes the Morse index. The existence of infinitely many solutions as well as property (a) is a classical result of Ambrosetti and Rabinowitz [5] based on the symmetric mountain pass theorem. The fact that the solutions are nodal and properties (b)–(d) has been proved in [10,17] where co-homological linking properties are being used; see also [61], where the existence of nodal solutions is proved using the Lusternik–Schnirelmann theory. It is not known whether $\operatorname{nod}(u_k)\to\infty$ as $k\to\infty$. In fact, it is not even known whether $\operatorname{nod}(u_k)\geqslant 3$ for some k. In the following, we sketch the proof from [61] for the existence of infinitely many nodal solutions using a variant of the symmetric mountain pass theorem.

PROOF OF THEOREM 2.24. Set $\lambda_i = \lambda_i(-\Delta + a(x))$, $E_i = \ker(-\Delta + a(x) - \lambda_i)$, $Y_k = \bigoplus_{i=1}^k E_i$ and $Z_k = \bigoplus_{i=k}^\infty E_i$. Then it is easy to check that, for all k with $\lambda_k > 0$, there are $\rho_k > r_k > 0$ such that $\sup_{Y_k \cap B_{\rho_k}(0)} \Phi(u) \leqslant a_k := 0$ and $b_k := \inf_{Z_k \cap \partial B_{r_k}(0)} \Phi(u) > 0$ and $b_k \to \infty$. Let $X = \mathcal{C}^1(\overline{\Omega}) \cap E$. Then $Y_k \subset X$ for any $k \geqslant 1$. Let $P_X^\pm$ be as in Example 2.4. Then $P_X^\pm$ are invariant sets of the negative gradient flow. Moreover, for $k \geqslant 2$, $Z_k \cap P_X^\pm = \{0\}$. This can be seen by noting that, for all $u \in P^\pm \setminus \{0\}$, $\int_\Omega u\phi_1 \,\mathrm{d}x \neq 0$ while for $u \in Z_k$, $\int_\Omega u\phi_1 \,\mathrm{d}x = 0$, where ϕ_1 is the first eigenfunction of the Laplacian operator on Ω. As a consequence of the Borsuk–Ulam theorem, the inclusion $(B_k, \partial B_k) \hookrightarrow (E, \Phi^0 \cup P^+ \cup P^-)$ is not nullhomotopic; here $B_k = \{u \in Y_k\colon \|u\| \leqslant \rho_k\}$. Lemma 2.3 or the deformation lemma from [61] or the proof of [61], Theorem 3.2, yield a critical point outside of $P^+ \cup P^-$. In fact, it follows that

$$c_k := \inf_{h \in \Gamma_k} \sup_{u \in h(B_k) \setminus (P_X^+ \cup P_X^-)} \Phi(u) \geqslant b_k$$

is a critical value with a critical point in $E \setminus (P^+ \cup P^-)$; here $\Gamma_k = \{h \in C(B_k, X)\colon h$ is odd, and $h(u) = u$ if $\|u\| = \rho_k\}$. □

The idea in the above proof can be adapted for many other boundary value problems with different nonlinearity; see [61] for more examples. See also [92] for an abstract version of a variational principle in the presence of invariant sets of the flows. Nodal solutions for nonlinear eigenvalue problems with superlinear nonlinearity have been studied in [47, 61,63].

Without symmetry of the domain or the nonlinearity it is not known whether the superlinear problems considered here have "many", even infinitely many solutions. Now we discuss the role of the domain. If Ω is a radial domain and $f = f(|x|, u)$ is radially symmetric in x then one can look for radially symmetric solutions $u(x) = v(|x|)$. Rewriting (2.1) as an ordinary differential equation for $v(r)$, ODE methods can be applied. In this setting there exist solutions with any prescribed number of nodal domains for a large class of superlinear nonlinearities.

The following recent result gives a lower bound on the number of solutions when the domain is large in the sense that it contains balls of large sizes.

THEOREM 2.25. *Suppose $f = f(u)$ is independent of $x \in \Omega$ and satisfies* (f_0') *with* $p < 2^*$, (f_1'), (f_2), *and $f'(0) < 0$. Then there exists $R > 0$ such that if Ω contains a ball of radius R then* (2.1) *has at least* 3 *nodal solutions u_1, u_2, u_3 with* $\mathrm{nod}(u_1) = \mathrm{nod}(u_2) = 2$, $2 \leqslant \mathrm{nod}(u_3) \leqslant 3$.

The proof can be found in [18]. This result is in some sense of the singularly perturbed type; cf. Remark 1.26. It yields 3 nodal solutions of the singularly perturbed problem (1.15) on any domain. As mentioned in Remark 1.26 there is a huge literature about singularly perturbed problems, which we cannot present in this chapter.

2.7. *Perturbed symmetric functionals*

It is natural to ask whether the solutions continue to exist if an odd function is perturbed, e.g., $f(x,u) = |u|^{p-2}u + g(x)$. This subsection is devoted to semilinear elliptic problems of the type

$$\begin{cases} -\Delta u = |u|^{p-2}u + h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.6}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth and bounded domain, $2 < p < 2^*$ and $h \in L^2(\Omega)$ is fixed. Solutions of (2.6) are critical points of

$$\Phi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x - \frac{1}{p}\int_\Omega |u|^p \,\mathrm{d}x - \int_\Omega hu \,\mathrm{d}x, \quad u \in H_0^1(\Omega).$$

When $h = 0$ and $p < 2^*$ the corresponding functional

$$\Phi_0(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x - \frac{1}{p}\int |u|^p \,\mathrm{d}x, \quad u \in H_0^1(\Omega),$$

is odd and has infinitely many critical points. The following result is due to Bahri and Lions [9] improving earlier work of Bahri and Berestycki [7] and, independently, Struwe [86].

THEOREM 2.26. *If* $2 < p < (2N-2)/(N-2)$ *then, for every* $h \in L^2(\Omega)$, *problem* (2.6) *has infinitely many solutions.*

The proof is based on the idea that Φ is a kind of perturbation of Φ_0. The symmetric linking which is based on the Borsuk–Ulam theorem and used in the proof of Theorem 2.24 applies to Φ_0 and yields a nonsymmetric linking for Φ. A different approach to perturbed symmetric functions is due to Bolle; see [27,28]. It is still not known whether (2.6) has infinitely many solutions for any $2 < p < 2^*$ and any h. However, Bahri proved in [6] the following generic existence result for the full range of p.

THEOREM 2.27. *If* $2 < p < 2^*$ *then the set of* $h \in H^{-1}(\Omega)$ *such that problem* (2.6) *has infinitely many solutions is a dense residual set in* $H^{-1}(\Omega)$.

3. Problems on the entire space

In this section we consider the problem

$$\begin{cases} -\Delta u + V(x)u = f(x,u), & x \in \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{3.1}$$

Depending upon the potential function $V(x)$ we review the existence of signed solutions, nodal solutions and multiplicity of solutions as well as their qualitative properties in some cases. Let

$$\Phi : E \to \mathbb{R}, \qquad \Phi(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2\,\mathrm{d}x - \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x$$

be the functional associated to (3.1). As usual, $F(x,t) = \int_0^t f(x,s)\,\mathrm{d}s$. The nonlinearity f will always satisfy (f_0) and is subcritical, so that Φ is defined on the space

$$E = \left\{u \in H^1(\mathbb{R}^N) \colon \int_{\mathbb{R}^N} V(x)u^2\,\mathrm{d}x < \infty\right\}$$

and is of class C^1. If V is bounded then $E = H^1(\mathbb{R}^N)$ and the Palais–Smale condition does not hold. On the contrary, if $V(x) \to \infty$ as $|x| \to \infty$ then Φ does satisfy the Palais–Smale condition. This compact case is most closely related to the bounded domain case in terms of the results and methods and will be dealt with in Section 3.1. If V and f are radially symmetric it is natural to look for radially symmetric solutions. Constraining E to the space E_{rad} of radial functions $u = u(|x|)$ compactness is recovered. We deal with the symmetric case in Section 3.2. In fact, in both Sections 3.1 and 3.2 weaker conditions than those just mentioned will be considered.

In the remaining subsections we deal with bounded, nonradial potentials, in particular with potentials which are periodic in the x_i-variables. Since f is subcritical this case may be called locally compact, a notion going back to Lions' seminal work [64]. Locally the problems have compactness, and the compactness is only lost from the mass going to infinity. Lions' concentration–compactness principle is an important tool in dealing with this and will be used. In Section 3.3 we consider potentials having a potential well whose steepness is controlled by a parameter. In Section 3.4 we present two results on the existence of a ground state solution for bounded potentials. In the periodic case, one can construct bound states having multibumps. The first result of this type is due to Coti Zelati and Rabinowitz [48] who constructed positive multibump solutions. Here we present the modified approach from [68] which can also be used to obtain nodal multibump-type solutions and to control the number of nodal domains. So far we always assumed $\inf V > 0$. Finally we discuss the case where V is periodic and negative somewhere. The periodicity implies that the spectrum of $-\Delta + V$ is purely continuous and consists of a disjoint union of closed intervals. In Section 3.6 we deal with the case where $-\Delta + V$ has essential spectrum below 0.

3.1. *The compact case*

We consider the space $E = \{u \in H^1(\mathbb{R}^N)\colon \int_{\mathbb{R}^N} V(x)u^2\,\mathrm{d}x < \infty\}$ together with the scalar product

$$\langle u, v\rangle_E = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v + \int_{\mathbb{R}^N} V(x)uv.$$

In general the embedding from E into $L^p(\mathbb{R}^N)$ is not compact, for example, when V is bounded so that $E = H^1(\mathbb{R}^N)$. We present general conditions on V which guarantee that the embedding is compact.

(V_0) $V \in C(\mathbb{R}^N, \mathbb{R})$ and $\inf_{\mathbb{R}^N} V(x) > 0$.

(V_1) There exists $r_0 > 0$ such that, for any $M > 0$,

$$\lim_{|y|\to\infty} m\big(\big\{x \in \mathbb{R}^N\colon\ |x - y| \leqslant r_0,\ V(x) \leqslant M\big\}\big) = 0,$$

where m denotes the Lebesgue measure in $\mathbb{R}^N$.

Without loss of generality we assume $\inf_{\mathbb{R}^N} V(x) = 1$.

LEMMA 3.1. *Under* (V_0) *and* (V_1), *the embedding from E into $L^p(\mathbb{R}^N)$ is compact for* $2 \leqslant p < 2^*$.

PROOF. Let (u_n) be bounded in E and assume $u_n \to 0$ weakly in E. We have to show that $u_n \to 0$ in $L^p(\mathbb{R}^N)$ for $2 \leqslant p < 2^*$. By the interpolation inequality we only need to consider $p = 2$. The Sobolev embedding theorem implies $u_n \to 0$ in L^2_{loc}. Thus it suffices to show that, for any $\varepsilon > 0$, there is $R > 0$ such that $\int_{B^c_R(0)} u_n^2 < \varepsilon$; here $B^c_R(0) = \mathbb{R}^N \setminus B_R(0)$. Let (y_i) be a sequence of points in $\mathbb{R}^N$ satisfying $\mathbb{R}^N \subset \bigcup_{i=1}^{\infty} B_{r_0}(y_i)$ and such that each point x is contained in at most 2^N such balls $B_{r_0}(y_i)$. Let $A_{R,M} = \{x \in B^c_R \mid V(x) \geqslant M\}$ and $B_{R,M} = \{x \in B^c_R \mid V(x) < M\}$. Then

$$\int_{A_{R,M}} u_n^2 \leqslant \frac{1}{M} \int_{\mathbb{R}^N} V(x) u_n^2,$$

and this can be made arbitrarily small by choosing M large. Choose q such that $2q \leqslant 2^*$ and let $q' = q/(q-1)$ be the dual exponent. Then for fixed $M > 0$,

$$\begin{aligned}
\int_{B_{R,M}} u_n^2 &\leqslant \sum_{i=1}^{\infty} \int_{B_{R,M}\cap B_{r_0}(y_i)} u_n^2 \\
&\leqslant \sum_{i=1}^{\infty} \biggl(\int_{B_{R,M}\cap B_{r_0}(y_i)} \bigl(|u_n|^{2q}\bigr)\biggr)^{1/q} \bigl(m\bigl(B_{R,M} \cap B_{r_0}(y_i)\bigr)\bigr)^{1/q'} \\
&\leqslant \varepsilon_R \sum_{i=1}^{\infty} \biggl(\int_{B_{R,M}\cap B_{r_0}(y_i)} |u_n|^{2q}\biggr)^{1/q} \\
&\leqslant C\varepsilon_R \sum_{i=1}^{\infty} \int_{B_{r_0}(y_i)} \bigl(|\nabla u_n|^2 + u_n^2\bigr) \\
&\leqslant C\varepsilon_R 2^N \int_{\mathbb{R}^N} \bigl(|\nabla u_n|^2 + |u_n|^2\bigr),
\end{aligned}$$

where $\varepsilon_R = \sup_{y_i}(m(B_{R,M} \cap B_{r_0}(y_i)))^{1/q'}$. Assumption (V_1) states that $\varepsilon_R \to 0$ as $R \to \infty$. Thus we may make this term small by choosing R large. □

Concerning the nonlinearity f we recall the following conditions.

(f_0) $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with $f(x,t) = o(t)$ as $t \to 0$. There exist $C > 0$ and $p \leqslant 2N/(N-2)$ such that $|f(x,t)| \leqslant C(|t| + |t|^{p-1})$ for all $x \in \mathbb{R}^N$, $t \in \mathbb{R}$.

(f_1) For every $x \in \mathbb{R}^N$, the function $\mathbb{R} \setminus \{0\} \to \mathbb{R}$, $t \mapsto f(x,t)/|t|$, is strictly increasing.

(f_2) There exists $\theta > 2$ such that $0 < \theta F(x,t) \leqslant f(x,t)t$ for all $x \in \mathbb{R}^N$, all $t \in \mathbb{R}$.

The first result is concerned with minimizers of Φ on the Nehari manifold

$$\mathcal{N} = \{u \in E \setminus \{0\}:\ \Phi'(u)u = 0\}.$$

In fact, we find a least energy positive and a least energy negative solution, that is, we minimize Φ on

$$\mathcal{N}_\pm = \{u \in \mathcal{N}:\ \pm u \geqslant 0\}.$$

THEOREM 3.2. *If* (V_0), (V_1), (f_0) *with* $p < 2^*$, (f_1) *and* (f_2) *hold then* $\inf\{\Phi(u):\ u \in \mathcal{N}_\pm\}$ *is achieved, hence* (3.1) *has a least energy positive solution and a least energy negative solution.*

PROOF. Using the compact embedding lemma (Lemma 3.1) the proof proceeds as the one of Theorem 1.23. □

Next we consider the nodal Nehari set

$$\begin{aligned}\mathcal{S} &= \{u \in E:\ u^+ \in \mathcal{N}, u^- \in \mathcal{N}\}\\ &= \{u \in E:\ u^+ \neq 0 \neq u^-, \Phi'(u)u^+ = 0 = \Phi'(u)u^-\}\end{aligned}$$

and find a least energy nodal solution.

THEOREM 3.3. *Suppose* (V_0), (V_1), (f_0) *with* $p < 2^*$, (f_1) *and* (f_2) *hold. Then* $\inf \Phi|_{\mathcal{S}}$ *is achieved by a nodal solution that has exactly two nodal domains.*

PROOF. Using the compact embedding Lemma 3.1 the proof proceeds as the one of Theorem 2.17. □

REMARK 3.4. As commented in Remark 2.18, (f_2) can be replaced by the weaker condition (f_3). This is done in [67].

THEOREM 3.5. *Assume* (V_0), (V_1), (f_0) *with* $p < 2^*$, (f_1) *and* (f_2). *If f is odd in u, then* (3.1) *has an unbounded sequence of nodal solutions u_k such that* $2 \leqslant \mathrm{nod}(u_k) \leqslant k$.

Details of the proofs of these and related results can be found in [12].

REMARK 3.6. Under the stronger condition that $\int_{\mathbb{R}^N} V(x)^{-1}\,\mathrm{d}x < \infty$, a nonlinearity with a combination of convex and concave terms was considered recently by Liu and Wang in [69] in which multiplicity of nodal solutions was given.

3.2. *The radially symmetric case*

We consider the following class of symmetric functions. We fix a decomposition $N = N_1 + \cdots + N_k$ with $N_i \geqslant 2$. For $x \in \mathbb{R}^N$, we write $x = (x_1, \ldots, x_k)$ with $x_i \in \mathbb{R}^{N_i}$ and define

$$H^1_{\mathrm{sym}}(\mathbb{R}^N) = \{u \in H^1(\mathbb{R}^N) \colon u(x) = u(|x_1|, \ldots, |x_k|)\}.$$

If $k = 1$ then $H^1_{\mathrm{sym}}(\mathbb{R}^N)$ consists precisely of the radial H^1-functions.

THEOREM 3.7. *The embedding of* $H^1_{\mathrm{sym}}(\mathbb{R}^N)$ *into* $L^p(\mathbb{R}^N)$ *is compact for* $2 < p < 2^*$.

For the proof we need the following lemma due to Lions [64].

LEMMA 3.8. *Let* (u_n) *be a bounded sequence in* $H^1(\mathbb{R}^N)$. *Assume that there is* $r > 0$ *and* $2 \leqslant q \leqslant 2^*$ *such that*

$$\limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_r(y)} |u_n|^q \,\mathrm{d}x = 0.$$

Then $\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^p \,\mathrm{d}x = 0$ *for* $2 < p < 2^*$. *If* $q = 2^*$, p *can be taken as* 2^*.

Now we give the proof of Theorem 3.7.

PROOF OF THEOREM 3.7. Let (u_n) be a bounded sequence in $H^1_{\mathrm{sym}}(\mathbb{R}^N)$ which we may assume converges weakly to 0 in $H^1_{\mathrm{sym}}(\mathbb{R}^N)$. We want to show that (u_n) converges to 0 in $L^p(\mathbb{R}^N)$. By Lemma 3.8, if u_n does not converge to zero in $L^p(\mathbb{R}^N)$ there are $r > 0$, $\delta > 0$ and $y_n \in \mathbb{R}^N$ such that

$$\liminf_{n\to\infty} \int_{B_r(y_n)} |u_n|^p \geqslant \delta.$$

Since the embedding $H^1(\Omega) \hookrightarrow L^p(\Omega)$ is compact for bounded domains Ω, we have $|y_n| \to \infty$. Up to a subsequence we may assume that there is $1 \leqslant i \leqslant k$ such that the ith components $y_{n,i} \in \mathbb{R}^{N_i}$ of (y_n) go to infinity. Using the radial symmetry in $\mathbb{R}^{N_i}$, as $n \to \infty$ we find more and more disjoint balls of radius r on which the L^p norm of u_n is the same as $\int_{B_r(y_n)} |u_n|^p$, contradicting the fact that u_n is bounded in $H^1_{\mathrm{sym}}(\mathbb{R}^N)$ and therefore in $L^p(\mathbb{R}^N)$. □

(S) $V(x)$ and $f(x,u)$ are radially symmetric in each of the variables $x_i \in \mathbb{R}^{N_i}$, $i = 1,\dots,k$.

If (S) holds then a critical point of Φ constrained to $H^1_{\mathrm{sym}}(\mathbb{R}^N)$ is a critical point of Φ by the principle of symmetric criticality. Thus, with the compactness Theorem 3.7 available we easily obtain existence results as in the last section. The proofs are very similar and we omit them.

THEOREM 3.9. *Assume* (V_0), (f_0) *with* $p < 2^*$, (f_1), (f_2) *and* (S). *Then the infimum of* Φ *on* $\mathcal{N}^{\pm} \cap H^1_{\mathrm{sym}}(\mathbb{R}^N)$ *is achieved, hence* (3.1) *has a least energy positive solution and a least energy negative solution in* $H^1_{\mathrm{sym}}(\mathbb{R}^N)$.

THEOREM 3.10. *Assume* (V_0), (f_0) *with* $p < 2^*$, (f_1), (f_2) *and* (S). *Then* (3.1) *has a nodal solution in* $H^1_{\mathrm{sym}}(\mathbb{R}^N)$ *which has exactly two nodal domains.*

THEOREM 3.11. *Assume* (V_0), (f_0) *with* $p < 2^*$, (f_1), (f_2) *and* (S). *If* f *is odd in* u, *then* (3.1) *has an unbounded sequence of nodal solutions* $u_k \in H^1_{\mathrm{sym}}(\mathbb{R}^N)$ *with* $2 \leqslant \mathrm{nod}(u_k) \leqslant k$.

REMARK 3.12. If the equation is autonomous then radial solutions have been obtained by Berestycki and Lions [25]. The autonomous case has been further investigated by many authors. For more recent results, see the paper by Jeanjean and Tanaka [55] and the references therein. Nonradial solutions have been obtained in dimensions $N = 4$ or $N \geqslant 6$ by Bartsch and Willem [20] provided f is odd and V and f are radial functions of x. In dimension $N = 5$ a nonradial solution can be obtained combining the idea from [20] with the concentration–compactness method; see the paper [71] by Lorca and Ubilla. Whether or not a nonradial solution exists in dimension $N = 3$ is open.

3.3. *The steep potential well case*

In this section, we consider potentials depending on a parameter that controls the depth of the potential well. More precisely, we want to find bound states of the equation

$$(\mathrm{S}_\lambda) \qquad -\Delta u + \big(\lambda a(x) + 1\big)u = f(x,u) \quad \text{in } \mathbb{R}^N.$$

We require the following assumptions on the potential function $V_\lambda(x) = \lambda a(x) + 1$.

(a_1) $a : \mathbb{R}^N \to \mathbb{R}$ is continuous, $a \geqslant 0$, $\Omega := \operatorname{int} a^{-1}(0) \neq \emptyset$, has smooth boundary and $\overline{\Omega} = a^{-1}(0)$.

(a_2) There exist $M_0 > 0$ and $r_0 > 0$ such that

$$\lim_{|y|\to\infty} m\big(\big\{x \in \mathbb{R}^N \colon |x - y| \leqslant r_0,\, a(x) \leqslant M_0\big\}\big) < \infty,$$

where m denotes the Lebesgue measure on $\mathbb{R}^N$.

The set Ω in assumption (a_1) is the bottom of the potential well. It is allowed that Ω is unbounded. It is also allowed that a is bounded. In view of the results and the proofs from the last section we need to study the compactness of the problem first. On the space $E = \{u \in H^1(\mathbb{R}^N)\colon \int_{\mathbb{R}^N} a(x)u^2\,\mathrm{d}x < \infty\}$ we use the norms

$$\|u\|_\lambda^2 = \int_{\mathbb{R}^N} \big(|\nabla u|^2 + \big(\lambda a(x) + 1\big)u^2\big)\,\mathrm{d}x \quad \text{for } \lambda \geqslant 0,$$

which are equivalent norms on E. We shall occasionally write E_λ for the Hilbert space E equipped with the norm $\|\cdot\|_\lambda$. The functional associated to (S_λ) is given by

$$\begin{aligned}\Phi_\lambda(u) &= \frac{1}{2}\int_{\mathbb{R}^N} \big(|\nabla u|^2 + \lambda\big(a(x)+1\big)u^2\big)\,\mathrm{d}x - \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x \\ &= \frac{1}{2}\|u\|_\lambda^2 - \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x. \end{aligned} \tag{3.2}$$

If (f_0) holds we have $\Phi_\lambda \in C^1(E_\lambda, \mathbb{R})$ for any $\lambda \geqslant 0$. We have the following parameter-dependent compactness result for Φ_λ.

PROPOSITION 3.13. *Suppose* (a_1), (a_2), (f_0) *and* (f_2). *For any $C_0 > 0$ there exists $\Lambda_0 > 0$ such that Φ_λ satisfies the* $(\mathrm{PS})_c$*-condition for all $\lambda \geqslant \Lambda_0$ and all $c \leqslant C_0$.*

Since the proof of Proposition 3.13 is rather technical we just refer the reader to [13,16]. The result should also be compared with Theorem 3.1 in which global compactness is given.

THEOREM 3.14. *Assume* (a_1), (a_2), (f_0) *with $p < 2^*$ and* (f_2). *Then for λ large,* (S_λ) *has a positive solution and a negative solution.*

PROOF. If, in addition, (f_1) holds then one obtains the positive and the negative solution by minimizing on the positive (resp. negative) Nehari manifold as before, using the compactness result from Proposition 3.13. If (f_1) does not hold one may apply the mountain pass theorem instead of minimizing. □

THEOREM 3.15. *Assume* (a_1), (a_2), (f_0) *with $p < 2^*$,* (f_1) *and* (f_2). *Then for λ large,* (S_λ) *has a nodal solution which has exactly two nodal domains.*

PROOF. Here we minimize on the nodal Nehari set associated to the problem and observe that at the minimal level the Palais–Smale condition holds for λ large by Proposition 3.13. □

Details of the proofs of these two results can be found in [13,14].

THEOREM 3.16. *Assume* (a_1), (a_2), (f_0') *with $p < 2^*$,* (f_1) *and* (f_2). *If f is odd in u, then, for any integer k, there is Λ_k such that for $\lambda \geqslant \Lambda_k$, equation* ($S_\lambda$) *has k pairs of* *nodal solutions u_j, $j = 1, \dots, k$, such that the number of nodal domains of u_j is bounded by $j + 1$.*

For the existence of multiple nodal solutions we need to use a combination of a minimax procedure and invariant sets of the gradient flow as was done in [12]. We define $K_\lambda(u) := (-\Delta + V_\lambda)^{-1} f(\cdot, u)$, and consider the flow on E associated to

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\eta_\lambda^t(u) = -\eta_\lambda^t(u) + K_\lambda\big(\eta_\lambda^t(u)\big), \\ \eta_\lambda^0(u) = u. \end{cases}$$

We claim that the cones $P^\pm = \{u \in E\colon \pm u \geqslant 0\}$ are K_λ-attractive in the sense of Definition 2.6. This is in fact independent of λ. For any $M \subset E$ and $\varepsilon > 0$, M_ε denotes the closed ε-neighborhood of M, i.e.,

$$M_\varepsilon := \big\{u \in E\colon \operatorname{dist}_\lambda(u, M) \leqslant \varepsilon\big\}.$$

LEMMA 3.17. *Assume* (a$_1$), (a$_2$) *and* (f$_0$). *Then there exists $\varepsilon_0 > 0$ such that*

$$K_\lambda\big(\big(P^\pm\big)_\varepsilon\big) \subset \operatorname{int}\big(\big(P^\pm\big)_\varepsilon\big) \quad \textit{for all } 0 < \varepsilon \leqslant \varepsilon_0, \lambda \geqslant 0,$$

so $P^\pm$ is K_λ-attractive uniformly in λ. Consequently,

$$\eta_\lambda^t\big(\big(P^\pm\big)_\varepsilon\big) \subset \operatorname{int}\big(\big(P^\pm\big)_\varepsilon\big) \quad \textit{for all } t > 0, 0 < \varepsilon \leqslant \varepsilon_0 \textit{ and } \lambda \geqslant 0.$$

PROOF. We write $V_\lambda(x) = \lambda a(x) + 1$. For $u \in E$, we denote $v = K_\lambda(u)$ and $u^+ = \max\{0, u\}$, $u^- = \min\{0, u\}$. Note that, for any $u \in E$ and $2 \leqslant p \leqslant 2^*$,

$$\|u^-\|_{L^p} = \inf_{w \in P^+} \|u - w\|_{L^p}. \tag{3.3}$$

Since

$$\|v^-\|_\lambda^2 = (v, v^-)_\lambda = \int_{\mathbb{R}^N} (\nabla v \cdot \nabla v^- + V_\lambda v v^-)\,\mathrm{d}x = \int_{\mathbb{R}^N} f(x, u) v^-\,\mathrm{d}x,$$

the fact that $v^+ \in P^+$ and $v - v^+ = v^-$ implies

$$\operatorname{dist}_\lambda\big(v, P^+\big) \cdot \|v^-\|_\lambda \leqslant \|v^-\|_\lambda^2 \leqslant \int_{\mathbb{R}^N} f(x, u^-) v^-\,\mathrm{d}x. \tag{3.4}$$

As a consequence of (f$_0$) there exists a $\delta > 0$ and $C_1 > 0$ such that

$$f(x, s) \geqslant (1 - \delta)s + C_1|s|^{2^*-2}s \quad \text{for } s \leqslant 0.$$

Thus

$$\begin{aligned}
&\int_{\mathbb{R}^N} f(x, u^-)v^- \,\mathrm{d}x \\
&\quad \leqslant \int_{\mathbb{R}^N} \big((1-\delta)u^- + C_1|u^-|^{2^*-2}u^-\big)v^- \,\mathrm{d}x \\
&\quad \leqslant (1-\delta)\|u^-\|_{L^2}\|v^-\|_{L^2} + C_1\|u^-\|_{L^{2^*}}^{2^*-1}\|v^-\|_{L^{2^*}}. \qquad (3.5)
\end{aligned}$$

From the Sobolev embedding and (3.3)–(3.5),

$$\begin{aligned}
&\mathrm{dist}_\lambda\big(v, P^+\big) \cdot \|v^-\|_\lambda \\
&\quad \leqslant (1-\delta) \inf_{w\in P^+} \|u-w\|_{L^2}\|v^-\|_\lambda + C_2 \inf_{w\in P^+} \|u-w\|_{L^{2^*}}^{2^*-1}\|v^-\|_\lambda,
\end{aligned}$$

which implies (if $\|v^-\|_\lambda \neq 0$)

$$\begin{aligned}
\mathrm{dist}_\lambda\big(v, P^+\big) &\leqslant (1-\delta) \inf_{w\in P^+} \|u-w\|_{L^2} + C_2 \inf_{w\in P^+} \|u-w\|_{L^{2^*}}^{2^*-1} \\
&\leqslant (1-\delta) \inf_{w\in P^+} \|u-w\|_\lambda + C_3 \inf_{w\in P^+} \|u-w\|_\lambda^{2^*-1} \\
&= (1-\delta)\,\mathrm{dist}_\lambda\big(u, P^+\big) + C_3\big(\mathrm{dist}_\lambda\big(u, P^+\big)\big)^{2^*-1}.
\end{aligned}$$

Therefore, there exists $\varepsilon_0 > 0$ such that if $\mathrm{dist}_\lambda(u, P^+) \leqslant \varepsilon_0$ then

$$\mathrm{dist}_\lambda\big(v, P^+\big) < \mathrm{dist}_\lambda\big(u, P^+\big). \qquad \square$$

The proof of Theorem 3.16 is based on Lusternik–Schnirelmann methods similar to those used in the proof of Theorem 2.24 and the above compactness result.

REMARK 3.18. Theorem 3.16 still holds when one replaces the potential $\lambda a + 1$ by $\lambda a + a_0$ with $a_0 \in L^\infty(\mathbb{R}^N)$ and such that $-\Delta + a_0$ is invertible. In this case 0 is a saddle point instead of a local minimum of the functional. The invariant sets given in Lemma 3.17 do not work, but one can use the invariant sets constructed in [66] in this case.

3.4. *Ground state solutions for bounded potentials*

In this section we consider bounded potentials V which cause some compactness problems. We shall present two results on the existence of a ground state solution. In order to keep the presentation simple we only deal with the case of a homogeneous nonlinearity. Thus we want to find a ground state solution $u \in E = H^1(\mathbb{R}^N)$ of the equation

$$-\Delta u + V(x)u = |u|^{p-2}u, \qquad (3.6)$$

where $2 < p < 2^*$. We shall first treat a periodic potential and assume

(V_3) V is 1-periodic in each of its variables.
The case where V is T_i-periodic in x_i, $i = 1, \dots, k$, can be easily reduced to the period 1 case.

THEOREM 3.19. *Assume* (V_0) *and* (V_3). *Then* (3.6) *has a least energy positive solution.*

PROOF. We consider the following minimization problem on $E = H^1(\mathbb{R}^N)$:

$$m := \inf_{u \in M} \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 \,\mathrm{d}x, \qquad M = \bigl\{u \in E\colon \|u\|_{L^p} = 1\bigr\}.$$

The Sobolev inequality implies $m > 0$. A minimizer of m gives a solution of (3.6) by a simple scaling factor. We show next that m is achieved. Let (u_n) be a minimizing sequence for m. Then up to a subsequence $u_n \to u$ weakly in E. Since $\|u_n\|_{L^p} = 1$, by Lemma 3.8 there is $r > 0$, $\delta > 0$ and $(y_n) \subset \mathbb{R}^N$ such that

$$\liminf_{n\to\infty} \int_{B_r(y_n)} |u_n|^p \geqslant \delta.$$

By increasing r we may assume that all y_n have integer coordinates. Setting $v_n(x) = u_n(x + y_n)$ we see that (v_n) is also a minimizing sequence for m because V is periodic. After passing to a subsequence we may assume that $v_n \to v$ weakly. It follows from

$$\liminf_{n\to\infty} \int_{B_r(0)} |v_n|^p \geqslant \delta$$

that $v \neq 0$. If $\|v\|_{L^p} = 1$ we are done. If $\|v\|_{L^p} < 1$ we produce a contradiction as follows. By the Brezis–Lieb lemma we have

$$1 = \|v_n - v\|_{L^p}^p + \|v\|_{L^p}^p + \mathrm{o}(1) \quad \text{as } n \to \infty.$$

Set $a = \|v\|_{L^p}^p$ so that $\|v_n - v\|_{L^p}^p \to 1 - a$ as $n \to \infty$.

$$\begin{aligned} m + \mathrm{o}(1) &= \int_{\mathbb{R}^N} |\nabla v_n|^2 + V(x)|v_n|^2 \\ &= \int_{\mathbb{R}^N} |\nabla v|^2 + V(x)|v|^2 + \int_{\mathbb{R}^N} \bigl|\nabla(v_n - v)\bigr|^2 + V(x)|v_n - v|^2 + \mathrm{o}(1) \\ &\geqslant m\bigl(\|v\|_{L^p}^2 + \|v_n - v\|_{L^p}^2\bigr) + \mathrm{o}(1). \end{aligned}$$

Passing to the limit we get $m \geqslant m(a^{2/p} + (1-a)^{2/p})$, a contradiction. □

Now we consider nonperiodic potentials satisfying
(V_4) $\lim_{|x|\to\infty} V(x) = V_\infty := \sup_{\mathbb{R}^N} V(x) < \infty$.

THEOREM 3.20. *If* (V_0) *and* (V_4) *hold then* (3.6) *has a least energy positive solution.*

PROOF. We consider again the minimization problem on $E = H^1(\mathbb{R}^N)$,

$$m := \inf_{u\in M} \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 \, \mathrm{d}x$$

and compare it with

$$m_\infty := \inf_{u\in M} \int_{\mathbb{R}^N} |\nabla u|^2 + V_\infty u^2 \, \mathrm{d}x,$$

where $M = \{u \in E\colon \|u\|_{L^p} = 1\}$ is as before. The Sobolev inequality yields $m > 0$, and it is easy to see that a minimizer of m gives a solution of (3.6). If $V(x) \equiv V_\infty$ we may use the last theorem to get the existence of a positive solution. Thus we may assume $V(x) < V_\infty$ somewhere. Then using the minimizer of m_∞ as a test function we obtain $m < m_\infty$. We show next that m is in fact achieved on M. Let (u_n) be a minimizing sequence for m. Then up to a subsequence $u_n \to u$ weakly in E. Since $\|u_n\|_{L^p} = 1$, by Lemma 3.8 there exist $r > 0$, $\delta > 0$ and $(y_n) \subset \mathbb{R}^N$ such that

$$\liminf_{n\to\infty} \int_{B_r(y_n)} |u_n|^p \geqslant \delta.$$

We claim that there exists a bounded sequence (y_n). If this is not true, we have $u = 0$ and $u_n \to 0$ in $L^q_{\mathrm{loc}}(\mathbb{R}^N)$. Then

$$\begin{aligned} m + \mathrm{o}(1) &= \int_{\mathbb{R}^N} |\nabla u_n|^2 + V(x)|u_n|^2 \, \mathrm{d}x \\ &= \int_{\mathbb{R}^N} |\nabla u_n|^2 + V_\infty |u_n|^2 \, \mathrm{d}x + \int_{\mathbb{R}^N} \big(V(x) - V_\infty\big)|u_n|^2 \, \mathrm{d}x \\ &= \int_{\mathbb{R}^N} |\nabla u_n|^2 + V_\infty |u_n|^2 \, \mathrm{d}x + \mathrm{o}(1) \\ &\geqslant m_\infty + \mathrm{o}(1). \end{aligned}$$

Passing to the limit we have a contradiction with $m < m_\infty$. Thus we proved that (y_n) is bounded which implies $u \neq 0$. If $\|u\|_{L^p} = 1$ we are finished. If $\|u\|_{L^p} < 1$ we can produce a contradiction as in the proof of the last theorem. □

3.5. *More on periodic potentials*

In this subsection we describe the multibump type solutions for the periodic nonlinear Schrödinger equation constructed by Coti Zelati and Rabinowitz in [48] and recent generalizations of the results by Liu and Wang in [68],

$$-\Delta u + V(x)u = f(x,u). \tag{3.7}$$

Throughout this section we assume (V_3), (f_0'), (f_2) and

(V'_0) $V \in C^1(\mathbb{R}^N, \mathbb{R})$ and $\inf_{\mathbb{R}^N} V(x) > 0$.

(f_6) $f \in C^1(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ is 1-periodic in each of its x-variables.

The periodicity conditions imply that (3.7) is $\mathbb{Z}^N$-invariant. The weak solutions of (3.7) correspond to critical points of

$$\Phi(u) := \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right) \mathrm{d}x - \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x$$

in $E = H^1(\mathbb{R}^N)$. Define the mountain pass value c as

$$c = \inf_{g\in\Gamma} \sup_{t\in[0,1]} \Phi\bigl(g(t)\bigr),$$

where

$$\Gamma = \bigl\{g \in C\bigl([0,1], E\bigr) \bigm| g(0) = 0, \Phi\bigl(g(1)\bigr) < 0\bigr\}.$$

In the homogeneous case $f(x,u) = |u|^{p-2}u$ this is just the minimizer of Φ on the Nehari manifold. In [48] it was proved that (3.7) has infinitely many k-bump solutions. In particular, we have the following theorem.

THEOREM 3.21. *$\mathcal{K}_{kc-\alpha}^{kc+\alpha}/\mathbb{Z}^N$ is infinite for any $k \geqslant 2$, provided the following condition is satisfied*:

($*$) *there is $\alpha > 0$ such that $\mathcal{K}^{c+\alpha}/\mathbb{Z}^N$ is finite.*

Here $\mathcal{K}_a^b$ denotes the set of critical points of Φ between the levels a and b.

In [48,68] it is proved that $\mathcal{K}_{kc-\alpha}^{kc+\alpha}/\mathbb{Z}^N$ contains infinitely many nodal solutions. There one can also find estimates on the number of nodal domains of these multibump type solutions.

In the following we review the modified approach given by Liu and Wang in [68]. This modified approach gives the one sign solutions and the nodal solutions in a uniform fashion, and it is easier for obtaining estimates on the number of nodal domains compared with the original approach by Coti Zelati and Rabinowitz [48].

For $a > 0$, the a-neighborhood of a set $\mathcal{A} \subset E$ is defined by

$$N_a(\mathcal{A}) = \bigl\{u \in E\colon\ \|u - \mathcal{A}\| < a\bigr\},$$

whose closure and boundary are denoted by $\overline{N_a}(\mathcal{A})$ and $\partial N_a(\mathcal{A})$, respectively. For $j = (j_1, \ldots, j_N) \in \mathbb{Z}^N$, we define the translation on $\mathbb{R}^N$ by

$$\tau_j u(x) = u(x_1 + j_1, \ldots, x_N + j_N).$$

For a finite subset E_1 of E and an integer $l \geqslant 1$, we denote

$$\mathcal{T}_l(E_1) = \left\{\sum_{i=1}^{j} \tau_{k_i} v_i\colon\ 1 \leqslant j \leqslant l, v_i \in E_1, k_i \in \mathbb{Z}^N\right\}.$$

Consider the positive cone P^+ and the negative cone P^- in E defined by

$$P^{\pm} = \{u \in E\colon\ \pm u \geqslant 0\}.$$

Any $u \in \mathcal{K} \setminus (P^+ \cup P^-)$ will be a nodal solution of (3.7).

LEMMA 3.22. (i) *there is $\nu > 0$ such that $\|u\| \geqslant \nu$ for all $u \in \mathcal{K} \setminus \{0\}$;*
(ii) *there is $\underline{c} > 0$ such that $\Phi(u) \geqslant \underline{c}$ for all $u \in \mathcal{K} \setminus \{0\}$;*
(iii) *for all $u \in \mathcal{K} \setminus \{0\}$ with $\Phi(u) \leqslant b$: $\|u\| \leqslant (2\theta b/(\theta - 2))^{1/2}$;*
(iv) *for any $b > 0$, there is $\nu_1 > 0$ depending on b such that $\|u^{\pm}\|_{L^2(\mathbb{R}^N)} \geqslant \nu_1$ for all $u \in \mathcal{K} \setminus (P^+ \cup P^-)$ with $\Phi(u) \leqslant b$.*

See the proofs in [48] and [68].

Let $A : E \to E$ be given by $A(u) := (-\Delta + V)^{-1}[f(\cdot, u(\cdot))]$ for $u \in E$. Then the gradient of Φ has the form $\Phi'(u) = u - A(u)$. Note that the set of fixed points of A is the same as the set of critical points of Φ, which is $\mathcal{K}$. By the proof of [48], Proposition 2.1, $\Phi' : E \to E$ is locally Lipschitz continuous.

Using Lemma 3.17, the behavior of PS sequences in the whole space E as well as in $\overline{N}_a(P^{\pm})$ can be studied.

LEMMA 3.23. *Let $(u_m) \subset E$ be such that $\Phi(u_m) \to b > 0$ and $\Phi'(u_m) \to 0$. Then there is an $l \in \mathbb{N}$ (depending on b), $v_1, \dots, v_l \in \mathcal{K} \setminus \{0\}$, a subsequence of u_m and corresponding $k_m^i \in \mathbb{Z}^N$, $i = 1, \dots, l$, such that*

$$\left\| u_m - \sum_{i=1}^{l} \tau_{k_m^i} v_i \right\| \to 0 \quad \text{as } m \to \infty, \tag{3.8}$$

$$\sum_{i=1}^{l} \Phi(v_i) = b, \tag{3.9}$$

and for $i \neq j$,

$$\left| k_m^i - k_m^j \right| \to \infty \quad \text{as } m \to \infty. \tag{3.10}$$

Moreover, there exists an $a_1 \in (0, a_0]$ (depending on b) such that if $(u_m) \subset \overline{N}_{a_1}(P^+)$ ($N_{a_1}(P^-)$, resp.) then $v_1, \dots, v_l \in (\mathcal{K} \setminus \{0\}) \cap P^+$ ($(\mathcal{K} \setminus \{0\}) \cap P^-$, resp.).

For $a \in [0, a_1]$, we define

$$\Gamma_a^{\pm} = \{ g \in \mathcal{C}([0,1], \overline{N}_a(P^{\pm}))\colon\ g(0) = 0 \text{ and } \Phi(g(1)) < 0 \}$$

and

$$c_a^{\pm} = \inf_{g \in \Gamma_a^{\pm}} \max_{\theta \in [0,1]} \Phi(g(\theta)).$$

For $a = 0$, $\overline{N}_a(P^{\pm}) = P^{\pm}$. In this case, we denote $\Gamma^{\pm} = \Gamma_0^{\pm}$ and $c^{\pm} = c_0^{\pm}$. Then we can prove that $c_a^{\pm}$ is independent of a for a small.

LEMMA 3.24. *There exists $a_2 \in (0, a_1)$ such that $c_a^{\pm} = c^{\pm}$ for all $a \in (0, a_2]$.*

Denote $\mathcal{K}^i = \mathcal{K} \cap P^i$ for $i \in \{+, -\}$. We will also use the notation: $(\mathcal{K}^i)^b = \mathcal{K}^i \cap \Phi^b$, $(\mathcal{K}^i)_a^b = \mathcal{K}^i \cap \Phi_a^b$ and $\mathcal{K}^i(c^i) = \mathcal{K}(c^i) \cap P^i$ for $i \in \{+, -\}$. Instead of $(*)$, we need the following condition.

$(*)_{\pm}$ There is $\alpha > 0$ such that $(\mathcal{K}^{\pm})^{c^{\pm}+\alpha}/\mathbb{Z}^N$ is finite.

Choose a representative in E from each equivalent class in $(\mathcal{K}^i)^{c^i+\alpha}/\mathbb{Z}^N$ and denote the resulting set by $\mathcal{F}^i$, $i \in \{+, -\}$. Let $\underline{c} > 0$ be the number from Lemma 3.22 which satisfies $\Phi(u) \geqslant \underline{c}$ for all $u \in \mathcal{K} \setminus \{0\}$. Denote $l^{\pm} = [(c^{\pm} + \alpha)/\underline{c}]$. According to [48], Proposition 2.57, $\mu(\mathcal{T}_{l^{\pm}}(\mathcal{F}^{\pm})) = \inf\{\|u - w\|\colon u \neq w \in \mathcal{T}_{l^{\pm}}(\mathcal{F}^{\pm})\} > 0$. Using this a deformation lemma in $\overline{N}_a(P^{\pm})$ can be given. The proof of the deformation lemma is similar to the proof of [48]. However, we need to construct a descending flow of Φ which makes $\overline{N}_a(P^i)$ positively invariant so that the deformation is from $\overline{N}_a(P^i)$ to itself.

Then by using the descending flow, one obtains the following theorem which asserts the existence of one-bump positive and negative solutions at the mountain pass level. These one-bump solutions will be used later to construct multibump nodal solutions.

THEOREM 3.25. *Let $(*)_{\pm}$ be satisfied. Then $c^{\pm}$ are critical values of Φ and there are critical points $u^{\pm} \in \mathcal{K}^{\pm}$ such that $\Phi(u^{\pm}) = c^{\pm}$.*

REMARK 3.26. This theorem shows the existence of a solution near the mountain pass level regardless of whether $(*)_{\pm}$ is satisfied or not.

Now, by $(*)_{\pm}$, there is an $\alpha_1 \in (0, \alpha)$ such that

$$\left(\mathcal{K}^i\right)_{c^i-\alpha_1}^{c^i+\alpha_1} = \mathcal{K}^i\left(c^i\right).$$

LEMMA 3.27. *Let $(*)_{\pm}$ be satisfied. Then there exist finite sets $A^+ \subset \mathcal{K}^+(c^+)$ and $A^- \subset \mathcal{K}^-(c^-)$ having the property that for any $\bar{\varepsilon}_1 \leqslant \alpha_1/2$, $r_1 \leqslant \frac{1}{12}\mu(\mathcal{T}_{l^{\pm}}(\mathcal{F}^{\pm}))$ and $p \in \mathbb{N}$, there is an $\varepsilon_1 \in (0, \bar{\varepsilon}_1)$ and $g_1^{\pm} \in \Gamma^{\pm}$ such that*

(i) $\max_{\theta \in [0,1]} \Phi(g_1^{\pm}(\theta)) \leqslant c^{\pm} + \varepsilon_1/p$;

(ii) *if $\Phi(g_1^{\pm}(\theta)) > c^{\pm} - \varepsilon_1$ then $g_1^{\pm}(\theta) \in N_{r_1}(A^{\pm})$.*

Let $A = A^+ \cup A^-$ with $A^{\pm}$ given in Lemma 3.27. For any fixed integer $k \geqslant 2$ we fix two positive integers k^+ and k^- such that $k = k^+ + k^-$. Denote $\Lambda^+ = \{1, \ldots, k^+\}$, $\Lambda^- = \{k^+ + 1, \ldots, k\}$. Let $j_i \in \mathbb{Z}^N$ for $i = 1, \ldots, k$ be fixed such that $j_i \neq j_m$ for $i \neq m$ and if $v_i \in A^+$ for $i \in \Lambda^+$ and $v_i \in A^-$ for $i \in \Lambda^-$ then

$$\left\|\sum_{i=1}^{k} \tau_{j_i} v_i\right\| \geqslant \frac{k\nu}{2} \quad \text{and} \quad \left|\Phi\left(\sum_{i=1}^{k} \tau_{j_i} v_i\right) - \left(k^+c^+ + k^-c^-\right)\right| < \frac{\alpha}{2}.$$

Define

$$\mathcal{M}\big(j_1, \ldots, j_k, A, k^+, k^-\big) = \left\{ \sum_{i=1}^{k} \tau_{j_i} v_i \colon\ v_i \in A^+ \text{ for } i \in \Lambda^+, v_i \in A^- \text{ for } i \in \Lambda^- \right\}$$

and

$$b_k = k^+ c^+ + k^- c^-.$$

The main theorem in [68] reads as follows.

THEOREM 3.28. *Let* $(*)_\pm$ *be satisfied. Then there is an* $r_0 > 0$ *such that, for any* $r \in (0, r_0)$,

$$N_r\big(\mathcal{M}\big(l j_1, \ldots, l j_k, A, k^+, k^-\big)\big) \cap \big(\mathcal{K}_{b_k-\alpha}^{b_k+\alpha} / \mathbb{Z}^N\big) \neq \emptyset$$

for all but finitely many $l \in \mathbb{N}$.

We sketch the proof of Theorem 3.28. For $\theta = (\theta_1, \ldots, \theta_k) \in [0,1]^k$, let $0_i = (\theta_1, \ldots, \theta_{i-1}, 0, \theta_{i+1}, \ldots, \theta_k)$ and $1_i = (\theta_1, \ldots, \theta_{i-1}, 1, \theta_{i+1}, \ldots, \theta_k)$, $1 \leqslant i \leqslant k$. Let a_2 be as in Lemma 3.24 and $a \in [0, a_2]$ and define

$$\Gamma_k(a) = \big\{ G = g_1 + \cdots + g_k \colon\ g_i \text{ satisfies } (\mathrm{g}_1)\text{–}(\mathrm{g}_3), 1 \leqslant i \leqslant k \big\},$$

where

(g_1) $g_i \in C([0,1]^k, \overline{N}_a(P^\pm))$ for $i \in \Lambda^\pm$,

(g_2) $g_i(0_i) = 0$ and $\Phi(g_i(1_i)) < 0$, $1 \leqslant i \leqslant k$,

(g_3) there are bounded open sets $\mathcal{O}_i$, $1 \leqslant i \leqslant k$, such that $\overline{\mathcal{O}}_i \cap \overline{\mathcal{O}}_j = \emptyset$ if $i \neq j$ and $\operatorname{supp} g_i(\theta) \subset \mathcal{O}_i$ for all $\theta \in [0,1]^k$.

LEMMA 3.29. *Let* $(*)_\pm$ *be satisfied. Define*

$$b_k(a) = \inf_{G \in \Gamma_k(a)} \max_{\theta \in [0,1]^k} \Phi\big(G(\theta)\big).$$

Then $b_k(a) = b_k = k^+ c^+ + k^- c^-$ *for* $a \in (0, a_2]$.

From here on we argue by contradiction. Thus assume that there are infinitely many l such that the conclusion of the theorem is false. The proofs go with the construction of a special G, which is chosen as close to the level $b_k(a)$ as possible, the deformation of this special G under the gradient flow, modification of the deformed G so that finally we get a map in the family $\Gamma_k(a)$ satisfying $\max \Phi(G) < b_k$, which is a contradiction with the definition of $b_k(a)$. We refer for the details to [68].

Next we state a result of [48,68] which is used in establishing more detailed nodal properties of these solutions.

THEOREM 3.30. *In the above theorem, if r is sufficiently small and l sufficiently large, we may replace the r-neighborhood in E with r-neighborhood in $X := \overline{C^1(\mathbb{R}^N)}$.*

The proof of this uses some elliptic estimates.

THEOREM 3.31. *Suppose $(*)_\pm$ holds. For multibump nodal solutions of* (3.7), *the number of nodal domains is bounded by the number of bumps. In particular, the two-bump nodal solutions have exactly two nodal domains. Moreover, there are infinitely many, geometrically different, two-bump, nodal solutions which have exactly two nodal domains.*

The proof of this theorem is a consequence of the following lemma from [68].

LEMMA 3.32. *If r is small enough and l is large enough then, for the solutions u given in Theorem* 3.28, *the number of nodal domains is bounded above by the number of bumps of the solutions.*

The techniques for the above theorem can be used to establish the nodal property of the solutions in another two cases. First, we consider (3.7) on a cylinder domain $\Omega = \omega \times \mathbb{R}$ and we write $x = (x', x_N)$ with $x' = (x_1, \dots, x_{N-1})$, where ω is a bounded smooth domain in $\mathbb{R}^{N-1}$. We assume (V_0), (f_0'), (f_2), (f_3), and (V_5) and (f_6) are satisfied with the periodicity only in the x_N direction. The space will be $E = H_0^1(\Omega)$. Then we can still define the mountain pass values $c^\pm > 0$. The problem now is $\mathbb{Z}$ invariant.

$(*')_\pm$ There is $\alpha > 0$ such that $\mathcal{K}^{c^\pm+\alpha}/\mathbb{Z}$ is finite.

THEOREM 3.33. *Suppose $(*')_\pm$ holds. Then, for any integers $k \geqslant m \geqslant 2$,* (3.7) *has infinitely many, geometrically different, k-bump, nodal solutions in $I_{kc-\alpha}^{kc+\alpha}$ which have exactly m nodal domains. More precisely, given any positive integers $k_1, k_2, \dots, k_m$ such that $\sum_{i=1}^m k_i = k \geqslant 2$, there are infinitely many, geometrically different, k-bump, nodal solutions in $I_{kc-\alpha}^{kc+\alpha}$ which have exactly m nodal domains D_i, $i = 1, \dots, m$, such that $u|_{D_i}$ is a k_i-bump positive or negative solution.*

Another case can be considered is when (3.7) has different x-dependence in different directions. We assume (V_0), (f_0'), (f_2), (f_3) and

(V_5) V is periodic in x_N and radially symmetric in $(x_1, \dots, x_{N-1})$.

(f_7) $f \in C^1(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ is periodic in x_N and radially symmetric in $(x_1, \dots, x_{N-1})$.

Then $\mathbb{Z}$ acts via translations on the x_N-variable, and the problem is $\mathbb{Z}$-invariant. With $x = (x', x_N)$ and $x' = (x_1, \dots, x_{N-1})$, we work with the space

$$E = \left\{ u \in H^1(\mathbb{R}^N) \colon u(x', x_N) = u(|x'|, x_N), \int_{\mathbb{R}^N} V(x) u^2 \, \mathrm{d}x < \infty \right\},$$

that is, functions in E are radially symmetric in the first $N-1$ variables. We can still define the mountain pass values $c^\pm$ in the space E.

$(*'')_\pm$ There is $\alpha > 0$ such that $\mathcal{K}^{c^\pm+\alpha}/\mathbb{Z}$ is finite.

THEOREM 3.34. *Suppose* $(*'')_\pm$ *holds. Then for any integer* $k \geqslant 2$, (3.7) *has infinitely many, geometrically different,* k*-bump nodal solutions in* $\Phi_{kc-\alpha}^{kc+\alpha}$ *such that the numbers of their nodal domains are bounded between* $[k/2]+1$ *and* k. *In particular, there are nodal solutions such that the numbers of their nodal domains tend to infinity.*

We refer the details of the proofs to [68].

3.6. *Strongly indefinite potentials*

In this section we consider problem (3.1) when V and f are periodic in the x-variables but V is not positive anymore. For periodic V the operator $-\Delta + V$ has a band spectrum. More precisely, the spectrum of $-\Delta + V$ on $L^2(\mathbb{R}^N)$ is purely continuous, bounded below and consists of a disjoint union of closed intervals. We require:

(V_5) $V \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$ is 1-periodic in each of its variables and the linear operator,

$$H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N), \qquad u \mapsto -\Delta u + V(x)u,$$

is invertible.

By (V_5) the operator $-\Delta + V(x)$ is self-adjoint on $L^2(\mathbb{R}^N)$ and may have continuous spectrum on the negative real axis with 0 in a spectral gap.

THEOREM 3.35. *Suppose* (V_5) *and* (f_0) *with* $p < 2^*$, *and* (f_2) *hold. Then problem* (3.1) *has a nontrivial solution in* $H^1(\mathbb{R}^N)$.

Theorem 3.35 is due to Kryszewski and Szulkin [58]. An earlier version requiring a stronger growth condition on f has been obtained by Troestler and Willem [89]. If 0 is a left end point of a spectral gap, that is $0 \in \sigma(-\Delta + V)$ and $(0, \varepsilon) \subset \mathbb{R} \setminus \sigma(-\Delta + V)$ then a solution of (3.1) has been found in [11] under additional growth conditions on f. The solution then does not lie in $H^1(\mathbb{R}^N)$ but only in $H^2_{\mathrm{loc}}(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ for $p \leqslant q \leqslant 2^*$.

THEOREM 3.36. *Suppose* (V_5) *and* (f_0) *with* $p < 2^*$, *and* (f_2) *hold. If in addition* f *is odd in* u *then problem* (3.1) *has infinitely many geometrically different solutions in* $H^1(\mathbb{R}^N)$.

Here two solutions u_1, u_2 are said to be geometrically different if there does not exist $k \in \mathbb{Z}^n$ with $u_2(x) = u_1(x+k)$ for every $x \in \mathbb{R}^N$. Also Theorem 3.36 is due to [58] and has been extended in [11] to the case where 0 is a left end point of a spectral gap.

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CHAPTER 2

Nonconvex Problems of the Calculus of Variations and Differential Inclusions

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Contents

HANDBOOK OF DIFFERENTIAL EQUATIONS
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Abstract

We study existence of minimizers for problems of the type

$$\inf\left\{\int_{\Omega} f\big(x, u(x), Du(x)\big)\,\mathrm{d}x\colon\ u \in u_0 + W_0^{1,p}\big(\Omega;\mathbb{R}^N\big)\right\}, \tag{P}$$

where u_0 is a given function.

After recalling some basic facts about existence of minimizers when the function f is convex (quasiconvex), we turn our attention to the case where f is not convex (quasiconvex).

We start by presenting the general tool of relaxation, which gives generalized solutions of (P).

We next discuss some differential inclusions, where we look for solutions $u \in u_0 + W_0^{1,\infty}(\Omega;\mathbb{R}^N)$ of

$$Du(x) \in E \quad \text{a.e. in } \Omega,$$

where $E \subset \mathbb{R}^{N\times n}$ is a given compact set.

Finally combining the relaxation theorem and the study of differential inclusions, we give necessary and sufficient conditions for existence of classical minimizers of (P) as well as several examples.

1. Introduction

We discuss the existence of minimizers for the problem

$$\inf\left\{ I(u) = \int_{\Omega} f\big(x, u(x), Du(x)\big)\,\mathrm{d}x\colon\ u \in u_0 + W_0^{1,p}\big(\Omega; \mathbb{R}^N\big) \right\}, \tag{P}$$

where

- $\Omega \subset \mathbb{R}^n$ is a bounded open set, with Lipschitz boundary $\partial\Omega$;
- $u : \Omega \to \mathbb{R}^N$,

  $$u = u(x) = u(x_1, \ldots, x_n) = \big(u^1(x), \ldots, u^N(x)\big)$$

  (if $N = 1$ or, by abuse of language, if $n = 1$, we will say that it is *scalar* valued while if $N, n \geqslant 2$, we will speak of the *vector* valued case);
- Du denotes its Jacobian matrix, i.e.,

  $$Du = \left(\frac{\partial u^i}{\partial x_j}\right)_{1\leqslant j\leqslant n}^{1\leqslant i\leqslant N} = \begin{pmatrix} \frac{\partial u^1}{\partial x_1} & \cdots & \frac{\partial u^1}{\partial x_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial u^N}{\partial x_1} & \cdots & \frac{\partial u^N}{\partial x_n} \end{pmatrix} \in \mathbb{R}^{N\times n};$$

- $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ is continuous, $f = f(x, u, \xi)$;
- $1 \leqslant p \leqslant \infty$ and $W^{1,p}(\Omega; \mathbb{R}^N)$ denotes the usual space of Sobolev maps, where

  $$u^i,\ \frac{\partial u^i}{\partial x_j} \in L^p(\Omega), \quad i = 1, \ldots, N,\ j = 1, \ldots, n;$$

- $u_0 \in W^{1,p}(\Omega; \mathbb{R}^N)$ is a given map;
- $u \in u_0 + W_0^{1,p}(\Omega; \mathbb{R}^N)$, meaning that $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ and $u = u_0$ on $\partial\Omega$ in the Sobolev sense.

This problem is the fundamental problem of the calculus of variations and it has received a considerable attention since the time of Fermat, Newton, Bernoulli, Euler and all along the 19th and 20th centuries.

The most general way of proving existence of minimizers of (P), meaning to find $\bar{u} \in u_0 + W_0^{1,p}(\Omega; \mathbb{R}^N)$ so that

$$I(\bar{u}) \leqslant I(u)$$

among all admissible $u \in u_0 + W_0^{1,p}(\Omega; \mathbb{R}^N)$, is the so-called *direct methods of the calculus of variations*. These methods rely on some kind of convexity condition of the function $\xi \to f(x, u, \xi)$. There are numerous examples showing that in absence of convexity the problem (P) has no minimizers. At the moment let us quote three elementary examples where nonexistence occurs.

EXAMPLE 1. Let $N = n = 1$,

$$f(x, u, \xi) = f(\xi) = \mathrm{e}^{-\xi^2}$$

and

$$\inf\left\{ I(u) = \int_0^1 f\left(u'(x)\right) \mathrm{d}x\colon\ u \in W_0^{1,1}(0,1) \right\}. \tag{P}$$

EXAMPLE 2. Let $N = n = 1$,

$$f(x, u, \xi) = f(u, \xi) = u^4 + \left(\xi^2 - 1\right)^2$$

and

$$\inf\left\{ I(u) = \int_0^1 f\left(u(x), u'(x)\right) \mathrm{d}x\colon\ u \in W_0^{1,4}(0,1) \right\}. \tag{P}$$

This example is due to Bolza.

EXAMPLE 3. Let $n = 2$, $N = 1$, $\Omega = (0,1)^2$,

$$f(x, u, \xi) = f(\xi) = f(\xi_1, \xi_2) = \left(\xi_1^2 - 1\right)^2 + \xi_2^4$$

and

$$\inf\left\{ I(u) = \int_\Omega f\left(Du(x)\right) \mathrm{d}x\colon\ u \in W_0^{1,4}(\Omega) \right\}. \tag{P}$$

We now continue this introduction by discussing only the scalar case (i.e., when $N = 1$ or $n = 1$), the general vectorial case will be discussed in the next sections. We moreover, in order to simplify the presentation, consider the case where there is no dependence on lower-order terms, i.e., $f(x, u, \xi) = f(\xi)$.

When dealing with nonconvex problems, the *first step* is the *relaxation theorem*, established by L.C. Young, Mac Shane, Ekeland and others. This consists in replacing the problem (P) by the so-called relaxed problem

$$\inf\left\{ \overline{I}(u) = \int_\Omega Cf\left(Du(x)\right) \mathrm{d}x\colon\ u \in u_0 + W_0^{1,p}(\Omega) \right\}, \tag{QP}$$

where Cf is the *convex envelope* of f, namely

$$Cf = \sup\{g \leqslant f\colon\ g \text{ convex}\}.$$

Therefore the direct methods, which do not apply to (P), apply to (QP). It can be shown (cf. Theorem 15) that

$$\inf(\mathrm{P}) = \inf(\mathrm{QP})$$

and that minimizers of (P) are necessarily minimizers of (QP), the converse being false. In the three above examples we have

(i) $Cf(\xi) \equiv 0$, $\inf(\mathrm{P}) = \inf(\mathrm{QP}) = 0$ and any $u \in W_0^{1,1}(0,1)$ is a solution of (QP);

(ii) $Cf(u,\xi) = u^4 + [\xi^2 - 1]_+^2$, $\inf(\mathrm{P}) = \inf(\mathrm{QP}) = 0$ and $u \equiv 0$ is a solution of (QP);

(iii) $Cf(\xi) = [\xi_1^2 - 1]_+^2 + \xi_2^4$, $\inf(\mathrm{P}) = \inf(\mathrm{QP}) = 0$ and $u \equiv 0$ is a solution of (QP);

where, for $x \in \mathbb{R}$,

$$[x]_+ = \begin{cases} x & \text{if } x \geqslant 0, \\ 0 & \text{if } x < 0. \end{cases}$$

The *second step* in proving the existence of minimizers for (P) is to see if among all solutions of (QP), if any, at least one of them is also a solution of (P). This amounts in finding $\bar{u} \in u_0 + W_0^{1,p}(\Omega)$ so that

$$\int_\Omega Cf\big(D\bar{u}(x)\big)\,\mathrm{d}x = \inf(\mathrm{P}) = \inf(\mathrm{QP})$$

and at the same time in solving the first-order differential equation (called, following Dacorogna and Marcellini [31], *implicit partial differential equation*)

$$Cf\big(D\bar{u}(x)\big) = f\big(D\bar{u}(x)\big) \quad \text{a.e. } x \in \Omega.$$

After this brief and informal introduction, we discuss the organization of the article.

In Section 2 we discuss all the notions of convexity that are involved in the vector valued case, in particular the so-called *quasiconvexity*.

In Section 3 we present the relaxation theorem in the vector valued case, introducing all the needed generalization of the notion of convex envelope.

In Section 4 we give some existence theorems for implicit differential equations of the above type.

In Section 5 using the results of the two preceding sections, we discuss necessary and sufficient conditions for the existence of minimizers for nonconvex problems.

In Section 6 we show how to apply the abstract results to scalar problems; obtaining sharper theorems in the case of single integrals (i.e., $n = 1$).

In Section 7 we present several examples involving vector valued functions (i.e., $n, N \geqslant 2$) which are relevant for applications.

The subject is very large and we do not intend to be complete and we refer to the bibliography for more details. Let us quote some of the significant contributions to the subject.

The scalar case ($n = 1$ or $N = 1$) has been intensively studied notably by Aubert and Tahraoui [4–6], Bauman and Phillips [10], Buttazzo, Ferone and Kawohl [13], Celada and Perrotta [14,15], Cellina [16,17], Cellina and Colombo [18], Cesari [20,21], Cutri [22], Dacorogna [26], Ekeland [39], Friesecke [40], Fusco, Marcellini and Ornelas [41], Giachetti and Schianchi [42], Klötzler [47], Marcellini [50–52], Mascolo [54], Mascolo and Schianchi [56,57], Monteiro Marques and Ornelas [58], Ornelas [64], Raymond [67–69], Sychev [75], Tahraoui [76,77], Treu [78] and Zagatti [80].

The vectorial case has been investigated for some special examples notably by Allaire and Francfort [3], Cellina and Zagatti [19], Dacorogna and Ribeiro [35], Dacorogna and Tanteri [37], Mascolo and Schianchi [55], Müller and Sverak [61] and Raymond [70]. A more systematic study was achieved by Dacorogna and Marcellini in [27,31,32], as well as in Dacorogna, Pisante and Ribeiro [34].

We have always considered in the present article the two important restrictions:

- f does not depend on lower-order terms, i.e., $f(x,u,\xi)=f(\xi)$;
- the boundary datum u_0 is affine, i.e., there exists $\xi_0\in\mathbb{R}^{N\times n}$ so that

$$Du_0=\xi_0.$$

In the above-mentioned literature, some authors have considered either of these two more general cases. The results are then much less general and essentially apply only to the scalar case.

2. Preliminaries and notations

2.1. *The different notions of convexity*

We start with the different definitions of convexity that we will use throughout this chapter and we refer to Dacorogna [26] for more details.

DEFINITION 4. (i) A function $f:\mathbb{R}^{N\times n}\to\overline{\mathbb{R}}=\mathbb{R}\cup\{+\infty\}$ is said to be *rank one convex* if

$$f\big(\lambda\xi+(1-\lambda)\eta\big)\leqslant\lambda f(\xi)+(1-\lambda)f(\eta)$$

for every $\lambda\in[0,1]$, $\xi,\eta\in\mathbb{R}^{N\times n}$ with $\operatorname{rank}\{\xi-\eta\}\leqslant 1$.

(ii) A Borel measurable and locally integrable function $f:\mathbb{R}^{N\times n}\to\mathbb{R}$ is said to be *quasiconvex* if

$$f(\xi)\leqslant\frac{1}{\operatorname{meas}D}\int_D f\big(\xi+D\varphi(x)\big)\,\mathrm{d}x$$

for every bounded domain $D\subset\mathbb{R}^n$, for every $\xi\in\mathbb{R}^{N\times n}$ and for every $\varphi\in W_0^{1,\infty}(D;\mathbb{R}^N)$.

(iii) A Borel measurable and locally integrable function $f:\mathbb{R}^{N\times n}\to\mathbb{R}$ is said to be *quasiaffine* if f and $-f$ are quasiconvex.

(iv) A function $f:\mathbb{R}^{N\times n}\to\overline{\mathbb{R}}=\mathbb{R}\cup\{+\infty\}$ is said to be *polyconvex* if there exists $g:\mathbb{R}^{r(n,N)}\to\overline{\mathbb{R}}$ convex, such that

$$f(\xi)=g\big(T(\xi)\big),$$

where $T:\mathbb{R}^{N\times n}\to\mathbb{R}^{r(n,N)}$ is such that

$$T(\xi)=(\xi,\operatorname{adj}_2\xi,\dots,\operatorname{adj}_{n\wedge N}\xi).$$

In the preceding definition, $\operatorname{adj}_s\xi$ stands for the matrix of all $s\times s$ minors of the matrix $\xi\in\mathbb{R}^{N\times n}$, $2\leqslant s\leqslant n\wedge N=\min\{n,N\}$, and

$$\tau(n,N)=\sum_{s=1}^{n\wedge N}\sigma(s),$$

where

$$\sigma(s)=\binom{N}{s}\binom{n}{s}=\frac{N!\,n!}{(s!)^2(N-s)!(n-s)!}.$$

REMARK 5. (i) The concepts were introduced by Morrey [59,60], but the terminology is the one of Ball [7]; note however that Ball calls quasiaffine functions, null Lagrangians.

(ii) These notions are related through the following diagram

$$\begin{aligned}&f\text{ convex}\quad\Longrightarrow\quad f\text{ polyconvex}\\&\quad\Longrightarrow\quad f\text{ quasiconvex}\quad\Longrightarrow\quad f\text{ rank one convex.}\end{aligned}$$

In the scalar case, $n=1$ or $N=1$, these notions are all equivalent and reduce therefore to the usual notion of convexity. However in the vectorial case, $n,N\geqslant 2$, these concepts are all different, meaning that there are counterexamples to all the above implications. The last counter implication being known, thanks to the celebrated example from Sverak [72], only when $n\geqslant 2$ and $N\geqslant 3$; the case $N=2$, $n\geqslant 2$ being still open.

(iii) Note that in the case $N=n=2$, the notion of polyconvexity can be read as follows:

$$\begin{cases}\tau(n,N)=\tau(2,2)=5\quad\text{since }\sigma(1)=4,\ \sigma(2)=1,\\ T(\xi)=(\xi,\det\xi).\end{cases}$$

(iv) Observe that, if we adopt the tensorial notation, the definition of rank one convexity can be read as follows,

$$\varphi(t)=f(\xi+ta\otimes b)$$

is convex in t for every $\xi\in\mathbb{R}^{N\times n}$ and for every $a\in\mathbb{R}^N$, $b\in\mathbb{R}^n$, where we have denoted by

$$a\otimes b=\left(a^i b_\alpha\right)_{1\leqslant\alpha\leqslant n}^{1\leqslant i\leqslant N}.$$

(v) One should also note that in the definition of quasiconvexity if the inequality holds for a given domain $D\subset\mathbb{R}^n$, then it holds for every such domain D.

(vi) If the function $f:\mathbb{R}^{N\times n}\to\mathbb{R}$, i.e., f takes only finite values, is convex or polyconvex or quasiconvex or rank one convex, then it is continuous and even locally Lipschitz.

(vii) It can be shown that a quasiaffine function is necessary of the form

$$f(\xi)=\langle\alpha;T(\xi)\rangle+\beta$$

for some constants $\alpha\in\mathbb{R}^\tau$ and $\beta\in\mathbb{R}$ and where $\langle\cdot;\cdot\rangle$ stands for the scalar product in $\mathbb{R}^\tau$; which in the case $N=n=2$ reads as ($\alpha=(\alpha_{11},\alpha_{12},\alpha_{21},\alpha_{22},\alpha_5)\in\mathbb{R}^5$)

$$f(\xi)=\sum_{i,j=1}^{2}\alpha_{ij}\xi_{ij}+\alpha_5\det\xi+\beta.$$

(viii) An equivalent characterization of polyconvexity can be given in terms of the separation theorem (cf. Theorem 1.3 in Dacorogna [26], p. 107). A function $f:\mathbb{R}^{N\times n}\to\mathbb{R}$ is polyconvex if and only if for every $\xi\in\mathbb{R}^{N\times n}$ there exists $\lambda=\lambda(\xi)\in\mathbb{R}^{\tau(N,n)}$ so that

$$f(\xi+\eta)-f(\xi)-\langle\lambda;T(\xi+\eta)-T(\xi)\rangle\geqslant 0\quad\text{for every }\eta\in\mathbb{R}^{N\times n}. \tag{1}$$

(ix) When the function f depends on lower-order terms as in the Introduction, i.e., $f:\Omega\times\mathbb{R}^N\times\mathbb{R}^{N\times n}\to\mathbb{R}$ with $f=f(x,u,\xi)$, all the above notions are understood only with respect to the variable ξ, all the other variables being kept fixed. For example in the case of quasiconvex functions, one should read

$$f(x_0,u_0,\xi)\leqslant\frac{1}{\operatorname{meas}D}\int_D f\bigl(x_0,u_0,\xi+D\varphi(x)\bigr)\,\mathrm{d}x$$

for every bounded domain $D\subset\mathbb{R}^n$, for every $(x_0,u_0,\xi)\in\Omega\times\mathbb{R}^N\times\mathbb{R}^{N\times n}$ and for every $\varphi\in W_0^{1,\infty}(D;\mathbb{R}^N)$.

The important concept from the point of view of minimization in the calculus of variations is the notion of quasiconvexity. This condition is equivalent to the fact that the functional I, defined in the Introduction, is (*sequentially*) *weakly lower semicontinuous* in $W^{1,p}(\Omega;\mathbb{R}^N)$ meaning that

$$I(u)\leqslant\liminf_{\nu\to\infty}I(u_\nu)$$

for every sequence $u_\nu\rightharpoonup u$ in $W^{1,p}$.

Important examples of quasiconvex functions are the following.

(i) The *quadratic case*. Let M be a symmetric matrix in $\mathbb{R}^{(N\times n)\times(N\times n)}$ and

$$f(\xi)=\langle M\xi;\xi\rangle,$$

where $\xi\in\mathbb{R}^{N\times n}$ and $\langle\cdot;\cdot\rangle$ denotes the scalar product in $\mathbb{R}^{N\times n}$. Then

$$f\text{ quasiconvex}\quad\Longleftrightarrow\quad f\text{ rank one convex}.$$

(ii) The *Alibert–Dacorogna–Marcellini example* (cf. [2]). Here we have $N = n = 2$ and

$$f(\xi) = |\xi|^2\big(|\xi|^2 - 2\gamma \det \xi\big),$$

where $|\xi|$ stands for the Euclidean norm of the matrix and $\gamma \geqslant 0$. Then

$$f \text{ is convex} \iff \gamma \leqslant \gamma_c = \frac{2}{3}\sqrt{2},$$
$$f \text{ is polyconvex} \iff \gamma \leqslant \gamma_p = 1,$$
$$f \text{ is quasiconvex} \iff \gamma \leqslant \gamma_q, \quad \text{where } \gamma_q > 1,$$
$$f \text{ is rank one convex} \iff \gamma \leqslant \gamma_r = \frac{2}{\sqrt{3}}.$$

(iii) Let $f : \mathbb{R}^{N\times n} \to \mathbb{R}$, $\Phi : \mathbb{R}^{N\times n} \to \mathbb{R}$ be quasiaffine and $g : \mathbb{R} \to \mathbb{R}$ be such that

$$f(\xi) = g\big(\Phi(\xi)\big)$$

(in particular if $N = n$, one can take $\Phi(\xi) = \det \xi$), then

$$f \text{ polyconvex} \iff f \text{ quasiconvex}$$
$$\iff f \text{ rank one convex} \iff g \text{ convex.}$$

(iv) Let $N = n + 1$ and for $\xi \in \mathbb{R}^{(n+1)\times n}$, denote

$$\operatorname{adj}_n \xi = \big(\det \hat{\xi}^1, -\det \hat{\xi}^2, \ldots, (-1)^{k+1} \det \hat{\xi}^k, \ldots, (-1)^{n+2} \det \hat{\xi}^{n+1}\big),$$

where $\hat{\xi}^k$ is the $n \times n$ matrix obtained from ξ by suppressing the kth line (when $\xi = Du$, $\operatorname{adj}_n Du$ represents, geometrically, the normal to the hypersurface). Let $g : \mathbb{R}^{n+1} \to \mathbb{R}$ be such that

$$f(\xi) = g(\operatorname{adj}_n \xi)$$

then

$$f \text{ polyconvex} \iff f \text{ quasiconvex}$$
$$\iff f \text{ rank one convex} \iff g \text{ convex.}$$

(v) Let $0 \leqslant \lambda_1(\xi) \leqslant \cdots \leqslant \lambda_n(\xi)$ denote the *singular values* of a matrix $\xi \in \mathbb{R}^{n\times n}$, which are defined as the eigenvalues of the matrix $(\xi\xi^\top)^{1/2}$. The functions

$$\xi \to \sum_{i=\nu}^{n} \lambda_i(\xi) \quad \text{and} \quad \xi \to \prod_{i=\nu}^{n} \lambda_i(\xi), \qquad \nu = 1, \ldots, n,$$

are respectively convex and polyconvex (note that $\prod_{i=1}^{n} \lambda_i(\xi) = |\det \xi|$). In particular, the function $\xi \to \lambda_n(\xi)$ is convex and in fact is the operator norm.

2.2. *Some function spaces*

The following notations will be used throughout.

- For $1 \leqslant p \leqslant \infty$, we will let $W^{1,p}(\Omega; \mathbb{R}^N)$ be the space of maps $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ such that

$$u \in L^p(\Omega, \mathbb{R}^N) \quad \text{and} \quad Du = \left(\frac{\partial u^i}{\partial x_j}\right)_{1 \leqslant j \leqslant n}^{1 \leqslant i \leqslant N} \in L^p(\Omega, \mathbb{R}^{N \times n}).$$

- For $1 \leqslant p < \infty$, $W_0^{1,p}(\Omega; \mathbb{R}^N)$ will denote the closure of $C_0^\infty(\Omega; \mathbb{R}^N)$ with respect to the $\|\cdot\|_{W^{1,p}}$ norm.
- $W_0^{1,\infty}(\Omega; \mathbb{R}^N) = W^{1,\infty}(\Omega; \mathbb{R}^N) \cap W_0^{1,1}(\Omega; \mathbb{R}^N)$.
- $Aff_{\text{piec}}(\overline{\Omega}; \mathbb{R}^N)$ will stand for the subset of $W^{1,\infty}(\Omega; \mathbb{R}^N)$ consisting of piecewise affine maps.
- $C^1_{\text{piec}}(\overline{\Omega}; \mathbb{R}^N)$ will denote the subset of $W^{1,\infty}(\Omega; \mathbb{R}^N)$ consisting of piecewise C^1 maps.

2.3. *Statement of the problem*

We will be concerned with existence of minimizers for the problem

$$\inf\left\{\int_\Omega f\big(Du(x)\big)\,\mathrm{d}x\colon\ u \in u_{\xi_0} + W_0^{1,\infty}\big(\Omega; \mathbb{R}^N\big)\right\}, \tag{P}$$

where

- $\Omega \subset \mathbb{R}^n$ is a bounded open set with Lipschitz boundary,
- $u : \Omega \to \mathbb{R}^N$ and thus $Du \in \mathbb{R}^{N \times n}$,
- $f : \mathbb{R}^{N \times n} \to \mathbb{R}$ is lower semicontinuous, locally bounded and nonnegative,
- $\xi_0 \in \mathbb{R}^{N \times n}$ and u_{ξ_0} is an affine map such that $Du_{\xi_0} = \xi_0$.

The hypothesis $f \geqslant 0$ can be replaced, with no changes, by

$$f(\xi) \geqslant \big\langle \alpha; T(\xi)\big\rangle + \beta \quad \text{for every } \xi \in \mathbb{R}^{N \times n},$$

for some constants $\alpha \in \mathbb{R}^\tau$ and $\beta \in \mathbb{R}$ and where $\langle\cdot;\cdot\rangle$ stands for the scalar product in $\mathbb{R}^\tau$. This hypothesis is made to avoid to have to deal with quasiconvex envelopes $Qf \equiv -\infty$.

3. Relaxation theorems

We now present the relaxation theorem, which corresponds to the first step described in the Introduction. But before that we need to introduce the notions of envelopes corresponding to the different concepts of convexity that we introduced in the previous section. The reference book for this part is still Dacorogna [26].

3.1. *The different envelopes*

We now define

$$\begin{aligned} Cf &= \sup\{g \leqslant f:\ g \text{ convex}\},\\ Pf &= \sup\{g \leqslant f:\ g \text{ polyconvex}\},\\ Qf &= \sup\{g \leqslant f:\ g \text{ quasiconvex}\},\\ Rf &= \sup\{g \leqslant f:\ g \text{ rank one convex}\}, \end{aligned}$$

they are respectively the *convex*, *polyconvex*, *quasiconvex*, *rank one convex envelope* of f. In view of the results of the previous section, we have

$$Cf \leqslant Pf \leqslant Qf \leqslant Rf \leqslant f.$$

As already said, we will always assume, in the sequel, that $f \geqslant 0$. We then have the following characterizations of the different envelopes.

THEOREM 6. *Let* $f : \mathbb{R}^{N\times n} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$.

Part 1. *Let for any integer* s

$$\Lambda_s = \left\{\lambda = (\lambda_1, \ldots, \lambda_s):\ \lambda_i \geqslant 0 \textit{ and } \sum_{i=1}^{s} \lambda_i = 1\right\},$$

then

$$Cf(\xi) = \inf\left\{\sum_{i=1}^{Nn+1} t_i f(\xi_i):\ \xi = \sum_{i=1}^{Nn+1} t_i \xi_i,\ t \in \Lambda_{Nn+1}\right\},$$

$$Pf(\xi) = \inf\left\{\sum_{i=1}^{\tau+1} t_i f(\xi_i):\ T(\xi) = \sum_{i=1}^{\tau+1} t_i T(\xi_i),\ t \in \Lambda_{\tau+1}\right\}.$$

Part 2. *Let* $R_0 f = f$ *and define inductively for* i *an integer,*

$$R_{i+1} f(\xi) = \inf\Big\{t R_i f(\xi_1) + (1-t) R_i f(\xi_2):\ t \in [0,1],\ \xi = t\xi_1 + (1-t)\xi_2,\ \operatorname{rank}\{\xi_1 - \xi_2\} = 1\Big\},$$

then

$$Rf(\xi) = \inf_{i \in \mathbb{N}} R_i f(\xi).$$

THEOREM 7 (Dacorogna formula). *If $f : \mathbb{R}^{N \times n} \to \mathbb{R}$ is locally bounded and Borel measurable then*

$$Qf(\xi) = \inf\left\{ \frac{1}{\operatorname{meas} \Omega} \int_{\Omega} f\big(\xi + D\varphi(x)\big)\, \mathrm{d}x\colon\ \varphi \in W_0^{1,\infty}\big(\Omega; \mathbb{R}^N\big) \right\},$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain. In particular, the infimum is independent of the choice of the domain.

REMARK 8. (i) The representation formula for Cf is standard and follows from Carathéodory theorem. The inductive way of representing Rf was found by Kohn and Strang [48]. The formulas for Pf and Qf (and a similar to that of Kohn and Strang for Rf) were established by Dacorogna (cf. [26]).

(ii) Using the separation theorems one can establish other formulas for Cf and Pf, cf. [26].

3.2. *Some examples*

We now discuss some examples that will be used in Section 7. We start with the following theorem established by Dacorogna [26] (cf. also [23] and [24]).

THEOREM 9.

Part 1. *Let $f : \mathbb{R}^{N \times n} \to \mathbb{R}$, $\Phi : \mathbb{R}^{N \times n} \to \mathbb{R}$ be quasiaffine and $g : \mathbb{R} \to \mathbb{R}$ be such that*

$$f(\xi) = g\big(\Phi(\xi)\big)$$

(*in particular if $N = n$, one can take $\Phi(\xi) = \det \xi$*), *then*

$$Pf(\xi) = Qf(\xi) = Rf(\xi) = Cg\big(\Phi(\xi)\big).$$

Part 2. *Let $N = n + 1$, $f : \mathbb{R}^{N \times n} \to \mathbb{R}$ and $g : \mathbb{R}^{n+1} \to \mathbb{R}$ be such that*

$$f(\xi) = g(\operatorname{adj}_n \xi),$$

then

$$Pf(\xi) = Qf(\xi) = Rf(\xi) = Cg(\operatorname{adj}_n \xi).$$

The next result, established by Dacorogna, Pisante and Ribeiro [34], concerns functions depending on singular values. We let $N = n$ and we denote by $\lambda_1(\xi), \dots, \lambda_n(\xi)$ the singular values of $\xi \in \mathbb{R}^{n\times n}$ with $0 \leqslant \lambda_1(\xi) \leqslant \cdots \leqslant \lambda_n(\xi)$ (which are the eigenvalues of the matrix $(\xi\xi^\top)^{1/2}$) and by Q the set

$$Q = \left\{x = (x_2, \dots, x_{n-1}) \in \mathbb{R}^{n-2}\colon\ 0 \leqslant x_2 \leqslant \cdots \leqslant x_{n-1}\right\}$$

which is the natural set, where to consider $(\lambda_2(\xi), \dots, \lambda_{n-1}(\xi))$ for $\xi \in \mathbb{R}^{n\times n}$.

THEOREM 10. *Let $g : Q \times \mathbb{R} \to \mathbb{R}$, $g = g(x,s)$, be a function such that $x \to g(x,s)$ is continuous and bounded from below for all $s \in \mathbb{R}$. Let $f : \mathbb{R}^{n\times n} \to \mathbb{R}$ be defined by*

$$f(\xi) = g\big(\lambda_2(\xi), \dots, \lambda_{n-1}(\xi), \det\xi\big),$$

then

$$Pf(\xi) = Qf(\xi) = Rf(\xi) = Ch(\det\xi),$$

where $h : \mathbb{R} \to \mathbb{R}$ is given by $h(s) = \inf_{x\in Q} g(x,s)$.

REMARK 11. We remark that if some dependence on λ_1 or λ_n is allowed, then no simple and general expression for the envelopes is known; see [34], when there is dependence on λ_1, and Theorem 3.5 by Buttazzo, Dacorogna and Gangbo [12], when there is dependence on λ_n.

The next result concerns the Saint Venant–Kirchhoff energy function, which is particularly important in nonlinear elasticity. The function, up to rescaling, is given by, $\nu \in (0, 1/2)$ being a parameter,

$$f(\xi) = \left|\xi\xi^\top - I\right|^2 + \frac{\nu}{1-2\nu}\left(|\xi|^2 - n\right)^2$$

or in terms of the singular values, $0 \leqslant \lambda_1(\xi) \leqslant \cdots \leqslant \lambda_n(\xi)$, of $\xi \in \mathbb{R}^{n\times n}$,

$$f(\xi) = \sum_{i=1}^{n}\left(\lambda_i^2 - 1\right)^2 + \frac{\nu}{1-2\nu}\left(\sum_{i=1}^{n}\lambda_i^2 - n\right)^2.$$

Le Dret and Raoult [49] have computed the quasiconvex envelope when $n = 2$ or $n = 3$ and they have shown the following.

THEOREM 12. *If $n = 2$ or $n = 3$, then*

$$Qf(\xi) = Cf(\xi).$$

When $n = 2$ it is given by

$$Cf(\xi) = Pf(\xi) = Qf(\xi) = Rf(\xi) = \begin{cases} f(\xi) & \text{if } \xi \notin D_1 \cup D_2, \\ \frac{1}{1-\nu}\left(\lambda_2^2 - 1\right)^2 & \text{if } \xi \in D_2, \\ 0 & \text{if } \xi \in D_1, \end{cases}$$

where

$$\begin{aligned} D_1 &= \left\{\xi \in \mathbb{R}^{2\times 2}\colon\ (1-\nu)\left[\lambda_1(\xi)\right]^2 + \nu\left[\lambda_2(\xi)\right]^2 < 1 \text{ and } \lambda_2(\xi) < 1\right\} \\ &= \left\{\xi \in \mathbb{R}^{2\times 2}\colon\ \lambda_1(\xi) \leqslant \lambda_2(\xi) < 1\right\}, \\ D_2 &= \left\{\xi \in \mathbb{R}^{2\times 2}\colon\ (1-\nu)\left[\lambda_1(\xi)\right]^2 + \nu\left[\lambda_2(\xi)\right]^2 < 1 \text{ and } \lambda_2(\xi) \geqslant 1\right\}. \end{aligned}$$

The last example is related to a problem of optimal design and has been studied by Kohn and Strang [48].

THEOREM 13. *Let $n = N = 2$ and*

$$f(\xi) = \begin{cases} 1 + |\xi|^2 & \text{if } \xi \neq 0, \\ 0 & \text{if } \xi = 0. \end{cases}$$

Then $Pf = Qf = Rf$ and

$$Qf(\xi) = \begin{cases} 1 + |\xi|^2 & \text{if } |\xi|^2 + 2|\det\xi| \geqslant 1, \\ 2\left(|\xi|^2 + 2|\det\xi|\right)^{1/2} - 2|\det\xi| & \text{if } |\xi|^2 + 2|\det\xi| < 1. \end{cases}$$

REMARK 14. The above result is still valid when $N \geqslant 3$, it suffices to replace $\det\xi$ by $\operatorname{adj}_2\xi \in \mathbb{R}^{\binom{N}{2}}$.

3.3. *The main theorem*

We now turn our attention to the relaxation theorem. We recall our minimization problem

$$\inf\left\{I(u) = \int_\Omega f\left(Du(x)\right)\mathrm{d}x\colon\ u \in u_0 + W_0^{1,p}\left(\Omega;\mathbb{R}^N\right)\right\}, \tag{P}$$

where $1 \leqslant p \leqslant \infty$.

We define the *relaxed problem* associated to (P) to be

$$\inf\left\{\overline{I}(u) = \int_\Omega Qf\left(Du(x)\right)\mathrm{d}x\colon\ u \in u_0 + W_0^{1,p}\left(\Omega;\mathbb{R}^N\right)\right\}. \tag{QP}$$

THEOREM 15 (Relaxation theorem). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ be Borel measurable and nonnegative satisfying, for $1 \leqslant p < \infty$,*

$$0 \leqslant f(\xi) \leqslant \alpha_1\big(1 + |\xi|^p\big) \quad \textit{for every } \xi \in \mathbb{R}^{N\times n}, \tag{2}$$

where $\alpha_1 > 0$ is a constant and for $p = \infty$ it is assumed that f is locally bounded.

Let

$$Qf = \sup\{g \leqslant f\colon\ g \textit{ quasiconvex}\}$$

be the quasiconvex envelope of f. Then

$$\inf(\mathrm{P}) = \inf(\mathrm{QP}).$$

More precisely, for every $u \in W^{1,p}(\Omega;\mathbb{R}^N)$, there exists a sequence $\{u^\nu\}_{\nu=1}^\infty \subset u_0 + W_0^{1,p}(\Omega;\mathbb{R}^N)$ such that

$$\int_\Omega f\big(Du^\nu(x)\big)\,\mathrm{d}x \to \int_\Omega Qf\big(Du(x)\big)\,\mathrm{d}x \quad \textit{as } \nu \to \infty.$$

REMARK 16. (i) If we add in the theorem a coercivity condition

$$\alpha_2\big(-1 + |\xi|^p\big) \leqslant f(\xi) \leqslant \alpha_1\big(1 + |\xi|^p\big),$$

where $\alpha_2 > 0$ and $p > 1$, we can infer that (QP) has a minimizer and that the sequence $\{u^\nu\}_{\nu=1}^\infty$ further satisfies

$$u^\nu \rightharpoonup u \quad \text{in } W^{1,p}\big(\Omega;\mathbb{R}^N\big) \text{ as } \nu \to \infty.$$

(ii) The theorem remains also valid if the function f depends on lower-order terms, i.e., $f = f(x,u,\xi)$. The quasiconvex envelope is then to be understood as the quasiconvex envelope only with respect to the variable ξ, the other variables (x,u) being kept fixed.

PROOF OF THEOREM 15. We divide the proof into two steps.

Step 1. We start with an approximation of the given function u. Let $\varepsilon > 0$ be arbitrary, we can then find disjoint open sets $\Omega_1, \dots, \Omega_k \subset \Omega$, $\xi_1, \dots, \xi_k \in \mathbb{R}^{N\times n}$, γ independent of ε and $v \in u + W_0^{1,p}(\Omega;\mathbb{R}^N)$ such that

$$\begin{cases} \operatorname{meas}\big[\Omega - \bigcup_{i=1}^k \Omega_i\big] \leqslant \varepsilon, \\ \|u\|_{W^{1,p}}, \|v\|_{W^{1,p}} \leqslant \gamma, \quad \|u - v\|_{W^{1,1}} \leqslant \varepsilon, \\ Dv(x) = \xi_i \quad \text{if } x \in \Omega_i. \end{cases} \tag{3}$$

By taking ε smaller if necessary we can also assume, using the continuity of Qf and the growth condition on f, that

$$\int_\Omega \left|Qf\big(Du(x)\big) - Qf\big(Dv(x)\big)\right| \mathrm{d}x \leqslant \varepsilon, \tag{4}$$

$$0 \leqslant \int_{\Omega - \bigcup_{i=1}^k \Omega_i} \left[f\big(Dv(x)\big) - Qf\big(Dv(x)\big)\right] \mathrm{d}x \leqslant \varepsilon. \tag{5}$$

Indeed let us discuss the case $1 \leqslant p < \infty$, the case $p = \infty$ being easy. As is well known (cf. Lemma 2.2 in [26], p. 156) any quasiconvex function is locally Lipschitz continuous and if it satisfies (2), then there exists $\beta > 0$ such that

$$\left|Qf(Du) - Qf(Dv)\right| \leqslant \beta\big(1 + |Du|^{p-1} + |Dv|^{p-1}\big)|Du - Dv|.$$

Using the Hölder inequality we obtain

$$\begin{aligned}&\int_\Omega \left|Qf(Du) - Qf(Dv)\right| \mathrm{d}x \\ &\quad\leqslant \beta\left[\int_\Omega \left[\big(1 + |Du|^{p-1} + |Dv|^{p-1}\big)\right]^{p/(p-1)}\right]^{(p-1)/p}\left[\int_\Omega |Du - Dv|^p\right]^{1/p}\end{aligned}$$

and (4) follows therefore from (3). The inequality (5) follows from (3) and a classical property of the integrals (cf. Lemma 1.4 in [26], p. 19).

Step 2. Now use Theorem 7 on every Ω_i to find $\varphi_i \in W_0^{1,\infty}(\Omega_i; \mathbb{R}^N)$

$$\begin{aligned}&\frac{1}{\operatorname{meas}\Omega_i}\int_{\Omega_i} f\big(\xi_i + D\varphi_i(x)\big)\,\mathrm{d}x \\ &\quad\geqslant Qf(\xi_i) \geqslant -\varepsilon + \frac{1}{\operatorname{meas}\Omega_i}\int_{\Omega_i} f\big(\xi_i + D\varphi_i(x)\big)\,\mathrm{d}x.\end{aligned}$$

Setting

$$w(x) = \begin{cases} v(x) + \varphi_i(x) & \text{if } x \in \Omega_i,\ i = 1, \ldots, k, \\ v(x) & \text{if } x \in \Omega - \bigcup_{i=1}^k \Omega_i, \end{cases}$$

we get that $w \in u + W_0^{1,p}(\Omega; \mathbb{R}^N)$ and (using (5))

$$0 \leqslant \int_{\bigcup_{i=1}^k \Omega_i} \left[f\big(Dw(x)\big) - Qf\big(Dv(x)\big)\right] \mathrm{d}x \leqslant \varepsilon \operatorname{meas}\left[\bigcup_{i=1}^k \Omega_i\right],$$

$$\begin{aligned}0 &\leqslant \int_{\Omega - \bigcup_{i=1}^k \Omega_i} \left[f\big(Dw(x)\big) - Qf\big(Dv(x)\big)\right] \mathrm{d}x \\ &= \int_{\Omega - \bigcup_{i=1}^k \Omega_i} \left[f\big(Dv(x)\big) - Qf\big(Dv(x)\big)\right] \mathrm{d}x \leqslant \varepsilon.\end{aligned}$$

In other words, combining these inequalities, we have proved that

$$0 \leqslant \int_{\Omega} \left[f\left(Dw(x)\right) - Qf\left(Dv(x)\right)\right] \mathrm{d}x \leqslant \varepsilon(1 + \operatorname{meas} \Omega).$$

Invoking (4), we find

$$\left| \int_{\Omega} \left[f\left(Dw(x)\right) - Qf\left(Du(x)\right)\right] \mathrm{d}x \right| \leqslant \varepsilon(2 + \operatorname{meas} \Omega).$$

Setting $\varepsilon = 1/\nu$ with $\nu \in \mathbb{N}$ and $u^{\nu} = w$, we have indeed obtained the theorem. □

We now discuss the history of this theorem (for precise references see [26]).

In the case $N = n = 1$, this result has been proved by L.C. Young and then generalized by others to the scalar case, $N = 1$ or $n = 1$, notably by Berliochi and Lasry, Ekeland, Ioffe and Tihomirov and Marcellini and Sbordone. Note that in this context

$$Qf = Cf = f^{**},$$

where Cf is the usual convex envelope of f. The problem (QP) can then be rewritten as

$$\inf\left\{ I^{**}(u) = \int_{\Omega} f^{**}\left(Du(x)\right) \mathrm{d}x\colon\ u \in u_0 + W_0^{1,p}\left(\Omega; \mathbb{R}^N\right)\right\}. \tag{P**}$$

The result for the vectorial case (i.e., $N, n > 1$, recall also that, in general, we now have $Qf > Cf$) was established by Dacorogna in [25]. Following a different approach it was later also proved by Acerbi and Fusco [1].

In the present context the equivalence between (QP) and (P**) is not any more valid, one has in general

$$\inf(\mathrm{P}) = \inf(\mathrm{QP}) > \inf\left(\mathrm{P}^{**}\right).$$

The inequality is, in general, strict as in the simple example where $N = n \geqslant 2$ and $f(\xi) = (\det \xi)^2$. We indeed have

$$f(\xi) = Qf(\xi) = (\det \xi)^2 \quad \text{and} \quad f^{**}(\xi) \equiv 0.$$

Therefore if $\det Du_0 > 0$, then, using the Jensen inequality, we have

$$\begin{aligned} \inf(\mathrm{P}) &= \inf(\mathrm{QP}) \\ &\geqslant \operatorname{meas} \Omega \left(\frac{1}{\operatorname{meas} \Omega} \int_{\Omega} \det Du_0(x)\, \mathrm{d}x \right)^2 \\ &> 0 = \inf\left(\mathrm{P}^{**}\right). \end{aligned}$$

Closely related to this approach is the notion of parametrized or Young measure, that we do not discuss here.

4. Implicit partial differential equations

4.1. *Introduction*

We now discuss the existence of solutions, $u \in W^{1,\infty}(\Omega; \mathbb{R}^N)$, for the Dirichlet problem involving differential inclusions of the form

$$\begin{cases} Du(x) \in E & \text{a.e. in } \Omega, \\ u(x) = \varphi(x), & x \in \partial\Omega, \end{cases}$$

where φ is a given function and $E \subset \mathbb{R}^{N\times n}$ is a given compact set.

To relate this study with what we said in Section 1, one should imagine that

$$E = \{\xi \in \mathbb{R}^{N\times n}\colon\ Qf(\xi) = f(\xi)\}$$

and therefore the differential inclusion is equivalent to the implicit partial differential equation

$$Qf\big(Du(x)\big) = f\big(Du(x)\big) \quad \text{a.e. } x \in \Omega.$$

In the scalar case ($n = 1$ or $N = 1$) a sufficient condition for solving the problem is

$$D\varphi(x) \in E \cup \operatorname{int}\operatorname{co} E \quad \text{a.e. in } \Omega,$$

where $\operatorname{int}\operatorname{co} E$ stands for the interior of the convex hull of E. This fact was observed by several authors, with different proofs and different levels of generality; notably in [11,17, 28,29,31,38,40]. It should be noted that this sufficient condition is also necessary, when properly reformulated.

When turning to the vectorial case ($n, N \geqslant 2$) the problem becomes considerably harder and no result with such a degree of elegancy and generality is available. The first general results were obtained by Dacorogna and Marcellini (see the References, in particular, [31]). At the same time Müller and Sverak [61] introduced the method of convex integration of Gromov in this framework, obtaining also similar existence results.

4.2. *The different convex hulls*

We recall the main notations that we will use throughout the present section and we refer, if necessary, for more details to Dacorogna and Marcellini [31].

Classically the *convex hull* of a given set E is the smallest convex set that contains E and it is denoted by $\operatorname{co} E$. We will now do the same with the other notions of convexity that we have seen earlier. This is not as straightforward as it may seem and there is not a general agreement on the exact definitions. We will not enter in abstract considerations and we will use as definition of the different hulls a consequence of these abstract definitions.

NOTATION 17. We let, for $E \subset \mathbb{R}^{N\times n}$,

$$\overline{\mathcal{F}}_E = \left\{ f : \mathbb{R}^{N\times n} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \colon\ f|_E \leqslant 0 \right\},$$
$$\mathcal{F}_E = \left\{ f \colon\ \mathbb{R}^{N\times n} \to \mathbb{R} \colon\ f|_E \leqslant 0 \right\}.$$

We then have respectively the *convex*, *polyconvex*, *rank one convex* and (closure of the) *quasiconvex hull* defined by

$$\operatorname{co} E = \left\{ \xi \in \mathbb{R}^{N\times n} \colon\ f(\xi) \leqslant 0 \text{ for every convex } f \in \overline{\mathcal{F}}_E \right\},$$
$$\operatorname{Pco} E = \left\{ \xi \in \mathbb{R}^{N\times n} \colon\ f(\xi) \leqslant 0 \text{ for every polyconvex } f \in \overline{\mathcal{F}}_E \right\},$$
$$\operatorname{Rco} E = \left\{ \xi \in \mathbb{R}^{N\times n} \colon\ f(\xi) \leqslant 0 \text{ for every rank one convex } f \in \overline{\mathcal{F}}_E \right\},$$
$$\overline{\operatorname{Qco}} E = \left\{ \xi \in \mathbb{R}^{N\times n} \colon\ f(\xi) \leqslant 0 \text{ for every quasiconvex } f \in \mathcal{F}_E \right\}.$$

We should point out that by replacing $\overline{\mathcal{F}}_E$ by $\mathcal{F}_E$ in the definitions of $\operatorname{co} E$ and $\operatorname{Pco} E$ we get their closures denoted by $\overline{\operatorname{co}}\, E$ and $\overline{\operatorname{Pco}}\, E$. However if we do so in the definition of $\operatorname{Rco} E$ we get a larger set than the closure of $\operatorname{Rco} E$. We should also draw the attention that some authors call the set

$$\left\{ \xi \in \mathbb{R}^{N\times n} \colon\ f(\xi) \leqslant 0 \text{ for every rank one convex } f \in \overline{\mathcal{F}}_E \right\}$$

the lamination convex hull, while they reserve the name of rank one convex hull to the set

$$\left\{ \xi \in \mathbb{R}^{N\times n} \colon\ f(\xi) \leqslant 0 \text{ for every rank one convex } f \in \mathcal{F}_E \right\}.$$

We think however that our terminology is more consistent with the classical definition of convex hull.

In general we have, for any set $E \subset \mathbb{R}^{N\times n}$,

$$E \subset \operatorname{Rco} E \subset \operatorname{Pco} E \subset \operatorname{co} E,$$
$$\overline{E} \subset \overline{\operatorname{Rco}} E \subset \overline{\operatorname{Qco}} E \subset \overline{\operatorname{Pco}} E \subset \overline{\operatorname{co}} E.$$

4.3. *Some examples of convex hulls*

We now give several examples that will be used in the applications of Sections 6 and 7. Let us start with the scalar case.

EXAMPLE 18 (Convex Hamiltonian). Let $F : \mathbb{R}^n \to \mathbb{R}$ be convex and let

$$E = \left\{ \xi \in \mathbb{R}^n \colon\ F(\xi) = 0 \right\},$$

then

$$\operatorname{co} E = \left\{ \xi \in \mathbb{R}^n \colon\ F(\xi) \leqslant 0 \right\}.$$

EXAMPLE 19 (Nonconvex Hamiltonian). Consider, for $\xi \in \mathbb{R}^n$, the nonconvex Hamiltonian

$$F(\xi) = \sum_{i=1}^{n} \left[(\xi_i)^2 - 1\right]^2$$

and

$$E = \left\{\xi \in \mathbb{R}^n\colon\ F(\xi) = 0\right\},$$

then

$$\operatorname{co} E = [-1, 1]^n.$$

We now turn to some examples in the vectorial case. The following result is due to Dacorogna and Tanteri (cf. [36] and also [31]), it concerns singular values. We recall that we denote by $0 \leqslant \lambda_1(\xi) \leqslant \cdots \leqslant \lambda_n(\xi)$ the singular values of a matrix $\xi \in \mathbb{R}^{n\times n}$, which are defined as the eigenvalues of the matrix $(\xi\xi^\top)^{1/2}$.

THEOREM 20. *Let*

$$E = \left\{\xi \in \mathbb{R}^{n\times n}\colon\ \lambda_i(\xi) = a_i,\ i = 1, \ldots, n\right\},$$

where $0 < a_1 \leqslant \cdots \leqslant a_n$. *The following equalities then hold*

$$\operatorname{co} E = \left\{\xi \in \mathbb{R}^{n\times n}\colon\ \sum_{i=\nu}^{n} \lambda_i(\xi) \leqslant \sum_{i=\nu}^{n} a_i,\ \nu = 1, \ldots, n\right\},$$

$$\operatorname{Pco} E = \overline{\operatorname{Qco}} E = \operatorname{Rco} E = \left\{\xi \in \mathbb{R}^{n\times n}\colon\ \prod_{i=\nu}^{n} \lambda_i(\xi) \leqslant \prod_{i=\nu}^{n} a_i,\ \nu = 1, \ldots, n\right\},$$

$$\operatorname{int}\operatorname{Rco} E = \left\{\xi \in \mathbb{R}^{n\times n}\colon\ \prod_{i=\nu}^{n} \lambda_i(\xi) < \prod_{i=\nu}^{n} a_i,\ \nu = 1, \ldots, n\right\},$$

where $\operatorname{int}\operatorname{Rco} E$ *stands for the interior of the rank one convex hull of* E.

This result admits some extensions (it corresponds in the theorem below to $\alpha = -\beta$), cf. Dacorogna and Tanteri [37] and Dacorogna and Ribeiro [35].

THEOREM 21. *Let* $\alpha \leqslant \beta$, $0 < a_2 \leqslant \cdots \leqslant a_n$ *be constants so that*

$$a_2 \prod_{i=2}^{n} a_i \geqslant \max\left\{|\alpha|, |\beta|\right\}.$$

Let

$$E = \left\{\xi \in \mathbb{R}^{n\times n}\colon\ \det\xi \in \{\alpha,\beta\},\ \lambda_i(\xi) = a_i,\ i = 2,\dots,n\right\},$$

then

$$\begin{aligned}\operatorname{Pco} E &= \overline{\operatorname{Qco}} E = \operatorname{Rco} E \\ &= \left\{\xi \in \mathbb{R}^{n\times n}\colon\ \det\xi \in [\alpha,\beta],\ \prod_{i=\nu}^{n}\lambda_i(\xi) \leqslant \prod_{i=\nu}^{n} a_i,\ \nu = 2,\dots,n\right\}.\end{aligned}$$

In particular if $\alpha = \beta$,

$$\begin{aligned}\operatorname{Pco} E &= \overline{\operatorname{Qco}} E = \operatorname{Rco} E \\ &= \left\{\xi \in \mathbb{R}^{n\times n}\colon\ \det\xi = \alpha,\ \prod_{i=\nu}^{n}\lambda_i(\xi) \leqslant \prod_{i=\nu}^{n} a_i,\ \nu = 2,\dots,n\right\}.\end{aligned}$$

REMARK 22. It is interesting to note some formal analogy between the above result (with $\alpha = \beta$) and some classical theorems of Weyl, Horn and Thompson (see [44], [45], p. 171, or [53]). Their result states that if we denote, as above, the singular values of a given matrix $\xi \in \mathbb{R}^{n\times n}$ by $0 \leqslant \lambda_1(\xi) \leqslant \cdots \leqslant \lambda_n(\xi)$, and its eigenvalues, which are complex in general, by $\mu_1(\xi),\dots,\mu_n(\xi)$ and if we order them by their modulus $(0 \leqslant |\mu_1(\xi)| \leqslant \cdots \leqslant |\mu_n(\xi)|)$ then the following result holds

$$\begin{aligned}&\prod_{i=\nu}^{n}\left|\mu_i(\xi)\right| \leqslant \prod_{i=\nu}^{n}\lambda_i(\xi), \qquad \nu = 2,\dots,n,\\ &\prod_{i=1}^{n}\left|\mu_i(\xi)\right| = \prod_{i=1}^{n}\lambda_i(\xi),\end{aligned}$$

for any matrix $\xi \in \mathbb{R}^{n\times n}$.

We will see several other examples in the next subsections.

4.4. *An existence theorem*

We start with the following definition introduced by Dacorogna and Marcellini [30] (cf. also [31]), which is the key condition to get existence of solutions.

DEFINITION 23 (Relaxation property). Let $E, K \subset \mathbb{R}^{N\times n}$. We say that K has the *relaxation property* with respect to E if for every bounded open set $\Omega \subset \mathbb{R}^n$, for every affine function u_ξ satisfying

$$Du_\xi(x) = \xi \in K,$$

there exists a sequence $u_\nu \in Aff_{\mathrm{piec}}(\overline{\Omega}; \mathbb{R}^N)$,

$$u_\nu \in u_\xi + W_0^{1,\infty}\big(\Omega; \mathbb{R}^N\big), \qquad Du_\nu(x) \in E \cup K \quad \text{a.e. in } \Omega,$$
$$u_\nu \overset{*}{\rightharpoonup} u_\xi \quad \text{in } W^{1,\infty}, \qquad \int_\Omega \operatorname{dist}\big(Du_\nu(x); E\big)\,\mathrm{d}x \to 0 \quad \text{as } \nu \to \infty.$$

REMARK 24. (i) It is interesting to note that in the scalar case ($n = 1$ or $N = 1$) then $K = \operatorname{int}\operatorname{co} E$ has the relaxation property with respect to E.

(ii) In the vectorial case we have that if K has the relaxation property with respect to E, then necessarily

$$K \subset \overline{\mathrm{Qco}}\, E.$$

Indeed first recall that the definition of quasiconvexity implies that, for every quasiconvex $f \in \mathcal{F}_E$,

$$f(\xi)\,\operatorname{meas}\Omega \leqslant \int_\Omega f\big(Du_\nu(x)\big)\,\mathrm{d}x.$$

Combining this last result with the fact that $\{Du_\nu\}$ is uniformly bounded, the fact that any quasiconvex function is continuous and the last property in the definition of the relaxation property, we get the inclusion $K \subset \overline{\mathrm{Qco}}\, E$.

The main theorem is the following one.

THEOREM 25. *Let $\Omega \subset \mathbb{R}^n$ be open. Let $E, K \subset \mathbb{R}^{N\times n}$ be such that E is compact and K is bounded. Assume that K has the relaxation property with respect to E. Let $\varphi \in Aff_{\mathrm{piec}}(\overline{\Omega}; \mathbb{R}^N)$ be such that*

$$D\varphi(x) \in E \cup K \quad \text{a.e. in } \Omega.$$

Then there exists (a dense set of) $u \in \varphi + W_0^{1,\infty}(\Omega; \mathbb{R}^N)$ such that

$$Du(x) \in E \quad \text{a.e. in } \Omega.$$

REMARK 26. (i) According to Chapter 10 in [31], the boundary datum φ can be more general if we make the following extra hypotheses:

- in the scalar case, if K is open, φ can be even taken in $W^{1,\infty}(\Omega; \mathbb{R}^N)$, with $D\varphi(x) \in E \cup K$ (cf. Corollary 10.11 in [31]);
- in the vectorial case, if the set K is open, φ can be taken in $C^1_{\mathrm{piec}}(\overline{\Omega}; \mathbb{R}^N)$ (cf. Corollary 10.15 or Theorem 10.16 in [31]), with $D\varphi(x) \in E \cup K$. While if K is open and convex, φ can be taken in $W^{1,\infty}(\Omega; \mathbb{R}^N)$ provided

$$D\varphi(x) \in C \quad \text{a.e. in } \Omega,$$

where $C \subset K$ is compact (cf. Corollary 10.21 in [31]).

(ii) In the scalar case (cf. Theorem 29) the hypothesis on the compactness of E can be dropped.

(iii) This theorem was first proved by Dacorogna and Marcellini [30] (cf. also Theorem 6.3 in [31]) under the further hypothesis that

$$E = \left\{\xi \in \mathbb{R}^{N\times n}\colon\ F_i(\xi) = 0,\ i = 1, 2, \ldots, I\right\},$$

where $F_i : \mathbb{R}^{N\times n} \to \mathbb{R}$, $i = 1, 2, \ldots, I$, are quasiconvex. This hypothesis was later removed by Sychev [74] using the theory of convex integration (see also Müller and Sychev [63]). Kirchheim in [46] pointed out that using a classical result (Theorem 38) then the proof of Dacorogna and Marcellini was still valid without the extra hypothesis on E. Kirchheim's idea, combined with the proof of [31], was then used by Dacorogna and Pisante [33] and we will follow this last approach.

PROOF OF THEOREM 25. We let $\overline{V}$ be the closure in $L^\infty(\Omega; \mathbb{R}^N)$ of

$$V = \left\{u \in Aff_{\text{piec}}\left(\overline{\Omega}; \mathbb{R}^N\right)\colon\ u = \varphi \text{ on } \partial\Omega \text{ and } Du(x) \in E \cup K\right\}.$$

V is nonempty since $\varphi \in V$. Let, for $k \in \mathbb{N}$,

$$V^k = \operatorname{int}\left\{u \in \overline{V}\colon\ \int_\Omega \operatorname{dist}\left(Du(x); E\right) \mathrm{d}x \leqslant \frac{1}{k}\right\},$$

where the "int" stands for the interior of the set. We claim that V^k, in addition to be open, is dense in the complete metric space $\overline{V}$. Postponing the proof of the last fact for the end of the proof, we conclude by Baire category theorem that

$$\bigcap_{k=1}^{\infty} V^k \subset \left\{u \in \overline{V}\colon\ \operatorname{dist}\left(Du(x), E\right) = 0 \text{ a.e. in } \Omega\right\} \subset \overline{V}$$

is dense, and hence nonempty, in $\overline{V}$. The result then follows since E is compact.

We now show that V^k is dense in $\overline{V}$. So let $u \in \overline{V}$ and $\varepsilon > 0$ be arbitrary. We wish to find $v \in V^k$ so that

$$\|u - v\|_{L^\infty} \leqslant \varepsilon.$$

We recall (cf. Section 4.6) that

$$\omega_D(\alpha) = \lim_{\delta\to 0} \sup_{v,w \in B_\infty(\alpha,\delta)} \|Dv - Dw\|_{L^1(\Omega)},$$

where

$$B_\infty(\alpha, \delta) = \left\{u \in \overline{V}\colon\ \|u - \alpha\|_{L^\infty} < \delta\right\}.$$

- We start by finding $\alpha \in \overline{V}$, a point of continuity of the operator D, so that

$$\|u - \alpha\|_{L^\infty} \leqslant \frac{\varepsilon}{3}.$$

  This is always possible by virtue of Corollary 40. In particular, we have that the oscillation $\omega_D(\alpha)$ of the gradient operator at α is zero.
- We next approximate $\alpha \in \overline{V}$ by $\beta \in V$ so that

$$\|\beta - \alpha\|_{L^\infty} \leqslant \frac{\varepsilon}{3} \quad \text{and} \quad \omega_D(\beta) < \frac{1}{2k}.$$

  This is possible since, by Proposition 37, we know that for every $\varepsilon > 0$ the set

$$\Omega_D^\varepsilon := \{u \in \overline{V} \colon \omega_D(u) < \varepsilon\}$$

  is open in $\overline{V}$.
- Finally we use the relaxation property on every piece where $D\beta$ is constant and we then construct $v \in V$, by patching all the pieces together, such that

$$\|\beta - v\|_{L^\infty} \leqslant \frac{\varepsilon}{3}, \qquad \omega_D(v) < \frac{1}{2k} \quad \text{and} \quad \int_\Omega \operatorname{dist}\bigl(Dv(x); E\bigr)\,\mathrm{d}x < \frac{1}{k}.$$

  Moreover, since $\omega_D(v) < \frac{1}{2k}$ we can find $\delta = \delta(k, v) > 0$ so that

$$\|v - \psi\|_{L^\infty} \leqslant \delta \quad \Longrightarrow \quad \|Dv - D\psi\|_{L^1} \leqslant \frac{1}{2k}$$

  and hence,

$$\int_\Omega \operatorname{dist}\bigl(D\psi(x); E\bigr)\,\mathrm{d}x \leqslant \int_\Omega \operatorname{dist}\bigl(Dv(x); E\bigr)\,\mathrm{d}x + \|Dv - D\psi\|_{L^1} < \frac{1}{k}$$

  for every $\psi \in B_\infty(v, \delta)$; which implies that $v \in V^k$.

Combining these three facts we have indeed obtained the desired density result. □

To conclude this subsection we give a sufficient condition that ensures the relaxation property. In concrete examples this condition is usually much easier to check than the relaxation property. We start with a definition.

DEFINITION 27 (Approximation property). Let $E \subset K(E) \subset \mathbb{R}^{N\times n}$. The sets E and $K(E)$ are said to have the *approximation property* if there exists a family of closed sets E_δ and $K(E_\delta)$, $\delta > 0$, such that

(1) $E_\delta \subset K(E_\delta) \subset \operatorname{int} K(E)$ for every $\delta > 0$;

(2) for every $\varepsilon > 0$ there exists $\delta_0 = \delta_0(\varepsilon) > 0$ such that $\operatorname{dist}(\eta; E) \leqslant \varepsilon$ for every $\eta \in E_\delta$ and $\delta \in [0, \delta_0]$;

(3) if $\eta \in \operatorname{int} K(E)$, then $\eta \in K(E_\delta)$ for every $\delta > 0$ sufficiently small.

We therefore have the following theorem (cf. Theorem 6.14 in [31], and for a slightly more flexible one see Theorem 6.15).

THEOREM 28. *Let $E \subset \mathbb{R}^{N\times n}$ be compact and* $\operatorname{Rco} E$ *has the approximation property with $K(E_\delta) = \operatorname{Rco} E_\delta$, then* $\operatorname{int} \operatorname{Rco} E$ *has the relaxation property with respect to E.*

4.5. *Some examples of existence of solutions*

We now give several examples of existence theorems that follow from the abstract ones.

The first one concerns the scalar case, where we can even get sharper results (cf. [11,17, 28,29,31,38,40]).

THEOREM 29. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $E \subset \mathbb{R}^n$. Let $\varphi \in W^{1,\infty}(\Omega)$ satisfy*

$$D\varphi(x) \in E \cup \operatorname{int} \operatorname{co} E \quad \text{a.e. } x \in \Omega \tag{6}$$

(*where* $\operatorname{int} \operatorname{co} E$ *stands for the interior of the convex hull of E*), *then there exists $u \in \varphi + W_0^{1,\infty}(\Omega)$ such that*

$$Du(x) \in E \quad \text{a.e. } x \in \Omega. \tag{7}$$

REMARK 30. The theorem is in fact much less restrictive than the abstract one, here we do not need, for example, E to be compact. For a proof we refer to Dacorogna and Marcellini [31].

We now show that (6) is in fact also a necessary condition, at least when φ is affine, for the general case see Section 2.4 in [31]. For the affine case the result is implicit in the above-mentioned articles, but we follow here Bandyopadhyay, Barroso, Dacorogna and Matias [9].

THEOREM 31. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $E \subset \mathbb{R}^n$, $\xi_0 \in \mathbb{R}^n$ and $u \in u_{\xi_0} + W_0^{1,\infty}(\Omega)$ (u_{ξ_0} being such that $Du_{\xi_0} = \xi_0$) so that*

$$Du(x) \in E \quad \text{a.e. } x \in \Omega,$$

then

$$\xi_0 \in E \cup \operatorname{int} \operatorname{co} E.$$

PROOF. Assume that $\xi_0 \notin E$, otherwise nothing is to be proved. It is easy to see that, by the Jensen inequality and since $Du(x) \in E$,

$$\xi_0 = \frac{1}{\operatorname{meas} \Omega} \int_\Omega Du(x)\, \mathrm{d}x \in \overline{\operatorname{co}} E.$$

Let us show that we cannot have $\xi_0 \in \partial(\overline{\text{co}}E)$. If we can prove this, we will deduce that $\xi_0 \in \operatorname{int}\overline{\text{co}}E$. Since $\operatorname{int}\overline{\text{co}}E = \operatorname{int}\text{co}\,E$ (cf. Theorem 6.3 in Rockafellar [71]), we will have the result.

If $\xi_0 \in \partial(\overline{\text{co}}E)$, we find from the separation theorem that there exists $\alpha \in \mathbb{R}^n$, $\alpha \neq 0$, such that

$$\langle \alpha; z - \xi_0 \rangle \geqslant 0 \quad \forall z \in \overline{\text{co}}E.$$

We therefore have that

$$\big\langle \alpha; Du(x) - \xi_0 \big\rangle \geqslant 0 \quad \text{a.e. } x \in \Omega.$$

Recalling that $u \in u_{\xi_0} + W_0^{1,\infty}(\Omega)$, we find that

$$\int_\Omega \big\langle \alpha; Du(x) - \xi_0 \big\rangle \, \mathrm{d}x = 0$$

which coupled with the above inequality leads to

$$\big\langle \alpha; Du(x) - \xi_0 \big\rangle = 0 \quad \text{a.e. } x \in \Omega.$$

Applying Lemma 57, we get that $u \equiv u_{\xi_0}$ and hence $\xi_0 \in E$, a contradiction with the hypothesis made at the beginning of the proof. Therefore $\xi_0 \notin \partial(\overline{\text{co}}E)$ as claimed and hence the theorem is proved. □

Theorem 29 applies to the following case.

COROLLARY 32 (Convex Hamiltonian). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $F : \mathbb{R}^n \to \mathbb{R}$ be convex and such that $\lim_{|\xi| \to \infty} F(\xi) = +\infty$. Let $\varphi \in W^{1,\infty}(\Omega)$ be such that*

$$F\big(D\varphi(x)\big) \leqslant 0 \quad \textit{a.e. } x \in \Omega.$$

Then there exists $u \in \varphi + W_0^{1,\infty}(\Omega)$ such that

$$F\big(Du(x)\big) = 0 \quad \textit{a.e. } x \in \Omega.$$

The next one deals with the singular values case that we have encountered in Section 4.3. The next theorem is due to Dacorogna and Ribeiro [35].

THEOREM 33 (Singular values). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $\alpha < \beta$ and $0 < a_2 \leqslant \cdots \leqslant a_n$ be such that*

$$\max\big\{|\alpha|, |\beta|\big\} \leqslant a_2 \prod_{i=2}^n a_i.$$

Let $\varphi \in C^1_{\text{piec}}(\overline{\Omega}; \mathbb{R}^n)$ *be such that, for almost every* $x \in \Omega$,

$$\alpha < \det D\varphi(x) < \beta, \qquad \prod_{i=\nu}^{n} \lambda_i\big(D\varphi(x)\big) < \prod_{i=\nu}^{n} a_i, \quad \nu = 2, \dots, n.$$

Then there exists $u \in \varphi + W_0^{1,\infty}(\Omega; \mathbb{R}^n)$ *so that, for almost every* $x \in \Omega$,

$$\det Du(x) \in \{\alpha, \beta\}, \qquad \lambda_\nu\big(Du(x)\big) = a_\nu, \quad \nu = 2, \dots, n.$$

REMARK 34. (i) If $\alpha = -\beta < 0$ and if we set

$$a_1 = \beta \left[\prod_{i=2}^{n} a_i \right]^{-1},$$

we recover the result of Dacorogna and Marcellini [31], namely that if

$$\prod_{i=\nu}^{n} \lambda_i\big(D\varphi(x)\big) < \prod_{i=\nu}^{n} a_i, \quad \nu = 1, \dots, n,$$

then there exists $u \in \varphi + W_0^{1,\infty}(\Omega; \mathbb{R}^n)$ so that

$$\lambda_\nu(Du) = a_\nu, \quad \nu = 1, \dots, n, \text{ a.e. in } \Omega.$$

(ii) If $\alpha = \beta \neq 0$ we can also prove, as in Dacorogna and Tanteri [37], that if

$$\det D\varphi(x) = \alpha, \qquad \prod_{i=\nu}^{n} \lambda_i\big(D\varphi(x)\big) < \prod_{i=\nu}^{n} a_i, \quad \nu = 2, \dots, n,$$

then there exists $u \in \varphi + W_0^{1,\infty}(\Omega; \mathbb{R}^n)$ so that

$$\lambda_\nu(Du) = a_\nu, \quad \nu = 2, \dots, n, \quad \text{and} \quad \det Du = \alpha \quad \text{a.e. in } \Omega.$$

4.6. *Appendix*

In this appendix we recall some well-known facts about the so-called functions of first class in the sense of Baire, with particular interest in their application to the gradient operator.

We start recalling some definitions.

DEFINITION 35. Let X, Y be metric spaces and $f : X \to Y$. We define the *oscillation* of f at $x_0 \in X$ as

$$\omega_f(x_0) = \lim_{\delta \to 0} \sup_{x, y \in B_X(x_0, \delta)} d_Y\big(f(y), f(x)\big),$$

where $B_X(x_0,\delta) := \{x \in X\colon\ d_X(x,x_0) < \delta\}$ is the open ball centered at x_0 and d_X, d_Y are the metric on the spaces X and Y, respectively.

DEFINITION 36. A function f is said to be of *first class* (in the sense of Baire) if it can be represented as the pointwise limit of an everywhere convergent sequence of continuous functions.

In the next proposition we recall some elementary properties of the oscillation function ω_f.

PROPOSITION 37. *Let X, Y be metric spaces, and $f : X \to Y$.*
(i) *f is continuous at $x_0 \in X$ if and only if $\omega_f(x_0) = 0$.*
(ii) *The set $\Omega_f^\varepsilon := \{x \in X\colon\ \omega_f(x) < \varepsilon\}$ is an open set in X.*

Using the notion of oscillation and Proposition 37 we can write the set $\mathcal{D}_f$ of all points at which a given function f is discontinuous as an F_σ set as follows

$$\mathcal{D}_f = \bigcup_{n=1}^{\infty} \left\{ x \in X\colon\ \omega_f(x) \geqslant \frac{1}{n} \right\}. \tag{8}$$

We therefore have the following Baire theorem for functions of first class (for a proof see Theorem 7.3 in Oxtoby [65], Yosida [79], p. 12, or Dacorogna and Pisante [33]).

THEOREM 38. *Let X, Y be metric spaces let X be complete and $f : X \to Y$. If f is a function of first class, then $\mathcal{D}_f$ is a set of first category.*

REMARK 39. From Theorem 38 and the Baire category theorem follows in particular that the set of points of continuity of a function of first class from a complete metric space X to any metric space Y, i.e., the set $\mathcal{D}_f^c$ complement of $\mathcal{D}_f$, is a dense G_δ set. Indeed for any $\varepsilon > 0$, the set

$$\Omega_f^\varepsilon := \left\{ x \in X\colon\ \omega_f(x) < \varepsilon \right\}$$

is open and dense in X.

In the proof of our main theorem we have used Theorem 38 applied to the following, quite surprising, special case of function of first class. This result was observed by Kirchheim [46] for complete sets of Lipschitz functions and the same argument gives in fact the result for general complete subsets $W^{1,\infty}(\Omega)$ functions.

COROLLARY 40. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $V \subset W^{1,\infty}(\Omega)$ be a nonempty complete space with respect to the L^∞ metric. Then the gradient operator $D : V \to L^p(\Omega;\mathbb{R}^n)$ is a function of first class for any $1 \leqslant p < \infty$.*

PROOF. For $h \neq 0$, we let

$$D^h = \left(D_1^h, \ldots, D_n^h\right): V \to L^p\left(\Omega; \mathbb{R}^n\right)$$

be defined, for every $u \in V$ and $x \in \Omega$, by

$$D_i^h u(x) = \begin{cases} \frac{u(x+he_i)-u(x)}{h} & \text{if } \operatorname{dist}\left(x, \Omega^c\right) > |h|, \\ 0 & \text{elsewhere,} \end{cases}$$

for $i = 1, \ldots, n$, where $e_1, \ldots, e_n$ stand for the vectors from the Euclidean basis.

The claim will follow once we will have proved that for any fixed h the operator D^h is continuous and that, for any sequence $h \to 0$,

$$\lim_{h\to 0} \left\| D_i^h u - D_i u \right\|_{L^p(\Omega)} = 0$$

for any $i = 1, \ldots, n$, $u \in V$.

The continuity of D^h follows easily by observing that for every $i = 1, \ldots, n$, $\varepsilon > 0$ and $u, v \in V$ we have that

$$\begin{aligned} &\left\| D_i^h u - D_i^h v \right\|_{L^p(\Omega)} \\ &\quad \leqslant \frac{1}{|h|} \left(\int_{\Omega_h} \left| u(x) - v(x) + u(x + he_i) - v(x + he_i) \right|^p \mathrm{d}x \right)^{1/p} \\ &\quad \leqslant \frac{2(\operatorname{meas} \Omega)^{1/p}}{|h|} \|u - v\|_{L^\infty(\Omega)}, \end{aligned}$$

where $\Omega_h = \{x \in \Omega\colon \operatorname{dist}(x, \Omega^c) > |h|\}$.

For the second claim we start observing that, for any $x \in \Omega_h$ and for any $u \in V$, we have

$$\left| u(x + he_i) - u(x) \right| \leqslant \|D_i u\|_{L^\infty(\Omega)} |h|.$$

This implies that

$$\left\| D_i^h u \right\|_{L^\infty(\Omega)} \leqslant \|D_i u\|_{L^\infty(\Omega)} < +\infty.$$

Moreover, by Rademacher theorem, for any sequence $h \to 0$,

$$\lim_{h\to 0} D_i^h u(x) = D_i u(x) \quad \text{a.e. } x \in \Omega.$$

The result follows by Lebesgue dominated convergence theorem. □

5. Existence of minimizers

5.1. *Introduction*

We now discuss the existence of minimizers for the problem

$$\inf\left\{\int_{\Omega} f\big(Du(x)\big)\,\mathrm{d}x\colon\ u\in u_{\xi_0}+W_0^{1,\infty}\big(\Omega;\mathbb{R}^N\big)\right\}, \tag{P}$$

where $\Omega\subset\mathbb{R}^n$ is a bounded open set with Lipschitz boundary, $u:\Omega\to\mathbb{R}^N$, $f:\mathbb{R}^{N\times n}\to\mathbb{R}$ is lower semicontinuous, locally bounded and nonnegative and u_{ξ_0} is a given affine map (i.e., $Du_{\xi_0}=\xi_0$, where $\xi_0\in\mathbb{R}^{N\times n}$ is a fixed matrix).

If the function f is quasiconvex, i.e.,

$$\int_U f\big(\xi+D\varphi(x)\big)\,\mathrm{d}x\geqslant f(\xi)\,\mathrm{meas}(U)$$

for every bounded domain $U\subset\mathbb{R}^n$, $\xi\in\mathbb{R}^{N\times n}$ and $\varphi\in W_0^{1,\infty}(U;\mathbb{R}^N)$, then the problem (P) trivially has u_{ξ_0} as a minimizer. We also recall that in the scalar case ($n=1$ or $N=1$), quasiconvexity and ordinary convexity are equivalent.

We now study the case where f fails to be quasiconvex. The first step in dealing with such problems is the relaxation theorem (cf. Theorem 15). It has as a direct consequence (cf. Theorem 41) that (P) has a solution $\bar{u}\in u_{\xi_0}+W_0^{1,\infty}(\Omega;\mathbb{R}^N)$ if and only if

$$f\big(D\bar{u}(x)\big)=Qf\big(D\bar{u}(x)\big)\quad \text{a.e. } x\in\Omega,$$

$$\int_{\Omega} Qf\big(D\bar{u}(x)\big)\,\mathrm{d}x=Qf(\xi_0)\,\mathrm{meas}\,\Omega,$$

where Qf is the *quasiconvex envelope* of f, namely

$$Qf=\sup\{g\leqslant f\colon\ g \text{ quasiconvex}\}.$$

The problem is then to discuss the existence or nonexistence of a $\bar{u}$ satisfying the two equations. The two equations are not really of the same nature. The first one is what we called in Section 4 an implicit partial differential equation. The second one is more geometric in nature and has to do with some "*quasiaffinity*" of the quasiconvex envelope Qf.

In the present section we will discuss some abstract necessary and sufficient conditions for the existence of minimizers for (P) and in Sections 6 and 7 we will see several examples. We will follow in the present section the approach of Dacorogna, Pisante and Ribeiro [34].

5.2. *Sufficient conditions*

With the help of the relaxation theorem and of Theorem 25, we are now in a position to discuss some existence results for the problem (P). The following theorem (cf. [27]) is elementary and gives a necessary and sufficient condition for existence of minima. It will be crucial in several of our arguments.

THEOREM 41. *Let Ω, f and u_{ξ_0} be as above, in particular, $Du_{\xi_0} = \xi_0$. The problem* (P) *has a solution if and only if there exists $\bar{u} \in u_{\xi_0} + W_0^{1,\infty}(\Omega; \mathbb{R}^N)$ such that*

$$f\big(D\bar{u}(x)\big) = Qf\big(D\bar{u}(x)\big) \quad \text{a.e. } x \in \Omega, \tag{9}$$

$$\int_\Omega Qf\big(D\bar{u}(x)\big)\,\mathrm{d}x = Qf(\xi_0)\operatorname{meas}\Omega. \tag{10}$$

PROOF. By the relaxation theorem and since u_{ξ_0} is affine, we have

$$\inf(\mathrm{P}) = \inf(\mathrm{QP}) = Qf(\xi_0)\operatorname{meas}\Omega.$$

Moreover, since we always have $f \geqslant Qf$ and we have a solution of (9) satisfying (10), we get that $\bar{u}$ is a solution of (P). The fact that (9) and (10) are necessary for the existence of a minimum for (P) follows in the same way. □

The previous theorem explains why the set

$$K = \big\{\xi \in \mathbb{R}^{N\times n}\colon\ Qf(\xi) < f(\xi)\big\}$$

plays a central role in the existence theorems that follow. In order to ensure (9) we will have to consider differential inclusions of the form studied in the previous section, namely: find $\bar{u} \in u_{\xi_0} + W_0^{1,\infty}(\Omega; \mathbb{R}^N)$ such that

$$D\bar{u}(x) \in \partial K \quad \text{a.e. } x \in \Omega.$$

In order to deal with the second condition (10) we will have to impose some hypotheses of the type "Qf is *quasiaffine* on K".

The main abstract theorem is the following one.

THEOREM 42. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $\xi_0 \in \mathbb{R}^{N\times n}$, $f : \mathbb{R}^{N\times n} \to \mathbb{R}$, a lower semicontinuous, locally bounded and nonnegative function and let*

$$K = \big\{\eta \in \mathbb{R}^{N\times n}\colon\ Qf(\eta) < f(\eta)\big\}.$$

Assume that there exists $K_0 \subset K$ such that
- $\xi_0 \in K_0$,
- *K_0 is bounded and has the relaxation property with respect to $\overline{K}_0 \cap \partial K$,*
- *Qf is quasiaffine on $\overline{K}_0$.*

Let $u_{\xi_0}(x) = \xi_0 x$. Then the problem

$$\inf\left\{I(u) = \int_\Omega f\big(Du(x)\big)\,\mathrm{d}x\colon\ u \in u_{\xi_0} + W_0^{1,\infty}\big(\Omega; \mathbb{R}^N\big)\right\} \tag{P}$$

has a solution $\bar{u} \in u_{\xi_0} + W_0^{1,\infty}(\Omega; \mathbb{R}^N)$.

REMARK 43. (i) Although this theorem applies only to functions f that takes only finite values, it can sometimes be extended to functions $f : \mathbb{R}^{N\times n} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$.

(ii) The last hypothesis in the theorem means that

$$\int_\Omega Qf\big(\xi + D\varphi(x)\big)\,\mathrm{d}x = Qf(\xi)\,\mathrm{meas}\,\Omega$$

for every $\xi \in \overline{K}_0$, every $\varphi \in W_0^{1,\infty}(\Omega; \mathbb{R}^N)$ with

$$\xi + D\varphi(x) \in \overline{K}_0 \quad \text{a.e. in } \Omega.$$

PROOF OF THEOREM 41. Since $\xi_0 \in K_0$ and K_0 is bounded and has the relaxation property with respect to $\overline{K}_0 \cap \partial K$, we can find, appealing to Theorem 25, a map $\bar{u} \in u_{\xi_0} + W_0^{1,\infty}(\Omega; \mathbb{R}^N)$ satisfying

$$D\bar{u} \in \overline{K}_0 \cap \partial K \quad \text{a.e. in } \Omega,$$

which means that (9) of Theorem 41 is satisfied. Moreover, since Qf is quasiaffine on $\overline{K}_0$, we have that (10) of Theorem 41 holds and thus the claim. □

The second hypothesis in the theorem is clearly the most difficult to verify, nevertheless there are some cases when it is automatically satisfied. For example, if K is bounded, we can take $K_0 = K$.

We will see that, in many applications, the set K turns out to be unbounded and in order to apply Theorem 42 we need to find some weaker conditions on K that guarantees the existence of a subset K_0 of K satisfying the requested properties. With this aim in mind we give the following notation and definition.

NOTATION 44. Let $K \subset \mathbb{R}^{N\times n}$ be open and $\lambda \in \mathbb{R}^{N\times n}$.

(i) For $\xi \in K$, we denote by $L_K(\xi, \lambda)$ the largest segment of the form $[\xi + t\lambda, \xi + s\lambda]$, $t < 0 < s$, so that $(\xi + t\lambda, \xi + s\lambda) \subset K$.

(ii) If $L_K(\xi, \lambda)$ is bounded, we denote by $t_-(\xi) < 0 < t_+(\xi)$ the elements so that $L_K(\xi, \lambda) = [\xi + t_-\lambda, \xi + t_+\lambda]$. They therefore satisfy

$$\xi + t_\pm\lambda \in \partial K \quad \text{and} \quad \xi + t\lambda \in K \quad \forall t \in (t_-, t_+).$$

(iii) If $H \subset K$, we let

$$L_K(H, \lambda) = \bigcup_{\xi \in H} L_K(\xi, \lambda).$$

DEFINITION 45. Let $K \subset \mathbb{R}^{N\times n}$ be open, $\xi_0 \in K$ and $\lambda \in \mathbb{R}^{N\times n}$.

(i) We say that K *is bounded at* ξ_0 *in the direction* λ if $L_K(\xi_0, \lambda)$ is bounded.

(ii) We say that K is *stably bounded at* ξ_0 *in the rank-one direction* $\lambda = \alpha \otimes \beta$ (with $\alpha \in \mathbb{R}^N$ and $\beta \in \mathbb{R}^n$) if there exists $\varepsilon > 0$ so that $L_K(\xi_0 + \alpha \otimes B_\varepsilon, \lambda)$ is bounded, where we have denoted by

$$\xi_0 + \alpha \otimes B_\varepsilon = \left\{\xi \in \mathbb{R}^{N\times n}\colon\ \xi = \xi_0 + \alpha \otimes b \text{ with } |b| < \varepsilon\right\}.$$

Clearly a bounded open set K is bounded at every point $\xi \in K$ and in any direction λ and consequently it is also stably bounded.

We now give an example of a globally unbounded set which is bounded in certain directions.

EXAMPLE 46. Let $N = n = 2$ and

$$K = \left\{\xi \in \mathbb{R}^{2\times 2}\colon\ \alpha < \det \xi < \beta\right\}.$$

The set K is clearly unbounded.

(i) If $\xi_0 = I$ then K is bounded, and even stably bounded, at ξ_0, in a direction of rank one, for example, with

$$\lambda = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \lambda = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

(ii) However if $\xi_0 = 0$, then K is unbounded in any rank one direction, but is bounded in any rank two direction.

In the following result we deal with sets K that are bounded in a rank-one direction only. This corollary says, roughly speaking, that if K is bounded at ξ_0 in a rank-one direction λ and this boundedness (in the same direction) is preserved under small perturbations of ξ_0 along rank one λ-compatible directions, then we can ensure the relaxation property required in the main existence theorem.

COROLLARY 47. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ a lower semicontinuous function, locally bounded and nonnegative and let $\xi_0 \in K$, where*

$$K = \left\{\xi \in \mathbb{R}^{N\times n}\colon\ Qf(\xi) < f(\xi)\right\}.$$

If there exist a rank-one direction $\lambda \in \mathbb{R}^{N\times n}$ such that

(i) *K is stably bounded at ξ_0 in the direction $\lambda = \alpha \otimes \beta$,*

(ii) *Qf is quasiaffine on the set* (*cf. Definition* 45) $L_K(\xi_0 + \alpha \otimes \overline{B_\varepsilon}, \lambda)$, *then the problem*

$$\inf\left\{I(u) = \int_\Omega f\big(Du(x)\big)\,\mathrm{d}x\colon\ u \in u_{\xi_0} + W_0^{1,\infty}\big(\Omega; \mathbb{R}^N\big)\right\} \tag{P}$$

has a solution $\bar{u} \in u_{\xi_0} + W_0^{1,\infty}(\Omega; \mathbb{R}^N)$.

PROOF. We divide the proof into two steps.

Step 1. Assume that $|\beta| = 1$, otherwise replace it by $\beta/|\beta|$, and let $\beta_k \in \mathbb{R}^n$, $k \geqslant n$, with $|\beta_k| = 1$, be such that

$$0 \in H := \operatorname{int} \operatorname{co}\{\beta, -\beta, \beta_3, \dots, \beta_k\} \subset B_1(0) = \{x \in \mathbb{R}^n \colon |x| < 1\}.$$

Let then, for $\varepsilon > 0$ as in the hypothesis,

$$K_0 := (\xi_0 + \alpha \otimes \varepsilon H) \cup \left[\partial K \cap L_K\left(\xi_0 + \alpha \otimes \varepsilon \overline{H}, \lambda\right)\right].$$

We therefore have that $\xi_0 \in K_0$ and, by hypothesis, that K_0 is bounded, since

$$K_0 \subset \overline{K}_0 \subset L_K\left(\xi_0 + \alpha \otimes \overline{B}_\varepsilon, \lambda\right).$$

Furthermore we have

$$\overline{K}_0 \cap \partial K = \partial K \cap L_K\left(\xi_0 + \alpha \otimes \varepsilon \overline{H}, \lambda\right).$$

In order to deduce the corollary from Theorem 42, we only need to show that K_0 has the relaxation property with respect to $\overline{K}_0 \cap \partial K$. This will be achieved in the next step.

Step 2. We now prove that K_0 has the relaxation property with respect to $\overline{K}_0 \cap \partial K$. Let $\xi \in K_0$ and let us find a sequence $u_\nu \in Aff_{\text{piec}}(\overline{\Omega}; \mathbb{R}^N)$ so that

$$\begin{aligned} &u_\nu \in u_\xi + W_0^{1,\infty}\left(\Omega; \mathbb{R}^N\right), && Du_\nu(x) \in \left(\overline{K}_0 \cap \partial K\right) \cup K_0 \quad \text{a.e. in } \Omega, \\ &u_\nu \overset{*}{\rightharpoonup} u_\xi \quad \text{in } W^{1,\infty}, && \int_\Omega \operatorname{dist}\left(Du_\nu(x); \overline{K}_0 \cap \partial K\right) \mathrm{d}x \to 0 \quad \text{as } \nu \to \infty. \end{aligned} \tag{11}$$

If $\xi \in \partial K \cap L_K(\xi_0 + \alpha \otimes \varepsilon \overline{H}, \lambda)$, nothing is to be proved; so we assume that $\xi \in \xi_0 + \alpha \otimes \varepsilon H$. By hypothesis (i), we can find $t_-(\xi) < 0 < t_+(\xi)$ so that

$$\xi_\pm := \xi + t_\pm \lambda \in \partial K \quad \text{and} \quad \xi + t\lambda \in K \quad \forall t \in (t_-, t_+)$$

and hence $\xi_\pm \in \overline{K}_0 \cap \partial K$. We moreover have that

$$\xi = \frac{-t_-}{t_+ - t_-}\xi_+ + \frac{t_+}{t_+ - t_-}\xi_- \quad \text{with } \xi_\pm \in \overline{K}_0 \cap \partial K. \tag{12}$$

Furthermore, since $\xi \in \xi_0 + \alpha \otimes \varepsilon H$, we can find $\gamma \in \varepsilon H$ such that

$$\xi = \xi_0 + \alpha \otimes \gamma.$$

The set H being open we have that $\overline{B}_\delta(\gamma) \subset \varepsilon H$, for every sufficiently small $\delta > 0$. Moreover, since for every $\delta > 0$, we have

$$0 \in \delta H = \operatorname{int} \operatorname{co}\{\pm\delta\beta, \delta\beta_3, \dots, \delta\beta_k\}$$

and since for every sufficiently small $\delta > 0$, we have

$$\pm\delta\beta \in \operatorname{co}\bigl\{\pm(t_+ - t_-)\beta\bigr\} \subset \operatorname{co}\bigl\{\pm(t_+ - t_-)\beta, \delta\beta_3, \ldots, \delta\beta_k\bigr\},$$

we get that

$$0 \in \delta H = \operatorname{int}\operatorname{co}\{\pm\delta\beta, \delta\beta_3, \ldots, \delta\beta_k\} \subset \operatorname{int}\operatorname{co}\bigl\{\pm(t_+ - t_-)\beta, \delta\beta_3, \ldots, \delta\beta_k\bigr\}.$$

We are therefore in a position to apply Lemma 48 to

$$\begin{aligned}
&a = \alpha, \qquad b = (t_+ - t_-)\beta, \qquad b_j = \delta\beta_j \quad \text{for } j = 3, \ldots, k,\ t = \frac{-t_-}{t_+ - t_-},\\
&A = \xi_+ = \xi + \frac{t_+}{t_+ - t_-}\alpha \otimes (t_+ - t_-)\beta = \xi + (1 - t)a \otimes b,\\
&B = \xi_- = \xi + \frac{t_-}{t_+ - t_-}\alpha \otimes (t_+ - t_-)\beta = \xi - ta \otimes b,
\end{aligned}$$

and find $u_\delta \in Aff_{\text{piec}}(\overline{\Omega}; \mathbb{R}^N)$, open sets $\Omega_+, \Omega_- \subset \Omega$, such that

$$\begin{cases}
\bigl|\operatorname{meas}(\Omega_+ \cup \Omega_-) - \operatorname{meas}\Omega\bigr| \leqslant \delta,\\
u_\delta(x) = u_\xi(x), \quad x \in \partial\Omega, \quad \text{and} \quad \bigl|u_\delta(x) - u_\xi(x)\bigr| \leqslant \delta, \quad x \in \Omega,\\
Du_\delta(x) = \xi_\pm \quad \text{a.e. in } \Omega_\pm,\\
Du_\delta(x) \in \xi + \{t_+\alpha \otimes \beta, t_-\alpha \otimes \beta, \alpha \otimes \delta\beta_3, \ldots, \alpha \otimes \delta\beta_k\} \quad \text{a.e. in } \Omega.
\end{cases} \tag{13}$$

Since $\xi_\pm \in \overline{K}_0 \cap \partial K$ and

$$\begin{aligned}
&\xi + \alpha \otimes \delta\beta_j \in \xi + \alpha \otimes \delta\overline{H} = \xi_0 + \alpha \otimes \bigl(\gamma + \delta\overline{H}\bigr)\\
&\quad \subset \xi_0 + \alpha \otimes \varepsilon H \subset K_0 \quad \text{for } j = 3, \ldots, k,
\end{aligned}$$

we deduce, by choosing $\delta = 1/\nu$ as $\nu \to \infty$, from (13), the relaxation property (12). This achieves the proof of Step 2 and thus of the corollary. □

We finally want to point out that, as a particular case of Corollary 47, we find the existence theorem (Theorem 3.1) proved by Dacorogna and Marcellini [27].

We have used the following result due to Müller and Sychev [63] and which is a refinement of a classical result.

LEMMA 48 (Approximation lemma). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $t \in [0, 1]$ and $A, B \in \mathbb{R}^{N\times n}$ such that*

$$A - B = a \otimes b$$

with $a \in \mathbb{R}^N$ *and* $b \in \mathbb{R}^n$. *Let* $b_3, \dots, b_k \in \mathbb{R}^n$, $k \geqslant n$, *such that* $0 \in \operatorname{int}\operatorname{co}\{b, -b, b_3, \dots, b_k\}$. *Let* φ *be an affine map such that*

$$D\varphi(x) = \xi_0 = tA + (1-t)B, \quad x \in \overline{\Omega}$$

(*i.e.*, $A = \xi_0 + (1-t)a \otimes b$ *and* $B = \xi_0 - ta \otimes b$). *Then, for every* $\varepsilon > 0$, *there exists a piecewise affine map* u *and there exist disjoint open sets* $\Omega_A, \Omega_B \subset \Omega$, *such that*

$$\begin{cases} |\operatorname{meas}\Omega_A - t\operatorname{meas}\Omega|, \ \left|\operatorname{meas}\Omega_B - (1-t)\operatorname{meas}\Omega\right| \leqslant \varepsilon, \\ u(x) = \varphi(x), \quad x \in \partial\Omega, \quad \text{and} \quad \left|u(x) - \varphi(x)\right| \leqslant \varepsilon, \quad x \in \Omega, \\ Du(x) = \begin{cases} A & \text{in } \Omega_A, \\ B & \text{in } \Omega_B, \end{cases} \\ Du(x) \in \xi_0 + \left\{(1-t)a \otimes b, -ta \otimes b, a \otimes b_3, \dots, a \otimes b_k\right\} \quad \text{a.e. in } \Omega. \end{cases}$$

5.3. *Necessary conditions*

Recall that we are considering the minimization problem

$$\inf\left\{I(u) = \int_\Omega f\big(Du(x)\big)\,\mathrm{d}x\colon\ u \in u_{\xi_0} + W_0^{1,\infty}\big(\Omega; \mathbb{R}^N\big)\right\}, \tag{P}$$

where Ω is a bounded open set of $\mathbb{R}^n$, u_{ξ_0} is affine, i.e., $Du_{\xi_0} = \xi_0$ and $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ is a lower semicontinuous, locally bounded and nonnegative function. In order to avoid the trivial case we will always assume that

$$Qf(\xi_0) < f(\xi_0).$$

Most nonexistence results for problem (P) follow by showing that the relaxed problem (QP) has a *unique* solution, namely u_{ξ_0}, which is by hypothesis not a solution of (P). This approach was strongly used in Marcellini [51], Dacorogna and Marcellini [27] and Dacorogna and Pisante [34]; we will follow here [34]. We should point out that we will give an example (see Proposition 78 in Section 7.5) related to minimal surfaces, where nonexistence occurs, while the relaxed problem has infinitely many solutions, none of them being a solution of (P).

The right notion in order to have uniqueness of the relaxed problem is the following definition.

DEFINITION 49. A quasiconvex function $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ is said to be *strictly quasiconvex* at $\xi_0 \in \mathbb{R}^{N\times n}$, if for some bounded domain $U \subset \mathbb{R}^n$ the following equality holds

$$\int_U f\big(\xi_0 + D\varphi(x)\big)\,\mathrm{d}x = f(\xi_0)\operatorname{meas}(U)$$

for some $\varphi \in W_0^{1,\infty}(U; \mathbb{R}^N)$, then necessarily $\varphi \equiv 0$.

We should observe that as in Remark 5(v) the notion of strict quasiconvexity is independent of the choice of the domain U, more precisely we have the following proposition.

PROPOSITION 50. *If a function $f:\mathbb{R}^{N\times n}\to\mathbb{R}$ is strictly quasiconvex at $\xi_0\in\mathbb{R}^{N\times n}$ for one bounded domain $U\subset\mathbb{R}^n$ it is so for any such domain.*

PROOF. Let $V\subset\mathbb{R}^n$ be a bounded domain and $\psi\in W_0^{1,\infty}(V;\mathbb{R}^N)$ be such that

$$\int_V f\big(\xi_0+D\psi(x)\big)\,\mathrm{d}x=f(\xi_0)\,\mathrm{meas}(V) \tag{14}$$

and let us conclude that we necessarily have $\psi\equiv 0$.

Choose first $a>0$ sufficiently large so that

$$V\subset Q_a=(-a,a)^n$$

and then define

$$v(x)=\begin{cases}\psi(x) & \text{if } x\in V,\\ 0 & \text{if } x\in Q_a-V,\end{cases}$$

so that $v\in W_0^{1,\infty}(Q_a;\mathbb{R}^N)$.

Let then $x_0\in U$ and choose ν sufficiently large so that

$$x_0+\frac{1}{\nu}Q_a=x_0+\left(-\frac{a}{\nu},\frac{a}{\nu}\right)^n\subset U.$$

Define next

$$\varphi(x)=\begin{cases}\frac{1}{\nu}v\big(\nu(x-x_0)\big) & \text{if } x\in x_0+\frac{1}{\nu}Q_a,\\ 0 & \text{if } x\in U-\big[x_0+\frac{1}{\nu}Q_a\big].\end{cases}$$

Observe that $\varphi\in W_0^{1,\infty}(U;\mathbb{R}^N)$ and

$$\begin{aligned}
&\int_U f\big(\xi_0+D\varphi(x)\big)\,\mathrm{d}x\\
&\quad=f(\xi_0)\,\mathrm{meas}\left(U-\left[x_0+\frac{1}{\nu}Q_a\right]\right)+\int_{[x_0+\frac{1}{\nu}Q_a]} f\Big(\xi_0+Dv\big(\nu(x-x_0)\big)\Big)\,\mathrm{d}x\\
&\quad=f(\xi_0)\left[\mathrm{meas}(U)-\frac{\mathrm{meas}(Q_a)}{\nu^n}\right]+\frac{1}{\nu^n}\int_{Q_a} f\big(\xi_0+Dv(y)\big)\,\mathrm{d}y\\
&\quad=f(\xi_0)\left[\mathrm{meas}(U)-\frac{\mathrm{meas}(Q_a)}{\nu^n}+\frac{\mathrm{meas}(Q_a-V)}{\nu^n}\right]\\
&\qquad+\frac{1}{\nu^n}\int_V f\big(\xi_0+D\psi(y)\big)\,\mathrm{d}y.
\end{aligned}$$

Appealing to (14), we deduce that

$$\int_U f\big(\xi_0 + D\varphi(x)\big)\,\mathrm{d}x = f(\xi_0)\,\mathrm{meas}(U).$$

Since f is strictly quasiconvex at $\xi_0 \in \mathbb{R}^{N\times n}$ for the domain U, we deduce that $\varphi \equiv 0$, which in turn implies that

$$v(y) \equiv 0 \quad \text{for every } y \in Q_a.$$

This finally implies that $\psi \equiv 0$ as claimed. □

We will see further some sufficient conditions that can ensure strict quasiconvexity, but let us start with the elementary following nonexistence theorem.

THEOREM 51. *Let $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ be lower semicontinuous, locally bounded and nonnegative, $\xi_0 \in \mathbb{R}^{N\times n}$ with $Qf(\xi_0) < f(\xi_0)$ and Qf be strictly quasiconvex at ξ_0. Then the relaxed problem* (QP) *has a unique solution, namely u_{ξ_0}, while* (P) *has no solution.*

PROOF. The fact that (QP) has only one solution follows by definition of the strict quasiconvexity of Qf and Proposition 50. Assume for the sake of contradiction that (P) has a solution $\bar{u} \in u_{\xi_0} + W_0^{1,\infty}(\Omega;\mathbb{R}^N)$. We should have from Theorem 41 that (writing $\bar{u}(x) = \xi_0 x + \varphi(x)$)

$$f\big(\xi_0 + D\varphi(x)\big) = Qf\big(\xi_0 + D\varphi(x)\big) \quad \text{a.e. } x \in \Omega,$$

$$\int_\Omega Qf\big(\xi_0 + D\varphi(x)\big)\,\mathrm{d}x = Qf(\xi_0)\,\mathrm{meas}\,\Omega.$$

Since Qf is strictly quasiconvex at ξ_0, we deduce from the last identity that $\varphi \equiv 0$. Hence we have, from the first identity, that $Qf(\xi_0) = f(\xi_0)$, which is in contradiction with the hypothesis. □

We now want to give some criteria that can ensure the strict quasiconvexity of a given function. The first one has been introduced by Dacorogna and Marcellini [27].

DEFINITION 52. A convex function $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ is said to be *strictly convex at $\xi_0 \in \mathbb{R}^{N\times n}$ in at least N directions* if there exists $\alpha = (\alpha^i)^{1\leqslant i\leqslant N} \in \mathbb{R}^{N\times n}$, $\alpha^i \neq 0$, for every $i = 1,\dots,N$, such that: if for some $\eta \in \mathbb{R}^{N\times n}$ the identity

$$\frac{1}{2} f(\xi_0 + \eta) + \frac{1}{2} f(\xi_0) = f\left(\xi_0 + \frac{1}{2}\eta\right)$$

holds, then necessarily

$$\langle \alpha^i ; \eta^i \rangle = 0, \quad i = 1,\dots,N.$$

In order to understand better the generalization of this notion to polyconvex functions (cf. Proposition 58), it might be enlightening to state the definition in the following way.

PROPOSITION 53. *Let $f:\mathbb{R}^{N\times n}\to\mathbb{R}$ be a convex function and, for $\xi\in\mathbb{R}^{N\times n}$, denote by $\partial f(\xi)$ the subdifferential of f at ξ. The two following conditions are then equivalent*:

(i) *f is strictly convex at $\xi_0\in\mathbb{R}^{N\times n}$ in at least N directions,*

(ii) *there exists $\alpha=(\alpha^i)^{1\leqslant i\leqslant N}\in\mathbb{R}^{N\times n}$ with $\alpha^i\neq 0$ for every $i=1,\dots,N$, so that whenever*

$$f(\xi_0+\eta)-f(\xi_0)-\langle\lambda;\eta\rangle=0$$

for some $\eta\in\mathbb{R}^{N\times n}$ and for some $\lambda\in\partial f(\xi_0)$, then

$$\langle\alpha^i;\eta^i\rangle=0,\quad i=1,\dots,N.$$

PROOF.

Step 1. We start with a preliminary observation that if

$$\frac{1}{2}f(\xi_0+\eta)+\frac{1}{2}f(\xi_0)=f\left(\xi_0+\frac{1}{2}\eta\right) \tag{15}$$

then, for every $t\in[0,1]$, we have

$$tf(\xi_0+\eta)+(1-t)f(\xi_0)=f(\xi_0+t\eta). \tag{16}$$

Let us show this under the assumption that $t>1/2$ (the case $t<1/2$ is handled similarly). We can therefore find $\alpha\in(0,1)$ such that

$$\frac{1}{2}=\alpha t+(1-\alpha)0=\alpha t.$$

From the convexity of f and by hypothesis, we obtain

$$\frac{1}{2}f(\xi_0+\eta)+\frac{1}{2}f(\xi_0)=f\left(\xi_0+\frac{1}{2}\eta\right)\leqslant\alpha f(\xi_0+t\eta)+(1-\alpha)f(\xi_0).$$

Assume, for the sake of contradiction, that

$$f(\xi_0+t\eta)<tf(\xi_0+\eta)+(1-t)f(\xi_0).$$

Combine then this inequality with the previous one to get

$$\begin{aligned}\frac{1}{2}f(\xi_0+\eta)+\frac{1}{2}f(\xi_0)&<\alpha\big[tf(\xi_0+\eta)+(1-t)f(\xi_0)\big]+(1-\alpha)f(\xi_0)\\&=\frac{1}{2}f(\xi_0+\eta)+\frac{1}{2}f(\xi_0)\end{aligned}$$

which is clearly a contradiction. Therefore the convexity of f and the above contradiction implies (16). This also implies that

$$f'(\xi_0, \eta) := \lim_{t \to 0^+} \frac{f(\xi_0 + t\eta) - f(\xi_0)}{t} = f(\xi_0 + \eta) - f(\xi_0).$$

Applying Theorem 23.4 in Rockafellar [71], combined with the fact that $\partial f(\xi_0)$ is nonempty and compact, we get that there exists $\lambda \in \partial f(\xi_0)$ so that $f(\xi_0 + \eta) - f(\xi_0) = \langle \lambda; \eta \rangle$ and hence

$$f(\xi_0 + t\eta) - f(\xi_0) - t\langle \lambda; \eta \rangle = 0 \quad \forall t \in [0, 1]. \tag{17}$$

We have therefore proved that (15) implies (17). Since the converse is obviously true, we conclude that they are equivalent.

Step 2. Let us show the equivalence of the two conditions.
(i) ⇒ (ii). We first observe that for any $\mu \in \mathbb{R}^{N \times n}$ we have

$$\begin{aligned} &\frac{1}{2} f(\xi_0 + \eta) + \frac{1}{2} f(\xi_0) - f\left(\xi_0 + \frac{1}{2}\eta\right) \\ &\quad = \frac{1}{2}\big[f(\xi_0 + \eta) - f(\xi_0) - \langle \mu; \eta \rangle\big] - \left[f\left(\xi_0 + \frac{1}{2}\eta\right) - f(\xi_0) - \frac{1}{2}\langle \mu; \eta \rangle\right]. \end{aligned} \tag{18}$$

Assume that, for $\lambda \in \partial f(\xi_0)$, we have

$$f(\xi_0 + \eta) - f(\xi_0) - \langle \lambda; \eta \rangle = 0.$$

From (18) applied to $\mu = \lambda$, from the definition of $\partial f(\xi_0)$ and from the convexity of f, we have

$$\begin{aligned} 0 &\leqslant \frac{1}{2} f(\xi_0 + \eta) + \frac{1}{2} f(\xi_0) - f\left(\xi_0 + \frac{1}{2}\eta\right) \\ &= -\left[f\left(\xi_0 + \frac{1}{2}\eta\right) - f(\xi_0) - \frac{1}{2}\langle \lambda; \eta \rangle\right] \leqslant 0. \end{aligned}$$

Using the above identity, we then are in the framework of (i) and we deduce that $\langle \alpha^i; \eta^i \rangle = 0$, $i = 1, \ldots, N$, and thus (ii).
(ii) ⇒ (i). Assume now that we have (15), namely

$$\frac{1}{2} f(\xi_0 + \eta) + \frac{1}{2} f(\xi_0) - f\left(\xi_0 + \frac{1}{2}\eta\right) = 0,$$

which, by Step 1, implies that there exists $\lambda \in \partial f(\xi_0)$ so that

$$f(\xi_0 + t\eta) - f(\xi_0) - t\langle \lambda; \eta \rangle = 0 \quad \forall t \in [0, 1].$$

We are therefore, choosing $t = 1$, in the framework of (ii) and we get $\langle \alpha^i; \eta^i \rangle = 0$, $i = 1, \ldots, N$, as wished. □

Of course any strictly convex function is strictly convex in at least N directions, but the above condition is much weaker. For example, in the scalar case, $N = 1$, it is enough that the function is not affine in a neighborhood of ξ_0, to guarantee the condition (see Corollary 55).

We now have the following result established by Dacorogna and Marcellini [27].

PROPOSITION 54. *If a convex function $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ is strictly convex at $\xi_0 \in \mathbb{R}^{N\times n}$ in at least N directions, then it is strictly quasiconvex at ξ_0.*

Theorem 51, combined with the above proposition, gives immediately a sharp result for the scalar case, namely the following corollary.

COROLLARY 55. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be lower semicontinuous, locally bounded and nonnegative, $\xi_0 \in \mathbb{R}^n$ with $Cf(\xi_0) < f(\xi_0)$ and Cf not affine in the neighborhood of ξ_0. Then* (P) *has no solution.*

REMARK 56. In the scalar case this result has been obtained by several authors, in particular, by Cellina [16], Friesecke [40] and Dacorogna and Marcellini [27]. It also gives (cf. Theorem 66), combined with the result of the preceding section, that, provided some appropriate boundedness is assumed, a necessary and sufficient condition for existence of minima for (P) is that f be affine on the connected component of $\{\xi : Cf(\xi) < f(\xi)\}$ that contains ξ_0.

Before proceeding with the proof of Proposition 54 we need the following elementary lemma.

LEMMA 57. *Let Ω be a bounded open set of $\mathbb{R}^n$ and $\varphi \in W_0^{1,\infty}(\Omega; \mathbb{R}^N)$ be such that*

$$\left\langle \alpha^i; D\varphi^i(x) \right\rangle = 0 \quad \text{a.e. } x \in \Omega,\ i = 1, \ldots, N,$$

for some $\alpha^i \neq 0$, $i = 1, \ldots, N$, then $\varphi \equiv 0$.

PROOF. Working component by component we can assume that $N = 1$ and therefore we will drop the indices. So let $\varphi \in W_0^{1,\infty}(\Omega)$ satisfy for some $\alpha \in \mathbb{R}^n$, $\alpha \neq 0$,

$$\left\langle \alpha; D\varphi(x) \right\rangle = 0 \quad \text{a.e. } x \in \Omega.$$

We then choose $\alpha_2, \ldots, \alpha_n \in \mathbb{R}^n$ so that $\{\alpha, \alpha_2, \ldots, \alpha_n\}$ generate a basis of $\mathbb{R}^n$. Let $a > 0$ and for m an integer,

$$Q_a^m = (-a, a)^m.$$

Let $x \in \Omega$ and let a and t be sufficiently small so that

$$x + \tau\alpha + \tau_2\alpha_2 + \cdots + \tau_n\alpha_n \in \Omega$$
$$\text{for every } \tau \in (0, t) \text{ and } (\tau_2, \ldots, \tau_n) \in Q_a^{n-1}.$$

Observe then that if $\varphi \in C_0^1(\Omega)$, then

$$\begin{aligned}
&\int_{Q_a^{n-1}} \big[\varphi(x + t\alpha + \tau_2\alpha_2 + \cdots + \tau_n\alpha_n) - \varphi(x + \tau_2\alpha_2 + \cdots + \tau_n\alpha_n)\big]\,\mathrm{d}\tau_2 \cdots \mathrm{d}\tau_n \\
&\quad = \int_{Q_a^{n-1}} \int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau}\big[\varphi(x + \tau\alpha + \tau_2\alpha_2 + \cdots + \tau_n\alpha_n)\big]\,\mathrm{d}\tau\,\mathrm{d}\tau_2 \cdots \mathrm{d}\tau_n \\
&\quad = \int_{Q_a^{n-1}} \int_0^t \big\langle D\varphi(x + \tau\alpha + \tau_2\alpha_2 + \cdots + \tau_n\alpha_n); \alpha\big\rangle\,\mathrm{d}\tau\,\mathrm{d}\tau_2 \cdots \mathrm{d}\tau_n.
\end{aligned}$$

By a standard regularization procedure the above identity also holds for any $\varphi \in W_0^{1,\infty}(\Omega)$. Since $\langle \alpha; D\varphi \rangle = 0$, we deduce that

$$\begin{aligned}
&\int_{Q_a^{n-1}} \big[\varphi(x + t\alpha + \tau_2\alpha_2 + \cdots + \tau_n\alpha_n) \\
&\qquad - \varphi(x + \tau_2\alpha_2 + \cdots + \tau_n\alpha_n)\big]\,\mathrm{d}\tau_2 \cdots \mathrm{d}\tau_n = 0.
\end{aligned}$$

Since φ is continuous, we deduce, by dividing by the measure of Q_a^{n-1} and letting $a \to 0$, that, for every t sufficiently small so that $x + t\alpha \in \Omega$,

$$\varphi(x + t\alpha) = \varphi(x).$$

Choosing t so that

$$x + \tau\alpha \in \Omega \quad \forall \tau \in [0, t) \quad \text{and} \quad x + t\alpha \in \partial\Omega,$$

we obtain the claim, namely

$$\varphi(x) = 0 \quad \forall x \in \Omega. \qquad \square$$

PROOF OF PROPOSITION 54. Assume that, for a certain bounded domain $U \subset \mathbb{R}^n$ and for some $\varphi \in W_0^{1,\infty}(U; \mathbb{R}^N)$, we have

$$\int_U f\big(\xi_0 + D\varphi(x)\big)\,\mathrm{d}x = f(\xi_0)\,\mathrm{meas}(U)$$

and let us show that $\varphi \equiv 0$.

Since f is convex and the above identity holds, we find

$$\begin{aligned} f(\xi_0)\,\mathrm{meas}(U) &= \int_U \left[\frac{1}{2} f(\xi_0) + \frac{1}{2} f\big(\xi_0 + D\varphi(x)\big)\right] \mathrm{d}x \\ &\geqslant \int_U f\left(\xi_0 + \frac{1}{2} D\varphi(x)\right) \mathrm{d}x \\ &\geqslant f(\xi_0)\,\mathrm{meas}(U), \end{aligned}$$

which implies that

$$\int_U \left[\frac{1}{2} f(\xi_0) + \frac{1}{2} f\big(\xi_0 + D\varphi(x)\big) - f\left(\xi_0 + \frac{1}{2} D\varphi(x)\right)\right] \mathrm{d}x = 0.$$

The convexity of f implies then that, for almost every x in U, we have

$$\frac{1}{2} f(\xi_0) + \frac{1}{2} f\big(\xi_0 + D\varphi(x)\big) - f\left(\xi_0 + \frac{1}{2} D\varphi(x)\right) = 0.$$

The strict convexity in at least N directions leads to

$$\big\langle \alpha^i; D\varphi^i(x)\big\rangle = 0 \quad \text{a.e. } x \in \Omega,\ i = 1, \ldots, N.$$

Lemma 57 gives the claim. □

We will now generalize Proposition 54. Since the notations in the next result are involved, we will first write the proposition when $N = n = 2$.

PROPOSITION 58. *Let $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ be polyconvex, $\xi_0 \in \mathbb{R}^{N\times n}$ and $\lambda = \lambda(\xi_0) \in \mathbb{R}^{\tau(N,n)}$ so that*

$$f(\xi_0 + \eta) - f(\xi_0) - \big\langle \lambda; T(\xi_0 + \eta) - T(\xi_0)\big\rangle \geqslant 0 \quad \text{for every } \eta \in \mathbb{R}^{N\times n}.$$

(i) *Let $N = n = 2$ and assume that there exist $\alpha^{1,1}, \alpha^{1,2}, \alpha^{2,2} \in \mathbb{R}^2$, $\alpha^{1,1} \neq 0$, $\alpha^{2,2} \neq 0$, $\beta \in \mathbb{R}$, so that if for some $\eta \in \mathbb{R}^{2\times 2}$ the following equality holds*

$$f(\xi_0 + \eta) - f(\xi_0) - \big\langle \lambda; T(\xi_0 + \eta) - T(\xi_0)\big\rangle = 0,$$

then necessarily

$$\big\langle \alpha^{2,2}; \eta^2\big\rangle = 0 \quad \text{and} \quad \big\langle \alpha^{1,1}; \eta^1\big\rangle + \big\langle \alpha^{1,2}; \eta^2\big\rangle + \beta \det \eta = 0.$$

Then f is strictly quasiconvex at ξ_0.

(ii) *Let $N, n \geqslant 2$ and assume that there exist, for every $\nu = 1, \dots, N$,*

$$\alpha^{\nu,\nu}, \alpha^{\nu,\nu+1}, \dots, \alpha^{\nu,N} \in \mathbb{R}^n, \quad \alpha^{\nu,\nu} \neq 0,$$
$$\beta^{\nu,s} \in \mathbb{R}^{\binom{n}{s}}, \quad 2 \leqslant s \leqslant n \wedge (N - \nu + 1),$$

so that if for some $\eta \in \mathbb{R}^{N\times n}$ the following equality holds

$$f(\xi_0 + \eta) - f(\xi_0) - \langle \lambda; T(\xi_0 + \eta) - T(\xi_0) \rangle = 0,$$

then necessarily

$$\sum_{s=\nu}^{N} \langle \alpha^{\nu,s}; \eta^s \rangle + \sum_{s=2}^{n\wedge(N-\nu+1)} \langle \beta^{\nu,s}; \mathrm{adj}_s(\eta^\nu, \dots, \eta^N) \rangle = 0, \quad \nu = 1, \dots, N.$$

Then f is strictly quasiconvex at ξ_0.

REMARK 59. (i) The existence of a λ as in the hypotheses of the proposition is automatically guaranteed by the polyconvexity of f (see (1) in Section 2, it corresponds in the case of a convex function to an element of $\partial f(\xi_0)$).

(ii) We have adopted the convention that if $l > k > 0$ are integers, then

$$\sum_{l}^{k} = 0.$$

EXAMPLE 60. Let $N = n = 2$ and consider the function

$$f(\eta) = (\eta_2^2)^2 + (\eta_1^1 + \det \eta)^2.$$

This function is trivially polyconvex and according to the proposition it is also strictly quasiconvex at $\xi_0 = 0$ (choose $\lambda = 0 \in \mathbb{R}^5$, $\alpha^{2,2} = (0, 1)$, $\alpha^{1,2} = (0, 0)$, $\alpha^{1,1} = (1, 0)$, $\beta = 1$).

PROOF OF PROPOSITION 58. We will prove the proposition only in the case $N = n = 2$, the general case being handled similarly.

Assume that, for a certain bounded domain $U \subset \mathbb{R}^2$ and for some $\varphi \in W_0^{1,\infty}(U; \mathbb{R}^2)$, we have

$$\int_U f(\xi_0 + D\varphi(x))\, \mathrm{d}x = f(\xi_0)\, \mathrm{meas}(U)$$

and let us prove that $\varphi \equiv 0$. This is equivalent, for every $\mu \in \mathbb{R}^{\tau(2,2)}$, to

$$\left[\int_U f(\xi_0 + D\varphi(x)) - f(\xi_0) - \langle \mu; T(\xi_0 + D\varphi(x)) - T(\xi_0) \rangle\right] \mathrm{d}x = 0.$$

Choosing $\mu = \lambda$ (λ as in the statement of the proposition) in the previous equation and using the polyconvexity of the function f, we get

$$f\big(\xi_0 + D\varphi(x)\big) - f(\xi_0) - \big\langle \lambda; T\big(\xi_0 + D\varphi(x)\big) - T(\xi_0)\big\rangle = 0 \quad \text{a.e. } x \in \Omega.$$

We hence infer that, for almost every $x \in \Omega$, we have

$$\big\langle \alpha^{2,2}; D\varphi^2\big\rangle = 0 \quad \text{and} \quad \big\langle \alpha^{1,1}; D\varphi^1\big\rangle + \big\langle \alpha^{1,2}; D\varphi^2\big\rangle + \beta \det D\varphi = 0.$$

Lemma 57, applied to the first equation, implies that $\varphi^2 \equiv 0$. Using this result in the second equation we get

$$\big\langle \alpha^{1,1}; D\varphi^1\big\rangle = 0$$

and hence, appealing once more to the lemma, we have the claim, namely $\varphi^1 \equiv 0$. □

Summarizing the results of Theorem 51, Propositions 54 and 58, we get the following corollary.

COROLLARY 61. *Let $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ be lower semicontinuous, locally bounded and nonnegative, $\xi_0 \in \mathbb{R}^{N\times n}$ with*

$$Qf(\xi_0) < f(\xi_0).$$

If either one of the two following conditions hold

(i) *$Qf(\xi_0) = Cf(\xi_0)$ and Cf is strictly convex at ξ_0 in at least N directions,*

(ii) *$Qf(\xi_0) = Pf(\xi_0)$ and Pf is strictly polyconvex at ξ_0* (*in the sense of Proposition* 58),

then (QP) *has a unique solution, namely u_{ξ_0}, while* (P) *has no solution.*

PROOF. The proof is almost identical under both hypotheses and so we will establish the corollary only in the first case. The result will follow from Theorem 51 if we can show that Qf is strictly convex at ξ_0. So assume that

$$\int_\Omega Qf\big(\xi_0 + D\varphi(x)\big)\,\mathrm{d}x = Qf(\xi_0)\,\mathrm{meas}\,\Omega$$

for some $\varphi \in W_0^{1,\infty}(\Omega; \mathbb{R}^N)$ and let us prove that $\varphi \equiv 0$. Using the Jensen inequality combined with the hypothesis $Qf(\xi_0) = Cf(\xi_0)$ and the fact that $Qf \geqslant Cf$, we find that the above identity implies

$$\int_\Omega Cf\big(\xi_0 + D\varphi(x)\big)\,\mathrm{d}x = Cf(\xi_0)\,\mathrm{meas}\,\Omega.$$

The hypotheses on Cf and Proposition 54 imply that $\varphi \equiv 0$, as wished. □

We now conclude this section with a different necessary condition that is based on the Carathéodory theorem.

Recall first that for any integer s, we let

$$\Lambda_s = \left\{ \lambda = (\lambda_1, \dots, \lambda_s)\colon\ \lambda_i \geqslant 0 \text{ and } \sum_{i=1}^{s} \lambda_i = 1 \right\}.$$

THEOREM 62. *If* (P) *has a solution* $\bar{u} \in u_{\xi_0} + W_0^{1,\infty}(\Omega; \mathbb{R}^N)$, *then there exist* $\mu \in \Lambda_{Nn+1}$ *and* $\xi_\nu \in \mathbb{R}^{N\times n}$, $|\xi_\nu| \leqslant \|\bar{u}\|_{W^{1,\infty}}$, $1 \leqslant \nu \leqslant Nn+1$, *such that*

$$Qf(\xi_0) \geqslant \sum_{\nu=1}^{Nn+1} \mu_\nu f(\xi_\nu) \quad \textit{and} \quad \xi_0 = \sum_{\nu=1}^{Nn+1} \mu_\nu \xi_\nu.$$

Moreover, if either $n = 1$ *or* $N = 1$, *the inequality becomes an equality, namely*

$$Cf(\xi_0) = \sum_{\nu=1}^{Nn+1} \mu_\nu f(\xi_\nu) \quad \textit{and} \quad \xi_0 = \sum_{\nu=1}^{Nn+1} \mu_\nu \xi_\nu.$$

REMARK 63. The theorem is just a curiosity in the vectorial case $n, N > 1$. However in the scalar case, $n > N = 1$, under some extra hypotheses (cf. Theorem 66), one of them being

$$\xi_0 \in \operatorname{int} \operatorname{co}\{\xi_1, \dots, \xi_{n+1}\},$$

it turns out that the necessary condition is also sufficient. But it is in the case $N \geqslant n = 1$ that it is particularly interesting since then this condition is also sufficient, cf. Theorem 64.

PROOF OF THEOREM 62. We decompose the proof into three steps.

Step 1. Let $\bar{u} \in u_{\xi_0} + W_0^{1,\infty}(\Omega; \mathbb{R}^N)$ be a solution of (P). It should therefore satisfy

$$\frac{1}{\operatorname{meas} \Omega} \int_\Omega f\big(D\bar{u}(x)\big)\, \mathrm{d}x = \inf(\mathrm{P}) = \inf(\mathrm{QP}) = Qf(\xi_0). \tag{19}$$

Let $r = \|\bar{u}\|_{W^{1,\infty}}$ and use the fact that f is locally bounded to find $R = R(r)$ so that

$$0 \leqslant f\big(D\bar{u}(x)\big) \leqslant R \quad \text{a.e. } x \in \Omega.$$

Denote by

$$K_r = \big\{ (\xi, y) \in \mathbb{R}^{N\times n} \times \mathbb{R}\colon\ |\xi| \leqslant r \text{ and } |y| \leqslant R \big\},$$
$$\operatorname{epi} f = \big\{ (\xi, y) \in \mathbb{R}^{N\times n} \times \mathbb{R}\colon\ f(\xi) \leqslant y \big\},$$
$$E = \operatorname{epi} f \cap K_r.$$

Note that since f is lower semicontinuous then epi f is closed and hence E is compact. Therefore its convex hull co E is also compact.

Observe that, for almost every $x \in \Omega$, we have

$$\big(D\bar{u}(x), f\big(D\bar{u}(x)\big)\big) \in E$$

and thus by the Jensen inequality and (19) we deduce that

$$\big(\xi_0, Qf(\xi_0)\big) = \frac{1}{\operatorname{meas}\Omega} \int_\Omega \big(D\bar{u}(x), f\big(D\bar{u}(x)\big)\big)\,\mathrm{d}x \in \operatorname{co} E.$$

Appealing to the Carathéodory theorem we can find $\lambda \in \Lambda_{Nn+2}$, $(\xi_i, y_i) \in E$, $1 \leqslant i \leqslant Nn+2$ (in particular, $f(\xi_i) \leqslant y_i$), such that

$$Qf(\xi_0) = \sum_{i=1}^{Nn+2} \lambda_i y_i \geqslant \sum_{i=1}^{Nn+2} \lambda_i f(\xi_i) \quad \text{and} \quad \xi_0 = \sum_{i=1}^{Nn+2} \lambda_i \xi_i.$$

(Note, in passing, that if f is continuous, we can replace in the above argument epi f by

$$\operatorname{graph} f = \big\{(x, y) \in \mathbb{R}^{N\times n} \times \mathbb{R}\colon\ f(x) = y\big\}$$

obtaining therefore equality instead of inequality in the above statement.)

Step 2. To obtain the theorem it therefore remains to show that one can take only $(Nn+1)$ elements. This is a classical procedure in convex analysis. The result is equivalent to showing that there exist μ_i, $1 \leqslant i \leqslant Nn+2$, such that

$$\begin{cases} \mu_i \geqslant 0, \qquad \sum_{i=1}^{Nn+2} \mu_i = 1, \quad \text{at least one of the } \mu_i = 0, \\ \sum_{i=1}^{Nn+2} \mu_i f(\xi_i) \leqslant \sum_{i=1}^{Nn+2} \lambda_i f(\xi_i), \qquad \xi_0 = \sum_{i=1}^{Nn+2} \mu_i \xi_i, \end{cases} \tag{20}$$

meaning in fact that $\mu \in \Lambda_{Nn+1}$ as wished.

Assume that $\lambda_i > 0$, $1 \leqslant i \leqslant Nn+2$, otherwise nothing is to be proved. Observe first that $\xi_0 \in \operatorname{co}\{\xi_1, \ldots, \xi_{Nn+2}\} \subset \mathbb{R}^{N\times n}$. Thus it follows from the Carathéodory theorem that there exist $\nu \in \Lambda_{Nn+2}$, with at least one of the $\nu_i = 0$ (i.e., $\nu \in \Lambda_{Nn+1}$), such that

$$\xi_0 = \sum_{i=1}^{Nn+2} \nu_i \xi_i.$$

Assume, without loss of generality, that

$$\sum_{i=1}^{Nn+2} \nu_i f(\xi_i) > \sum_{i=1}^{Nn+2} \lambda_i f(\xi_i); \tag{21}$$

otherwise choosing $\mu_i = \nu_i$ we would have immediately (20). Let

$$J = \big\{ i \in \{1, \dots, Nn+2\} \colon \lambda_i - \nu_i < 0 \big\}.$$

Observe that $J \neq \emptyset$, since otherwise $\lambda_i \geqslant \nu_i \geqslant 0$ for every i and since at least one of the $\nu_i = 0$, we would have a contradiction with $\sum \nu_i = \sum \lambda_i = 1$ and $\lambda_i > 0$ for every i. We then define

$$\gamma = \min_{i \in J} \left\{ \frac{\lambda_i}{\nu_i - \lambda_i} \right\}.$$

We clearly have that $\gamma > 0$. Finally let

$$\mu_i = \lambda_i + \gamma(\lambda_i - \nu_i), \quad 1 \leqslant i \leqslant Nn+2.$$

We immediately get that

$$\mu_i \geqslant 0, \qquad \sum_{i=1}^{Nn+2} \mu_i = 1, \quad \text{at least one of the } \mu_i = 0. \tag{22}$$

From (21) we obtain

$$\begin{aligned} \sum_{i=1}^{Nn+2} \mu_i f(\xi_i) &= \sum_{i=1}^{Nn+2} \lambda_i f(\xi_i) + \gamma \left(\sum_{i=1}^{Nn+2} \lambda_i f(\xi_i) - \sum_{i=1}^{Nn+2} \nu_i f(\xi_i) \right) \\ &\leqslant \sum_{i=1}^{Nn+2} \lambda_i f(\xi_i). \end{aligned}$$

The combination of the above with (22) (assuming for the sake of notation that $\mu_{Nn+2} = 0$) gives immediately

$$Qf(\xi_0) \geqslant \sum_{i=1}^{Nn+1} \mu_i f(\xi_i) \quad \text{and} \quad \xi_0 = \sum_{i=1}^{Nn+1} \mu_i \xi_i.$$

Step 3. The result for the scalar case follows from the fact that $Qf(\xi_0) = Cf(\xi_0)$ and from Theorem 6. □

6. The scalar case

We now see how to apply the above abstract considerations to the case where either $n = 1$ or $N = 1$. We recall that

$$\inf \left\{ I(u) = \int_\Omega f\big(Du(x)\big)\,\mathrm{d}x \colon u \in u_{\xi_0} + W_0^{1,\infty}\big(\Omega; \mathbb{R}^N\big) \right\}. \tag{P}$$

We will first treat the more elementary case where $n = 1$ and then the case $N = 1$.

6.1. *The case of single integrals*

In this very elementary case we can get much simpler and sharper results.

THEOREM 64. *Let $f : \mathbb{R}^N \to \mathbb{R}$ be nonnegative, locally bounded and lower semicontinuous. Let $a < b$, $\alpha, \beta \in \mathbb{R}^N$, $N \geqslant 1$, and*

$$\inf\left\{ I(u) = \int_a^b f\big(u'(x)\big)\,\mathrm{d}x \colon\ u \in X \right\}, \tag{P}$$

where

$$X = \big\{u \in W^{1,\infty}\big((a,b);\mathbb{R}^N\big)\colon\ u(a) = \alpha,\ u(b) = \beta\big\}.$$

The two following statements are then equivalent:

(i) *problem* (P) *has a minimizer,*

(ii) *there exist $\lambda_\nu \geqslant 0$ with $\sum_{\nu=1}^{N+1} \lambda_\nu = 1$, $\gamma_\nu \in \mathbb{R}^N$, $1 \leqslant \nu \leqslant N+1$, such that*

$$Cf\left(\frac{\beta - \alpha}{b - a}\right) = \sum_{\nu=1}^{N+1} \lambda_\nu f(\gamma_\nu) \quad \textit{and} \quad \frac{\beta - \alpha}{b - a} = \sum_{\nu=1}^{N+1} \lambda_\nu \gamma_\nu, \tag{23}$$

where $Cf = \sup\{g \leqslant f\colon\ g$ convex$\}$.

Furthermore, if (23) *is satisfied and if*

$$I_p = \left[a + (b-a)\sum_{\nu=1}^{p-1} \lambda_\nu, a + (b-a)\sum_{\nu=1}^{p} \lambda_\nu\right], \quad 1 \leqslant p \leqslant N+1,$$

then

$$\bar{u}(x) = \gamma_p(x-a) + (b-a)\sum_{\nu=1}^{p} \lambda_\nu(\gamma_\nu - \gamma_p) + \alpha, \quad x \in I_p,\ 1 \leqslant p \leqslant N+1,$$

is a solution of (P).

REMARK 65. (i) The sufficiency of (23) is implicitly or explicitly proved in the papers mentioned in the bibliography. The necessity is less known but is also implicit in the literature. The theorem as stated can be found in Dacorogna [25].

(ii) Recall that by the Carathéodory theorem (cf. Theorem 6) we always have

$$Cf\left(\frac{\beta-\alpha}{b-a}\right)=\inf\left\{\sum_{\nu=1}^{N+1}\lambda_\nu f(\gamma_\nu)\colon\ \sum_{\nu=1}^{N+1}\lambda_\nu\gamma_\nu=\frac{\beta-\alpha}{b-a}\right\}. \tag{24}$$

Therefore (23) states that a necessary and sufficient condition for existence of solutions is that the infimum in (24) be attained. Note also that if f is convex *or* f coercive (in the sense that $f(\xi)\geqslant a|\xi|^p+b$ with $p>1$, $a>0$) then the infimum in (24) is always attained.

(iii) Therefore if $f(x,u,\xi)=f(\xi)$, counterexamples to existence must be nonconvex and noncoercive; cf. Example 1, where

$$\inf\left\{I(u)=\int_0^1 \mathrm{e}^{-(u'(x))^2}\,\mathrm{d}x\colon\ u\in W_0^{1,\infty}(0,1)\right\}, \tag{P}$$

i.e., $f(\xi)=\mathrm{e}^{-\xi^2}$, then $Cf(\xi)\equiv 0$ and therefore, by the relaxation theorem,

$$\inf(\mathrm{P})=\inf(\mathrm{QP})=0.$$

However it is obvious that $I(u)\neq 0$ for every $u\in W_0^{1,\infty}(0,1)$ and hence the infimum of (P) is not attained.

(iv) A similar proof to that of Theorem 64 (see, for example, Marcellini [50]) shows that a sufficient condition to ensure existence of minima to

$$\inf\left\{I(u)=\int_a^b f\big(x,u'(x)\big)\,\mathrm{d}x\colon\ u\in X\right\} \tag{P}$$

is (23), where λ_ν and γ_ν are then measurable functions. Of course if f depends explicitly on u, the example of Bolza (cf. Example 2) shows that the theorem is then false.

PROOF OF THEOREM 64. It is easy to see that we can reduce our study to the case where

$$a=0,\qquad b=1\quad\text{and}\quad\alpha=0.$$

Sufficient condition. The sufficiency part is elementary. Let

$$\inf\left\{\overline{I}(u)=\int_0^1 Cf\big(u'(x)\big)\,\mathrm{d}x\colon\ u\in X\right\}, \tag{QP}$$

where now

$$X=\left\{u\in W^{1,\infty}\big((0,1);\mathbb{R}^N\big)\colon\ u(0)=0,\ u(1)=\beta\right\}.$$

Then $\tilde{u}(x)=\beta x$ is trivially a solution of (QP) and therefore

$$\inf(\mathrm{QP})=Cf(\beta).$$

Let now $\bar{u}$ be as in the statement of the theorem. Observe first that $\bar{u} \in W^{1,\infty}((0,1);\mathbb{R}^N)$ and $\bar{u}(0)=0$, $\bar{u}(1)=\beta$. We now compute

$$\begin{aligned}\overline{I}(\bar{u}) &= \int_0^1 f\big(\bar{u}'(x)\big)\,\mathrm{d}x = \sum_{p=1}^{N+1}\int_{I_p} f\big(\bar{u}'(x)\big)\,\mathrm{d}x = \sum_{p=1}^{N+1} f(\gamma_p)\operatorname{meas} I_p \\ &= \sum_{p=1}^{N+1}\lambda_p f(\gamma_p) = Cf(\beta) = \inf(\mathrm{QP}) \leqslant \inf(\mathrm{P}).\end{aligned}$$

Necessary condition. This has already been proved in Theorem 62. □

6.2. *The case of multiple integrals*

We now discuss the case $n > N = 1$. This is of course a more difficult case than the preceding one and no such simple result as Theorem 64 is available. However, we immediately have from Sections 5.2 and 5.3 (Theorem 29 and Corollary 55) the theorem stated below. For some historical comments on this theorem, see the remark following Corollary 55.

But let us first recall the problem and the notation. We have

$$\inf\left\{ I(u) = \int_\Omega f\big(Du(x)\big)\,\mathrm{d}x \colon\ u \in u_{\xi_0} + W_0^{1,\infty}(\Omega)\right\}, \tag{P}$$

where Ω is a bounded open set of $\mathbb{R}^n$, u_{ξ_0} is affine, i.e., $Du_{\xi_0} = \xi_0$ and $f:\mathbb{R}^n \to \mathbb{R}$ is a lower semicontinuous, locally bounded and nonnegative function. Let

$$Cf = \sup\{g \leqslant f\colon\ g \text{ convex}\}.$$

In order to avoid the trivial situation we assume that

$$Cf(\xi_0) < f(\xi_0).$$

We next set

$$K = \big\{\xi \in \mathbb{R}^n\colon\ Cf(\xi) < f(\xi)\big\}$$

and we assume that it is connected, otherwise we replace it by its connected component that contains ξ_0.

THEOREM 66.

Necessary condition. *If* (P) *has a minimizer, then Cf is affine in a neighborhood of ξ_0.*

Sufficient condition. *If there exists $E \subset \partial K$ such that $\xi_0 \in \operatorname{int}\operatorname{co} E$ and $Cf|_{E\cup\{\xi_0\}}$ is affine, then* (P) *has a solution.*

REMARK 67. (i) By $Cf|_{E\cup\{\xi_0\}}$ affine we mean that there exist $\alpha \in \mathbb{R}^n$, $\beta \in \mathbb{R}$ such that

$$Cf(\xi) = \langle \alpha; \xi \rangle + \beta \quad \text{for every } \xi \in E \cup \{\xi_0\}.$$

Usually one proves that Cf is affine on the whole of $\operatorname{co} E$.

(ii) The theorem applies, of course, to the case where $E = \partial K$ and Cf is affine on the whole of K (since K is open and $\xi_0 \in K \subset \operatorname{int}\operatorname{co} K$). However in many simple examples such as the one given below, it is not realistic to assume that $E = \partial K$.

PROOF OF THEOREM 66. The necessary part is just Corollary 55. We therefore discuss only the sufficient part. We use Theorem 29 to find $\bar{u} \in u_{\xi_0} + W_0^{1,\infty}(\Omega)$ such that

$$D\bar{u}(x) \in E \subset \partial K \quad \text{a.e. } x \in \Omega$$

and hence

$$f\big(D\bar{u}(x)\big) = Cf\big(D\bar{u}(x)\big) \quad \text{a.e. } x \in \Omega.$$

Then use the fact that $Cf|_{E\cup\{\xi_0\}}$ is affine to deduce that

$$\int_\Omega Cf\big(D\bar{u}(x)\big)\,\mathrm{d}x = Cf(\xi_0)\operatorname{meas}\Omega.$$

The conclusion then follows from Theorem 41. □

We now would like to give two simple examples. The first one generalizes Example 3.

EXAMPLE 68. Let $N = 1$, $n = 2$, $\Omega = (0,1)^2$, $u_0 \equiv 0$, $a \geqslant 0$ and

$$f(\xi) = \big(\xi_1^2 - 1\big)^2 + \big(\xi_2^2 - a^2\big)^2.$$

We find that

$$Cf(\xi) = \big[\xi_1^2 - 1\big]_+^2 + \big[\xi_2^2 - a^2\big]_+^2,$$

where

$$[x]_+ = \begin{cases} x & \text{if } x \geqslant 0, \\ 0 & \text{if } x < 0. \end{cases}$$

We therefore have that

$$K = \big\{\xi \in \mathbb{R}^2\colon\ \xi_1^2 < 1 \text{ or } \xi_2^2 < a^2\big\}$$

and note that it is unbounded and that Cf is not affine on the whole of K.

Let us discuss the two different cases.

Case 1: $a = 0$. This corresponds to Example 3. Then clearly Cf is not affine in the neighborhood of $\xi_0 = 0$, since it is strictly convex in the direction $e_2 = (0, 1)$. Hence (P) has no solution.

Case 2: $a > 0$. We let

$$E = \{\xi \in \mathbb{R}^2\colon\ |\xi_1| = 1 \text{ and } |\xi_2| = a\} \subset \partial K.$$

Note that $\xi_0 = 0 \in \operatorname{int}\operatorname{co} E$ and $Cf|_{\operatorname{co} E} \equiv 0$ is affine. Therefore the theorem applies and we obtain that (P) has a solution.

EXAMPLE 69. We conclude with the following example (cf. Marcellini [51] and Dacorogna and Marcellini [27]). Let $n \geqslant 2$ and

$$f(Du) = g(|Du|),$$

where $g : \mathbb{R} \to \mathbb{R}$ is lower semicontinuous, locally bounded and nonnegative with

$$g(0) = \inf\{g(t) : t \geqslant 0\}.$$

It is easy to see that $Cf = Cg$. Let

$$S = \{t \geqslant 0\colon\ Cg(t) < g(t)\},$$
$$K = \{\xi \in \mathbb{R}^n\colon\ Cf(\xi) < f(\xi)\} = \{\xi \in \mathbb{R}^n\colon\ |\xi| \in S\}.$$

Assume that $\xi_0 \in K$ and that S is connected, otherwise replace it by its connected component containing $|\xi_0|$.

We then have to consider two cases.

Case 1: *Cg is strictly increasing at $|\xi_0|$.* Then clearly Cf is not affine in any neighborhood of ξ_0 and hence (P) has no solution.

Case 2: *Cg is constant on S.* Assume that S is bounded, this can be guaranteed if, for example,

$$\lim_{t \to +\infty} \frac{g(t)}{t} = +\infty.$$

So let $|\xi_0| \in S = (\alpha, \beta)$ and choose in the sufficient part of the theorem

$$E = \{\xi \in \mathbb{R}^n\colon\ |\xi| = \beta\}$$

and apply the theorem to find a minimizer for (P).

7. The vectorial case

We now consider several examples of the form studied in the previous sections, namely

$$\inf\left\{I(u)=\int_{\Omega} f\big(Du(x)\big)\,\mathrm{d}x:\ u\in u_{\xi_0}+W_0^{1,\infty}\big(\Omega;\mathbb{R}^N\big)\right\},\tag{P}$$

where Ω is a bounded open set of $\mathbb{R}^n$, u_{ξ_0} is affine, i.e., $Du_{\xi_0}=\xi_0$ and $f:\mathbb{R}^{N\times n}\to\mathbb{R}$ is a lower semicontinuous, locally bounded and nonnegative function.

1. We consider in Section 7.1 the case where $N=n$ and

$$f(\xi)=g\big(\lambda_2(\xi),\dots,\lambda_{n-1}(\xi),\det\xi\big),$$

where $0\leqslant\lambda_1(\xi)\leqslant\cdots\leqslant\lambda_n(\xi)$ are the singular values of $\xi\in\mathbb{R}^{n\times n}$.

2. In Section 7.2 we deal with the case

$$f(\xi)=g\big(\Phi(\xi)\big),$$

where $\Phi:\mathbb{R}^{N\times n}\to\mathbb{R}$ is quasiaffine (so in particular, we can have, when $N=n$, $\Phi(\xi)=\det\xi$, as in the previous case).

3. We next discuss in Section 7.3 the Saint Venant–Kirchhoff energy functional. Up to rescaling, the function under consideration is (here $N=n$ and $\nu\in(0,1/2)$ is a parameter)

$$f(\xi)=\left|\xi\xi^{\top}-I\right|^2+\frac{\nu}{1-2\nu}\big(|\xi|^2-n\big)^2$$

or in terms of the singular values, $0\leqslant\lambda_1(\xi)\leqslant\cdots\leqslant\lambda_n(\xi)$, of $\xi\in\mathbb{R}^{n\times n}$,

$$f(\xi)=\sum_{i=1}^{n}\big(\lambda_i^2-1\big)^2+\frac{\nu}{1-2\nu}\left(\sum_{i=1}^{n}\lambda_i^2-n\right)^2.$$

4. In Section 7.4 we consider a problem of optimal design where $N=n=2$ and

$$f(\xi)=\begin{cases}1+|\xi|^2 & \text{if } \xi\neq 0,\\ 0 & \text{if } \xi=0.\end{cases}$$

5. In Section 7.5 we deal with the minimal surface case, namely when $N=n+1$ and $f(\xi)=g(\operatorname{adj}_n\xi)$.

6. Finally in Section 7.6 we discuss the problem of potential wells.

We recall that, throughout Section 7, the sets $O(n)$ and $SO(n)$ will denote respectively the set of *orthogonal* and *special orthogonal* matrices, more presicely,

$$O(n)=\big\{R\in\mathbb{R}^{n\times n}:\ RR^{\top}=1\big\},$$
$$SO(n)=\big\{R\in O(n):\ \det R=1\big\}.$$

7.1. *The case of singular values*

In this section we let $N = n$ and we denote by $\lambda_1(\xi), \dots, \lambda_n(\xi)$ the singular values of $\xi \in \mathbb{R}^{n\times n}$ with $0 \leqslant \lambda_1(\xi) \leqslant \cdots \leqslant \lambda_n(\xi)$ and by Q the set

$$Q = \left\{x = (x_2, \dots, x_{n-1}) \in \mathbb{R}^{n-2}\colon\ 0 \leqslant x_2 \leqslant \cdots \leqslant x_{n-1}\right\},$$

which is the natural set where to consider $(\lambda_2(\xi), \dots, \lambda_{n-1}(\xi))$ for $\xi \in \mathbb{R}^{n\times n}$.

The functions under consideration are of the form studied in Theorem 10, namely

$$f(\xi) = g\big(\lambda_2(\xi), \dots, \lambda_{n-1}(\xi), \det\xi\big),$$

and we have

$$Pf(\xi) = Qf(\xi) = Rf(\xi) = Ch(\det\xi),$$

where $h:\mathbb{R} \to \mathbb{R}$ is given by $h(s) = \inf_{x\in Q} g(x, s)$.

We next apply the theory of Section 5.2 to get the following existence result established by Dacorogna, Pisante and Ribeiro [34].

THEOREM 70. *Let*

$$f(\xi) = g\big(\lambda_2(\xi), \dots, \lambda_{n-1}(\xi)\big) + h(\det\xi),$$

where $g : Q \to \mathbb{R}$ *is nonnegative, continuous and verifies*

$$\inf g = g(m_2, \dots, m_{n-1}) \quad \textit{with } 0 < m_2 \leqslant \cdots \leqslant m_{n-1}$$

and $h:\mathbb{R} \to \mathbb{R}$ *is a nonnegative, lower semicontinuous and locally bounded function such that*

$$\lim_{|t|\to+\infty} \frac{h(t)}{|t|} = +\infty. \tag{25}$$

Then (P) *has a solution.*

PROOF. We note that, by Theorem 10, $Qf(\xi) = \inf g + Ch(\det\xi)$. Letting

$$K = \left\{\xi \in \mathbb{R}^{n\times n}\colon\ Qf(\xi) < f(\xi)\right\}$$

we see that

$$K = L_1 \cup L_2,$$

where

$$L_1 = \left\{\xi \in \mathbb{R}^{n\times n}\colon\ Ch(\det\xi) < h(\det\xi)\right\},$$
$$L_2 = \left\{\xi \in \mathbb{R}^{n\times n}\colon\ Ch(\det\xi) = h(\det\xi),\ \inf g < g\big(\lambda_2(\xi),\dots,\lambda_{n-1}(\xi)\big)\right\}.$$

We now prove the result. Clearly, if $\xi_0 \notin K$ then u_{ξ_0} is a solution of (P), so from now on we assume that $\xi_0 \in K$. There are three different cases to consider, one of them will be treated with Theorem 42 and the two others with Theorem 41.

Case 1: $\xi_0 \in L_1$. We first observe that hypothesis (25) allows us to write

$$S = \left\{t \in \mathbb{R}\colon\ Ch(t) < h(t)\right\} = \bigcup_{j\in\mathbb{N}} (\alpha_j, \beta_j),$$

Ch being affine in each interval (α_j, β_j); thus Qf is quasiaffine on each connected component of L_1 and

$$L_1 = \left\{\xi \in \mathbb{R}^{n\times n}\colon\ \det\xi \in \bigcup_{j\in\mathbb{N}} (\alpha_j, \beta_j)\right\}.$$

Let (α_j, β_j) be an interval as above such that $\det\xi_0 \in (\alpha_j, \beta_j)$. We get the result applying Theorem 42 with

$$K_0 = \left\{\xi \in \mathbb{R}^{n\times n}\colon\ \det\xi \in (\alpha_j, \beta_j),\ \prod_{i=\nu}^{n} \lambda_i(\xi) < \prod_{i=\nu}^{n} m_i,\ \ \nu = 2,\dots,n\right\},$$

where m_n is chosen sufficiently large so that

$$m_{n-1} \leqslant m_n, \tag{26}$$
$$\prod_{i=\nu}^{n} \lambda_i(\xi_0) < \prod_{i=\nu}^{n} m_i, \quad \nu = 2,\dots,n, \tag{27}$$
$$\max\left\{|\alpha_j|, |\beta_j|\right\} < m_2 \prod_{i=2}^{n} m_i. \tag{28}$$

Clearly $K_0 \subset L_1 \subset K$, moreover, (27) ensures that $\xi_0 \in K_0$ and (28) ensures the relaxation property of K_0 with respect to

$$E = \left\{\xi \in \mathbb{R}^{n\times n}\colon\ \det\xi \in \{\alpha_j, \beta_j\}, \lambda_\nu(\xi) = m_\nu,\ \nu = 2,\dots,n\right\} \subset \overline{K}_0 \cap \partial K$$

through Theorems 21 and 28 and the family of sets

$$E_\delta = \left\{\xi \in \mathbb{R}^{n\times n}\colon\ \det\xi \in \{\alpha_j + \delta, \beta_j - \delta\}, \lambda_i(\xi) = m_i - \delta,\ i = 2,\dots,n\right\}$$

(cf. the proof of Theorem 1.1 of Dacorogna and Ribeiro [35] for details). Consequently K_0 has the relaxation property with respect to $\overline{K}_0 \cap \partial K$.

Case 2: $\xi_0 \in L_2$ *and* $\det \xi_0 \neq 0$. We consider in this case the set

$$K_1 = \left\{ \xi \in \mathbb{R}^{n\times n}\colon \det \xi = \det \xi_0,\ \prod_{i=\nu}^{n} \lambda_i(\xi) < \prod_{i=\nu}^{n} m_i,\ \nu = 2, \dots, n \right\},$$

where m_n satisfies the conditions (26) and (27) of the first case (with strict inequality for the first one: $m_n > m_{n-1}$). It was shown by Dacorogna and Tanteri [37] that K_1 has the relaxation property with respect to

$$E = \left\{ \xi \in \mathbb{R}^{n\times n}\colon \det \xi = \det \xi_0,\ \lambda_\nu(\xi) = m_\nu,\ \nu = 2, \dots, n \right\},$$

and moreover, there exists $u \in u_{\xi_0} + W_0^{1,\infty}(\Omega, \mathbb{R}^n)$ such that $Du \in E$ a.e. in Ω. Since $Qf = f$ in E and $Qf(\xi_0) = Qf(Du)$, we can apply Theorem 41 and get the result.

Case 3: $\xi_0 \in L_2$ *and* $\det \xi_0 = 0$. We here just briefly outline the idea and we refer to Dacorogna, Pisante and Ribeiro [34] for details. Since any matrix $\xi \in \mathbb{R}^{n\times n}$ can be decomposed in the form RDQ, where $R, Q \in O(n)$ and $D = \operatorname{diag}(\lambda_1(\xi), \dots, \lambda_n(\xi))$ (cf. [45]) we can reduce ourselves to the case of $\xi_0 = \operatorname{diag}(\lambda_1(\xi_0), \dots, \lambda_n(\xi_0))$. In particular, as $\det \xi_0 = 0$, we have $\lambda_1(\xi_0) = 0$ and thus the first line of ξ_0 equal to zero. Let $m_n \geqslant m_{n-1}$ and define

$$K_1 = \left\{ \xi \in \mathbb{R}^{n\times n}\colon \xi^1 = 0,\ \prod_{i=\nu}^{n} \lambda_i(\xi) < \prod_{i=\nu}^{n} m_i,\ \nu = 2, \dots, n \right\},$$
$$E = \left\{ \xi \in \mathbb{R}^{n\times n}\colon \xi^1 = 0,\ \lambda_i(\xi) = m_i,\ i = 2, \dots, n \right\},$$

we get that K_1 has the relaxation property with respect to E. If we choose m_n sufficiently large such that $\xi_0 \in K_1$ we can apply Theorem 25 to get the existence of $u \in u_{\xi_0} + W_0^{1,\infty}(\Omega, \mathbb{R}^n)$ such that $Du \in E$. Finally, as $Qf = f$ in E and $Qf(\xi_0) = Qf(Du)$, applying Theorem 41, we conclude the proof. □

7.2. *The case of quasiaffine functions*

We next study the minimization problem

$$\inf\left\{ \int_\Omega g\big(\Phi\big(Du(x)\big)\big)\,\mathrm{d}x\colon u \in u_{\xi_0} + W_0^{1,\infty}\big(\Omega; \mathbb{R}^N\big) \right\}, \tag{P}$$

where Ω is a bounded open set of $\mathbb{R}^n$, $Du_{\xi_0} = \xi_0$ and

- $g : \mathbb{R} \to \mathbb{R}$ is a lower semicontinuous, locally bounded and nonnegative function,
- $\Phi : \mathbb{R}^{N\times n} \to \mathbb{R}$ is quasiaffine and nonconstant.

We recall that in particular we can have, when $N = n$, $\Phi(\xi) = \det \xi$.

The relaxed problem is then

$$\inf\left\{\int_\Omega Cg\big(\Phi\big(Du(x)\big)\big)\,\mathrm{d}x\colon\ u \in u_{\xi_0} + W_0^{1,\infty}\big(\Omega;\mathbb{R}^N\big)\right\}, \tag{QP}$$

where Cg is the convex envelope of g (here $f(\xi) = g(\Phi(\xi))$ and we get $Qf = Cg$, cf. Theorem 9).

The existence result is the following theorem.

THEOREM 71. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, $g : \mathbb{R} \to \mathbb{R}$ a nonnegative, lower semicontinuous and locally bounded function such that*

$$\lim_{|t|\to+\infty} \frac{g(t)}{|t|} = +\infty \tag{29}$$

and $u_{\xi_0}(x) = \xi_0 x$, with $\xi_0 \in \mathbb{R}^{N\times n}$. Then there exists $\bar{u} \in u_{\xi_0} + W_0^{1,\infty}(\Omega;\mathbb{R}^N)$ solution of (P).

REMARK 72. This result has first been established by Mascolo and Schianchi [55] and later by Dacorogna and Marcellini [27] for the case of the determinant. The general case is due to Cellina and Zagatti [19] and later to Dacorogna and Ribeiro [35]. Here we see that it can be obtained as a particular case of Theorem 42.

PROOF OF THEOREM 71. We will here only sketch the proof and we refer for details to Dacorogna and Ribeiro [35]. We first let

$$S = \big\{t \in \mathbb{R}\colon\ Cg(t) < g(t)\big\}.$$

From the hypothesis on g we can write

$$S = \bigcup_{j\in\mathbb{N}} (\alpha_j, \beta_j)$$

with Cg affine in each interval (α_j, β_j) and thus Qf is quasiaffine on each connected component of K, where

$$K = \big\{\xi \in \mathbb{R}^{N\times n}\colon\ \Phi(\xi) \in S\big\}.$$

If $\Phi(\xi_0) \notin S$ then u_{ξ_0} is a solution of (P). In the other case, $\Phi(\xi_0) \in (\alpha, \beta) \subset S$ for some α and β, and we apply Theorem 42 with

$$K_0 = \big\{\xi \in \mathbb{R}^{N\times n}\colon\ \Phi(\xi) \in (\alpha, \beta),\ \big|\xi_j^i\big| < c_j^i,\ i = 1, \dots, N,\ j = 1, \dots, n\big\},$$

where c^i_j are constants sufficiently large so that $\xi_0 \in K_0$ and satisfying

$$\inf\left\{\left|\Phi(\xi)\right|: \left|\xi^i_j\right| = c^i_j\right\} > \max\left\{|\alpha|, |\beta|\right\}.$$

This condition allows us to obtain the relaxation property of K_0 with respect to

$$\overline{K_0} \cap \partial K = \left\{\xi \in \mathbb{R}^{N\times n}: \Phi(\xi) \in \{\alpha, \beta\}, \left|\xi^i_j\right| \leqslant c^i_j, i = 1, \ldots, N, j = 1, \ldots, n\right\}.$$

The relaxation property is obtained using the approximation property (cf. Definition 27 and Theorem 28) considering the sets, here $\delta > 0$ is sufficiently small,

$$H_\delta = \left\{\xi \in \mathbb{R}^{N\times n}: \Phi(\xi) \in \{\alpha + \delta, \beta - \delta\},\right.$$
$$\left.\left|\xi^i_j\right| \leqslant c^i_j - \delta, i = 1, \ldots, N, j = 1, \ldots, n\right\}.$$

This concludes the proof of the theorem. □

The problem under consideration is sufficiently flexible that we could also proceed as in Dacorogna and Marcellini [27], using Corollary 47. Indeed if $D\Phi(\xi_0) \neq 0$ (in the case $\Phi(\xi) = \det \xi$ this means that $\operatorname{rank} \xi_0 \geqslant n - 1$), we can apply the corollary, since the connected component of K containing ξ_0 is bounded, in the neighborhood of ξ_0, in a direction of rank one. We do not discuss the details of this different approach.

7.3. *The Saint Venant–Kirchhoff energy*

The problem is now of the form

$$\inf\left\{\int_\Omega f\left(Du(x)\right) \mathrm{d}x: u \in u_{\xi_0} + W^{1,\infty}_0\left(\Omega; \mathbb{R}^n\right)\right\}, \tag{P}$$

where, upon rescaling, the function under consideration is, $\nu \in (0, 1/2)$ being a parameter,

$$f(\xi) = \left|\xi\xi^\top - I\right|^2 + \frac{\nu}{1 - 2\nu}\left(|\xi|^2 - n\right)^2$$

or in terms of the singular values, $0 \leqslant \lambda_1(\xi) \leqslant \cdots \leqslant \lambda_n(\xi)$, of $\xi \in \mathbb{R}^{n\times n}$,

$$f(\xi) = \sum_{i=1}^n \left(\lambda_i^2 - 1\right)^2 + \frac{\nu}{1 - 2\nu}\left(\sum_{i=1}^n \lambda_i^2 - n\right)^2.$$

According to Le Dret and Raoult [49] the quasiconvex envelope and the convex envelope coincide, at least when $n = 2$ or $n = 3$, i.e.,

$$Qf(\xi) = Cf(\xi).$$

In the case $n = 2$, it is given by

$$Qf(\xi) = \begin{cases} f(\xi) & \text{if } \xi \notin D_1 \cup D_2, \\ \frac{1}{1-\nu}\big(\lambda_2^2 - 1\big)^2 & \text{if } \xi \in D_2, \\ 0 & \text{if } \xi \in D_1, \end{cases}$$

where

$$\begin{aligned} D_1 &= \big\{\xi \in \mathbb{R}^{2\times 2}\colon\ (1-\nu)\big[\lambda_1(\xi)\big]^2 + \nu\big[\lambda_2(\xi)\big]^2 < 1 \text{ and } \lambda_2(\xi) < 1\big\} \\ &= \big\{\xi \in \mathbb{R}^{2\times 2}\colon\ \lambda_1(\xi) \leqslant \lambda_2(\xi) < 1\big\}, \\ D_2 &= \big\{\xi \in \mathbb{R}^{2\times 2}\colon\ (1-\nu)\big[\lambda_1(\xi)\big]^2 + \nu\big[\lambda_2(\xi)\big]^2 < 1 \text{ and } \lambda_2(\xi) \geqslant 1\big\}. \end{aligned}$$

The existence theorem is the following one.

THEOREM 73. *Let $\Omega \subset \mathbb{R}^2$, f and ξ_0 be as above.*
(i) *If $\xi_0 \notin D_2$ then* (P) *has a solution.*
(ii) *If $\xi_0 \in \operatorname{int} D_2$ then* (P) *has no solution.*

REMARK 74. The nonexistence part has been proved by Dacorogna and Marcellini [27].

PROOF OF THEOREM 73. (i) The case where $\xi_0 \notin D_1 \cup D_2$ corresponds to the trivial case, where $Qf(\xi_0) = f(\xi_0)$. So we now assume that $\xi_0 \in D_1$. Note that Qf is quasiaffine on D_1 (in fact $Qf(\xi) \equiv 0$). Apply then Theorem 33 (and the remark following it) to get $u \in u_{\xi_0} + W_0^{1,\infty}(\Omega; \mathbb{R}^2)$ such that

$$\lambda_1(Du) = \lambda_2(Du) = 1 \quad \text{a.e. in } \Omega.$$

This implies that $Qf(Du) = f(Du) = Qf(\xi_0) = 0$ and hence the claim follows from Theorem 41.

(ii) It was shown in [27], and we do not discuss here the details, that if $\xi_0 \in \operatorname{int} D_2$ then the function Qf is strictly quasiconvex at ξ_0 and therefore (P) has no solution. □

7.4. *An optimal design problem*

We now consider the case, studied by many authors following the pioneering work of Kohn and Strang [48], where

$$\inf\left\{\int_\Omega f\big(Du(x)\big)\,\mathrm{d}x\colon\ u \in u_{\xi_0} + W_0^{1,\infty}\big(\Omega; \mathbb{R}^2\big)\right\}, \tag{P}$$

$\Omega \subset \mathbb{R}^2$ is a bounded open set with Lipschitz boundary, $Du_{\xi_0} = \xi_0$ and

$$f(\xi) = \begin{cases} 1 + |\xi|^2 & \text{if } \xi \neq 0, \\ 0 & \text{if } \xi = 0. \end{cases}$$

It was shown by Kohn and Strang [48] that the quasiconvex envelope is then

$$Qf(\xi) = \begin{cases} 1 + |\xi|^2 & \text{if } |\xi|^2 + 2|\det \xi| \geqslant 1, \\ 2\big(|\xi|^2 + 2|\det \xi|\big)^{1/2} - 2|\det \xi| & \text{if } |\xi|^2 + 2|\det \xi| < 1. \end{cases}$$

The existence of minimizers for problem (P) was then established by Dacorogna and Marcellini in [27] and [31]. Later Dacorogna and Tanteri [37] gave a different proof which is more in the spirit of the present report and we follow here this last approach.

THEOREM 75. *Let $\Omega \subset \mathbb{R}^2$, f and ξ_0 be as above. Then a necessary and sufficient condition for* (P) *to have a solution is that one of the following conditions hold*:

(i) $\xi_0 = 0$ *or* $|\xi_0|^2 + 2|\det \xi_0| \geqslant 1$ (*i.e.*, $f(\xi_0) = Qf(\xi_0)$),

(ii) $\det \xi_0 \neq 0$.

PROOF. We do not discuss the details and in particular not the necessary part (see [27] for details). So we assume that we are in the nontrivial case

$$\det \xi_0 \neq 0 \quad \text{and} \quad |\xi_0|^2 + 2|\det \xi_0| < 1. \tag{30}$$

We just point out how to define the set K_0 of Theorem 42. We have (denoting by $\mathbb{R}_s^{2\times 2}$ the set of 2×2 symmetric matrices)

$$\begin{aligned} K &= \big\{\xi \in \mathbb{R}^{2\times 2}\colon |\xi|^2 + 2|\det \xi| < 1\big\} \setminus \{0\} \\ K_0 &= \big\{\xi \in \mathbb{R}_s^{2\times 2}\colon \det \xi > 0 \text{ and } \operatorname{trace} \xi \in (0,1)\big\}, \\ \overline{K_0} \cap \partial K &= \{0\} \cup \big\{\xi \in \mathbb{R}_s^{2\times 2}\colon \det \xi \geqslant 0 \text{ and } \operatorname{trace} \xi = 1\big\} \\ &= \big\{\xi \in \mathbb{R}_s^{2\times 2}\colon \det \xi \geqslant 0 \text{ and } \operatorname{trace} \xi \in \{0,1\}\big\}. \end{aligned}$$

Since f is invariant under rotations and symmetries and (30) holds, we can assume, without loss of generality, that $\xi_0 \in K_0$. Furthermore Qf is quasiaffine on K_0 ($Qf(\xi) = 2 \operatorname{trace} \xi - 2 \det \xi$), while it is not so on K. It remains to prove that K_0 has the relaxation property with respect to $\overline{K_0} \cap \partial K$; and this is easily established as in [37]. □

7.5. *The minimal surface case*

Following Dacorogna, Pisante and Ribeiro [34], we now deal with the case where $N = n + 1$ and

$$f(\xi) = g(\operatorname{adj}_n \xi).$$

The minimization problem is then

$$\inf\left\{\int_\Omega g\big(\operatorname{adj}_n\big(Du(x)\big)\big)\,\mathrm{d}x\colon u \in u_{\xi_0} + W_0^{1,\infty}\big(\Omega; \mathbb{R}^{n+1}\big)\right\}, \tag{P}$$

where Ω is a bounded open set of $\mathbb{R}^n$, $Du_{\xi_0} = \xi_0$ and $g : \mathbb{R}^{n+1} \to \mathbb{R}$ is a nonnegative, lower semicontinuous and locally bounded nonconvex function.

From Theorem 9 we have

$$Qf(\xi) = Cg(\mathrm{adj}_n \xi).$$

We next set

$$S = \left\{ y \in \mathbb{R}^{n+1} \colon Cg(y) < g(y) \right\}$$

and assume, in order to avoid the trivial situation, that $\mathrm{adj}_n \xi_0 \in S$. We also assume that S is connected, otherwise we replace it by its connected component that contains $\mathrm{adj}_n \xi_0$.

Observe that

$$K = \left\{ \xi \in \mathbb{R}^{(n+1)\times n} \colon Qf(\xi) < f(\xi) \right\} = \left\{ \xi \in \mathbb{R}^{(n+1)\times n} \colon \mathrm{adj}_n \xi \in S \right\}.$$

THEOREM 76. *If S is bounded, Cg is affine in S and* $\operatorname{rank} \xi_0 \geqslant n-1$, *then* (P) *has a solution.*

REMARK 77. The fact that Cg be affine in S is not a necessary condition for existence of minima, as seen in Proposition 78.

PROOF OF THEOREM 76. The result follows if we choose a convenient rank-one direction $\lambda = \alpha \otimes \beta \in \mathbb{R}^{(n+1)\times n}$ satisfying the hypothesis of Corollary 47. We remark that, since we suppose Cg affine in S, Qf is quasiaffine in $L_K(\xi_0 + \alpha \otimes B_\varepsilon, \lambda)$ (cf. Notation 44 and Definition 45) independently of the choice of λ. So we only have to prove that K is stably bounded at ξ_0 in a direction $\lambda = \alpha \otimes \beta$.

Firstly we observe that we can find (cf. Theorem 3.1.1 in [45]) $P \in O(n+1)$, $Q \in SO(n)$ and $0 \leqslant \lambda_1 \leqslant \cdots \leqslant \lambda_n$, so that

$$\xi_0 = PLQ, \quad \text{where } L = (\lambda_j \delta_{ij})_{1\leqslant j \leqslant n}^{1\leqslant i \leqslant n+1};$$

in particular when $n = 2$, we have

$$L = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \\ 0 & 0 \end{pmatrix}.$$

Since $\operatorname{rank} \xi_0 \geqslant n-1$ we have that $\lambda_2 > 0$. We also note that

$$\mathrm{adj}_n \xi_0 = \mathrm{adj}_n P \cdot \mathrm{adj}_n L \quad \text{and} \quad \mathrm{adj}_n L = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (-1)^n \lambda_1 \cdots \lambda_n \end{pmatrix}.$$

Without loss of generality we assume $\xi_0 = L$. We then choose $\lambda = \alpha \otimes \beta$, where $\alpha = (1, 0, \dots, 0) \in \mathbb{R}^{n+1}$ and $\beta = (1, 0, \dots, 0) \in \mathbb{R}^n$. We will see that $L_K(\xi_0 + \alpha \otimes B_\varepsilon, \lambda)$ is bounded for some $\varepsilon > 0$. Let $\eta \in L_K(\xi_0 + \alpha \otimes B_\varepsilon, \lambda)$ then we can write $\eta = \xi_0 + \alpha \otimes \gamma_\varepsilon + t\lambda$ for some $\gamma_\varepsilon \in B_\varepsilon$ and $t \in \mathbb{R}$. By definition of $L_K(\xi_0 + \alpha \otimes B_\varepsilon, \lambda)$ we have $\operatorname{adj}_n \eta \in \overline{S}$. Since S is bounded and

$$|\operatorname{adj}_n \eta| = \left|\lambda_1 + \gamma_\varepsilon^1 + t\right| \lambda_2 \cdots \lambda_n,$$

it follows, using the fact that rank $\xi_0 \geqslant n-1$, that $|t|$ is bounded by a constant depending on S, ξ_0 and ε. Consequently $|\eta| \leqslant |\xi_0| + |\alpha \otimes \gamma_\varepsilon| + |t||\lambda|$ is bounded for any fixed positive ε and we get the result. □

As already alluded in Section 5.3, we obtain now a result of nonexistence although the integrand of the relaxed problem is not strictly quasiconvex. We will consider the case where $N = 3$, $n = 2$ and $f : \mathbb{R}^{3\times 2} \to \mathbb{R}$ is given by

$$f(\xi) = g(\operatorname{adj}_2 \xi),$$

where $g : \mathbb{R}^3 \to \mathbb{R}$ is defined by

$$g(\nu) = \left(\nu_1^2 - 4\right)^2 + \nu_2^2 + \nu_3^2.$$

We therefore get $Qf(\xi) = Cg(\operatorname{adj}_2 \xi)$ and

$$Cg(\nu) = \left[\nu_1^2 - 4\right]_+^2 + \nu_2^2 + \nu_3^2,$$

where

$$[x]_+ = \begin{cases} x & \text{if } x \geqslant 0, \\ 0 & \text{if } x < 0. \end{cases}$$

We will choose the boundary datum as follows

$$u_{\xi_0}(x) = \begin{pmatrix} u_{\xi_0}^1(x) = \alpha_1 x_1 + \alpha_2 x_2 \\ u_{\xi_0}^2(x) = 0 \\ u_{\xi_0}^3(x) = 0 \end{pmatrix}$$

and hence

$$Du_{\xi_0}(x) = \xi_0 = \begin{pmatrix} \alpha_1 & \alpha_2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \operatorname{adj}_2 Du_{\xi_0}(x) = \operatorname{adj}_2 \xi_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The problem is then

$$\inf\left\{ I(u) = \int_\Omega f\left(Du(x)\right) \mathrm{d}x \colon\ u \in u_{\xi_0} + W_0^{1,\infty}\left(\Omega; \mathbb{R}^3\right) \right\}. \tag{P}$$

Note also that $Qf(\xi_0) = 0 < f(\xi_0) = 16$.

In terms of the preceding notations we have

$$S = \{y \in \mathbb{R}^3\colon\ Cg(y) < g(y)\} = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3\colon\ |y_1| < 2\},$$
$$K = \{\xi \in \mathbb{R}^{3\times 2}\colon\ Qf(\xi) < f(\xi)\} = \{\xi \in \mathbb{R}^{3\times 2}\colon\ \mathrm{adj}_2\xi \in S\}$$

and we observe that Cg is not affine on S, which in turn implies that Qf is not quasiaffine on K.

The following result shows that the hypothesis of strict quasiconvexity of Qf is not necessary for *nonexistence*.

PROPOSITION 78. (P) *has a solution if and only if* $u_{\xi_0} \equiv 0$. *Moreover,* Qf *is not strictly quasiconvex at any* $\xi_0 \in \mathbb{R}^{3\times 2}$ *of the form*

$$\xi_0 = \begin{pmatrix} \alpha_1 & \alpha_2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

PROOF.

Step 1. We first show that if (P) has a solution then $u_{\xi_0} \equiv 0$. If $u \in u_{\xi_0} + W_0^{1,\infty}(\Omega; \mathbb{R}^3)$ is a solution of (P) we necessarily have, denoting by $\nu(\xi) = \mathrm{adj}_2\xi$,

$$|\nu_1(Du)| = 2, \qquad \nu_2(Du) = \nu_3(Du) = 0$$

since

$$Qf(Du_{\xi_0}) = Cg(\mathrm{adj}_2 Du_{\xi_0}) = Cg(0) = 0.$$

The three equations read as

$$\begin{cases} \left|u^2_{x_1}u^3_{x_2} - u^2_{x_2}u^3_{x_1}\right| = 2, \\ u^1_{x_1}u^3_{x_2} - u^1_{x_2}u^3_{x_1} = 0, \\ u^1_{x_1}u^2_{x_2} - u^1_{x_2}u^2_{x_1} = 0. \end{cases} \tag{31}$$

Multiplying the second equation of (31) first by $u^2_{x_1}$, then by $u^2_{x_2}$, using the third equation of (31), we get

$$0 = u^2_{x_1}u^1_{x_1}u^3_{x_2} - u^2_{x_1}u^1_{x_2}u^3_{x_1} = u^2_{x_1}u^1_{x_1}u^3_{x_2} - u^1_{x_1}u^2_{x_2}u^3_{x_1} = u^1_{x_1}\left(u^2_{x_1}u^3_{x_2} - u^2_{x_2}u^3_{x_1}\right),$$
$$0 = u^2_{x_2}u^1_{x_1}u^3_{x_2} - u^2_{x_2}u^1_{x_2}u^3_{x_1} = u^2_{x_1}u^1_{x_2}u^3_{x_2} - u^2_{x_2}u^1_{x_2}u^3_{x_1} = u^1_{x_2}\left(u^2_{x_1}u^3_{x_2} - u^2_{x_2}u^3_{x_1}\right).$$

Combining these last equations with the first one of (31), we find

$$u^1_{x_1} = u^1_{x_2} = 0 \quad \text{a.e.}$$

We therefore find that any solution of (P) should have $Du^1 = 0$ a.e. and hence $u^1 \equiv$ constant on each connected component of Ω. Since u^1 agrees with $u^1_{\xi_0}$ on the boundary of Ω, we deduce that $u^1_{\xi_0} \equiv 0$ and thus $u_{\xi_0} \equiv 0$, as claimed.

Step 2. We next show that if $u_{\xi_0} \equiv 0$, then (P) has a solution. It suffices to choose $u^1 \equiv 0$ and to solve

$$\begin{cases} \left|u^2_{x_1} u^3_{x_2} - u^2_{x_2} u^3_{x_1}\right| = 2 & \text{a.e. in } \Omega, \\ u^2 = u^3 = 0 & \text{on } \partial\Omega. \end{cases}$$

This is possible by virtue of, for example, Corollary 7.30 in [31].

Step 3. We finally prove that Qf is not strictly quasiconvex at any $\xi_0 \in \mathbb{R}^{3\times 2}$ of the form given in the statement of the proposition. Indeed, let $0 < R_1 < R_2 < R$ and denote by B_R the ball centered at 0 and of radius R. Choose $\lambda, \mu \in C^\infty(B_R)$ such that

(1) $\lambda = 0$ on ∂B_R and $\lambda \equiv 1$ on B_{R_2},

(2) $\mu \equiv 0$ on $B_R \setminus \overline{B}_{R_2}$, $\mu \equiv 1$ on B_{R_1} and

$$\left|\mu^2 + \mu(x_1 \mu_{x_1} + x_2 \mu_{x_2})\right| < 2 \quad \text{for every } x \in B_R.$$

This last condition (which is a restriction only in $B_{R_2} \setminus \overline{B}_{R_1}$) is easily ensured by choosing appropriately R_1, R_2 and R.

We then choose $u(x) = u_{\xi_0}(x) + \varphi(x)$, where

$$\varphi^1(x) = -\lambda(x) u^1_{\xi_0}(x), \qquad \varphi^2(x) = \mu(x) x_1 \quad \text{and} \quad \varphi^3(x) = \mu(x) x_2.$$

We therefore have that $\varphi \in W_0^{1,\infty}(B_R; \mathbb{R}^3)$, $\operatorname{adj}_2 Du \equiv 0$ on $B_R \setminus \overline{B}_{R_2}$, while on B_{R_2} we have

$$\operatorname{adj}_2 Du = \left(\mu^2 + \mu(x_1 \mu_{x_1} + x_2 \mu_{x_2}), 0, 0\right).$$

We have thus obtained that $Cg(\operatorname{adj}_2 Du) \equiv 0$ and hence

$$Qf(\xi_0 + D\varphi) \equiv Qf(\xi_0) = 0.$$

This implies that (QP) has infinitely many solutions. However since φ does not vanish identically, we deduce that Qf is not strictly quasiconvex at any ξ_0 of the given form. □

7.6. *The problem of potential wells*

The general problem of potential wells has been intensively studied by many authors in conjunction with crystallographic models involving fine microstructures. The reference paper on the subject is Ball and James [8]. It has then been studied by many authors including Bhattacharya, Firoozye, James and Kohn, Dacorogna and Marcellini, De Simone and Dolzmann, Dolzmann and Müller, Ericksen, Firoozye and Kohn, Fonseca and Tartar, Kinderlehrer and Pedregal, Kohn, Luskin, Müller and Sverak, Pipkin, Sverak, and we refer to [31] for exact bibliographic references.

In mathematical terms the problem of *potential wells* can be described as follows. Find a minimizer of the problem

$$\inf\left\{\int_\Omega f\big(Du(x)\big)\,\mathrm{d}x\colon\ u\in u_{\xi_0}+W_0^{1,\infty}\big(\Omega;\mathbb{R}^n\big)\right\},\tag{P}$$

where $\Omega\subset\mathbb{R}^n$ is a bounded open set, u_{ξ_0} is an affine map with $Du_{\xi_0}=\xi_0$ and $f:\mathbb{R}^{n\times n}\to\mathbb{R}_+$ is such that

$$f(\xi)=0\quad\Longleftrightarrow\quad\xi\in E=\bigcup_{i=1}^m SO(n)A_i.$$

The m wells are $SO(n)A_i$, $1\leqslant i\leqslant m$ (and $SO(n)$ denotes the set of matrices U such that $U^\top U=UU^\top=I$ and $\det U=1$).

The interesting case is when

$$\xi_0\in\operatorname{int}\operatorname{Rco}E,$$

and we have then that

$$Qf(\xi_0)=0.$$

Therefore by the relaxation theorem we have

$$\inf(\mathrm{P})=\inf(\mathrm{QP})=0.$$

The existence of minimizers, since Qf is affine on $\operatorname{Rco}E$ (indeed $Qf\equiv 0$), for (P) is then reduced to finding a function $u\in u_{\xi_0}+W_0^{1,\infty}(\Omega;\mathbb{R}^n)$ so that

$$Du(x)\in E=\bigcup_{i=1}^m \mathrm{SO}(n)A_i.$$

The problem is relatively well understood only in the cases of two wells, i.e. $m=2$, and in dimension $n=2$. It is this case that we briefly discuss now. We therefore have now $A,B\in\mathbb{R}^{2\times 2}$ with $0<\det A<\det B$ and we want to find $u\in u_{\xi_0}+W_0^{1,\infty}(\Omega;\mathbb{R}^2)$, where $\Omega\subset\mathbb{R}^2$ is a bounded open set, satisfying

$$Du(x)\in SO(2)A\cup SO(2)B\quad\text{a.e. in }\Omega.$$

The first important result is to identify the set where the gradient of the boundary datum, ξ_0, should lie. This was resolved by Sverak [73] who showed that

$$\begin{aligned}\operatorname{Rco} E = \Big\{\xi \in \mathbb{R}^{2\times 2}\colon \ &\text{there exist } 0 \leqslant \alpha \leqslant \frac{\det B - \det \xi}{\det B - \det A},\\ &0 \leqslant \beta \leqslant \frac{\det \xi - \det A}{\det B - \det A} R, S \in SO(2),\\ &\text{so that } \xi = \alpha R A + \beta S B\Big\}\end{aligned}$$

while the interior is given by the same formulas with strict inequalities in the right-hand side.

We therefore have the following theorem.

THEOREM 79. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set*

$$\xi_0 \in \operatorname{int} \operatorname{Rco} E.$$

Then there exists $u \in u_{\xi_0} + W_0^{1,\infty}(\Omega; \mathbb{R}^2)$ such that

$$Du(x) \in E = SO(2)A \cup SO(2)B \quad \text{a.e. in } \Omega$$

and therefore (P) *has a solution.*

This result was proved by Müller and Sverak [61] using the so-called method of convex integration of Gromov [43] and by Dacorogna and Marcellini in [28] and [31] following the approach presented in Section 4.4, and we refer to [31] for details.

The case where $\det A = \det B > 0$ can also be handled (cf. Müller and Sverak [62], see also Dacorogna and Tanteri [37]), using the representation formula of Sverak [73], namely

$$\begin{aligned}\operatorname{Rco} E = \big\{\xi \in \mathbb{R}^{2\times 2}\colon \ &\text{there exist } R, S \in SO(2),\ 0 \leqslant \alpha, \beta \leqslant \alpha + \beta \leqslant 1,\\ &\det \xi = \det A = \det B \text{ so that } \xi = \alpha R A + \beta S B\big\}.\end{aligned}$$

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CHAPTER 3

Bifurcation and Related Topics in Elliptic Problems

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1. Introduction

Bifurcation theory provides a bridge between the *linear* world and the more complicated *nonlinear* world, and thus plays an important role in the study of various nonlinear problems. Nonlinear elliptic boundary value problems enjoy many nice properties that allow the use of a variety of powerful tools in nonlinear functional analysis. In the past three decades, bifurcation theory has been successfully combined with these tools to yield rather deep results for elliptic problems. Traditionally bifurcation analysis was based on local linearization techniques, but more and more global analysis is involved in modern bifurcation theory. A highlight is the global bifurcation theory of Krasnoselskii and Rabinowitz (see [Kr,Ra]), which is resulted from the use of topological degree theory, general set point theory and a linearization consideration. The final result, generally known as Rabinowitz's global bifurcation theorem (to be recalled later), has played a fundamental role in proving a great number of existence results for elliptic problems. Making use of the maximum principle, one can study elliptic problems in the framework of ordered Banach spaces. The extra order structure greatly strengthens these abstract tools. An excellent presentation of these techniques up to the late 1970s can be found in Amann's by now classical review article [Am].

In this chapter we intend to present some further results for elliptic boundary value problems, where bifurcation theory plays an important role in the proofs; we will focus on recent developments, well after [Am]. In Section 2 we discuss a new phenomenon, namely bifurcation from infinity caused by spatial "degeneracy" in the nonlinearity and determined by "boundary blow-up solutions". In Section 3 we combine bifurcation argument and order structure to study a system of elliptic equations, and demonstrate that, apart from multiplicity results, these techniques can be used to discuss the stability and the profiles of the solutions. In Section 4 we present some recent fine techniques in determining the exact number of positive solutions of elliptic equations over a ball; in particular, we give the proof of a long standing conjecture on the perturbed Gelfand equation. In Section 5 we discuss the usefulness of nodal properties of solutions in global bifurcation theory. Most of the problems discussed here have an open ending; related problems and open questions can be found in the remarks at the end of the sections or subsections.

The choice of the topics in this chapter is subjective, and the bibliography is by no means complete. For clarity and simplicity, we present most of our results for problems with a specific nonlinearity, with the hope that the interested reader can easily extend them to more general situations.

We have not tried to make this chapter entirely self-contained. Most of the results presented here are proved in full, but some of them are only stated, with the proofs referred to the relevant references. The proofs not given here are either very technical or less relevant to our main theme here. We assume that the reader is familiar with the standard theory for second-order linear elliptic equations (see [GT] and [PW]) and standard nonlinear functional analysis (see [De]).

Some classical bifurcation theorems. We recall several classical bifurcation theorems which form the corner stones for our analysis in this chapter; indeed, they are fundamental in the development of the modern bifurcation theory in general.

We first describe Rabinowitz's global bifurcation theorem. Let E be a Banach space and $\mathbb{R}^1$ denote the set of real numbers. We consider the nonlinear eigenvalue problem

$$u = \lambda Lu + H(\lambda, u), \quad (\lambda, u) \in \mathbb{R}^1 \times E, \tag{1.1}$$

where $L : E \to E$ is a compact linear map, $H : \mathbb{R}^1 \times E \to E$ is compact and continuous, and is $o(\|u\|)$ for u near 0 uniformly on bounded λ intervals. We assume that $H(\lambda, 0) \equiv 0$, and therefore we have the curve of trivial solutions $\{(\lambda, 0)\colon \lambda \in \mathbb{R}^1\}$. We are interested in the existence of *nontrivial* solutions $(\lambda, u) \in \mathbb{R}^1 \times E$ and will denote the closure of the set of nontrivial solutions of (1.1) by $\mathcal{S}$.

Let $r(L)$ denote the set of $\mu \in \mathbb{R}^1$ such that there exists $v \in E \setminus \{0\}$ with $v = \mu Lv$. It is well known that the possible bifurcations points for (1.1) with respect to the curve of trivial solutions lie in the set $\{(\mu, 0)\colon \mu \in r(L)\}$; moreover, if $\mu \in r(L)$ is of odd (algebraic) multiplicity, then $(\mu, 0)$ is a bifurcation point, see [Kr].

THEOREM 1.1 (Rabinowitz's global bifurcation theorem [Ra]). *Under the above assumptions, if $\mu \in r(L)$ is of odd* (*algebraic*) *multiplicity, then $\mathcal{S}$ possesses a maximal subcontinuum $\mathcal{S}_\mu$ such that $(\mu, 0) \in \mathcal{S}_\mu$ and $\mathcal{S}_\mu$ either*
(i) *meets infinity in $\mathbb{R}^1 \times E$, i.e., $\mathcal{S}_\mu$ is unbounded, or*
(ii) *meets $(\hat{\mu}, 0)$, where $\hat{\mu} \in r(L) \setminus \{\mu\}$.*

Next we recall two local bifurcation theorems due to Crandall and Rabinowitz [CR1,CR2]. Let E_1, E_2 be Banach spaces and $J = (a, b)$ an open interval in $\mathbb{R}^1$. Let $N(L)$ and $R(L)$ denote the null space and range of a linear map L between Banach spaces.

THEOREM 1.2 (Bifurcation from a simple eigenvalue [CR1]). *Let E_1, E_2 and J be as above. Suppose that U is a neighborhood of* 0 *in E_1, $\lambda_0 \in J$ and $F : J \times U \to E_2$ has the following properties*:
(a) $F(\lambda, 0) \equiv 0$ *for* $\lambda \in J$,
(b) *the partial derivatives F_λ, F_u and $F_{\lambda u}$ exist and are continuous,*
(c) $\dim N(F_u(\lambda_0, 0)) = \operatorname{codim} R(F_u(\lambda_0, 0)) = 1$,
(d) $F_{\lambda u}(\lambda_0, 0)u_0 \notin R(F_u(\lambda_0, 0))$, *where $u_0 \in E_1$ spans $N(F_u(\lambda_0, 0))$.*
Let Z be any complement of $\operatorname{span}\{u_0\}$ *in E_1. Then there exists an open interval J_0 containing* 0 *and continuously differentiable functions $\lambda : J_0 \to \mathbb{R}^1$ and $\psi : J_0 \to Z$ such that $\lambda(0) = \lambda_0$, $\psi(0) = 0$, and if $u(s) = su_0 + s\psi(s)$, then $F(\lambda(s), u(s)) = 0$. Moreover, the solution set of $F(\lambda, u) = 0$ near $(\lambda_0, 0)$ consists precisely of the curves $\{(\lambda(s), u(s))\colon s \in J_0\}$ and $\{(\lambda, 0)\colon \lambda \in J\}$.*

If the equation $F(\lambda, u) = 0$ in Theorem 1.2 can be written in the form of (1.1), then conditions (c) and (d) of Theorem 1.2 become

$$\dim N(\lambda_0 L - I) = \operatorname{codim} R(\lambda_0 L - I) = 1$$

and

$$u_0 \notin R(\lambda_0 L - I) \quad \text{if } N(\lambda_0 L - I) = \operatorname{span}\{u_0\}.$$

If $\lambda_0 \in r(L)$ has the above properties, then one says that $1/\lambda_0$ is a *simple eigenvalue* of L. Therefore Theorem 1.2 is usually known as the theorem of bifurcation from a simple eigenvalue; it provides a much better description of the *local* bifurcation branch.

Both Theorems 1.1 and 1.2 describe the situation that a nontrivial solution branch bifurcates from a trivial solution curve. The following theorem describes the situation that a solution curve "changes direction" in the (λ, u) space. As will be discussed later, this theorem plays a vital role in obtaining *exact multiplicity* results.

THEOREM 1.3 (Turning point theorem [CR2]). *Let E_1, E_2 and J be as in Theorem* 1.2. *Suppose that V is a neighborhood of v_0 in E_1, $\lambda_0 \in J$ and $F : J \times V \to E_2$ is continuously differentiable and has the following properties*:

(a) $F(\lambda_0, v_0) = 0$,

(b) $\dim N(F_u(\lambda_0, v_0)) = \operatorname{codim} R(F_u(\lambda_0, v_0)) = 1$,

(c) $F_\lambda(\lambda_0, v_0) \notin R(F_u(\lambda_0, v_0))$.

Let Z be any complement of $\operatorname{span}\{u_0\}$ *in E_1, where $u_0 \in E_1$ spans $N(F_u(\lambda_0, v_0))$. Then there exists an open interval J_0 containing* 0 *and continuously differentiable functions $\lambda : J_0 \to \mathbb{R}^1$ and $\tau : J_0 \to Z$ such that $\lambda(0) = \lambda_0$, $\lambda'(0) = 0$, $\tau(0) = \tau'(0) = 0$, and if $u(s) = v_0 + su_0 + \tau(s)$, then $F(\lambda(s), u(s)) = 0$. Moreover, the solution set of $F(\lambda, u) = 0$ near (λ_0, v_0) consists precisely of the curve $\{(\lambda(s), u(s)) \colon s \in J_0\}$. Furthermore, if F is k-times continuously differentiable* (*analytic*)*, so are $\lambda(s)$ and $\tau(s)$.*

Note that if we can somehow determine the sign of $\lambda''(0)$ in Theorem 1.3, then we would know in which direction the solution curve is bent near (λ_0, v_0). For example, if $\lambda''(0) < 0$, then the solution curve is bent to the left, i.e., for values of λ less than λ_0. In the case $\lambda''(0) = 0$, it is possible that the solution curve does not change direction at (λ_0, v_0) but behaves like the curve $x = y^3$ at $(0, 0)$ in the xy-plane.

2. Bifurcation from infinity by spatial degeneracy

In this section we use the problem

$$-\Delta u = \lambda u - b(x)|u|^{p-1}u, \quad u|_{\partial\Omega} = 0, \tag{2.1}$$

to demonstrate how bifurcation from infinity can be caused by $b(x)$ vanishing in a subset of the underlying domain Ω. We call this behavior of $b(x)$ a degeneracy in (2.1). Here Ω is a bounded smooth domain in $\mathbb{R}^N$, $p > 1$ and $b(x)$ is a continuous nonnegative function over $\overline{\Omega}$. Positive solutions of problem (2.1) can be regarded as steady-states of a biological species over the spatial region Ω, whose growth is governed by a *degenerate* logistic law. When $b(x)$ is replaced by $b(x) + \varepsilon$ with a small positive constant ε, (2.1) describes the steady-states of a species governed by a *classical* logistic law. We will make use of (2.1) to study the perturbed problem and reveal that, for small ε, the profile (or pattern) of the positive solutions for the perturbed problem can be determined rather completely. This is of interest in population biology; we will also use these results in Section 3.

2.1. *Bifurcation from infinity and boundary blow-up problems*

We consider positive solutions of (2.1); as will become clear soon, for this case, the theory is rather complete. We note that by the strong maximum principle, a nontrivial nonnegative solution of (2.1) must be strictly positive inside Ω. We assume that $\overline{\Omega}_0 := b^{-1}(0)$, where Ω_0 is an open connected set whose closure is contained in Ω, and both $\partial\Omega$ and $\partial\Omega_0$ are smooth (say, C^2). We denote by λ_1^{Ω} and $\lambda_1^{\Omega_0}$ the first eigenvalues of $-\Delta$ under Dirichlet boundary conditions over Ω and Ω_0, respectively. In general, we use $\lambda_1^D(\phi)$ to denote the first eigenvalue of $-\Delta + \phi$ over D under Dirichlet boundary conditions.

If (2.1) has a positive solution, then we write (2.1) in the form

$$-\Delta u + \big(b(x)u^{p-1}\big)u = \lambda u$$

and obtain that $\lambda_1^{\Omega}(b(x)u^{p-1}) = \lambda$. Using the well-known monotonicity properties of $\lambda_1^D(\phi)$, we deduce

$$\lambda_1^{\Omega} < \lambda_1^{\Omega}\big(b(x)u^{p-1}\big) < \lambda_1^{\Omega_0}\big(b(x)u^{p-1}\big) = \lambda_1^{\Omega_0}$$

since $b(x) = 0$ on Ω_0. Therefore we have the necessary condition,

$$\lambda_1^{\Omega} < \lambda < \lambda_1^{\Omega_0}, \tag{2.2}$$

for (2.1) to possess a positive solution. We claim that (2.2) is also a sufficient condition for the existence of a positive solution of (2.1). This can be proved by combining a global bifurcation argument with an a priori bound result. Indeed, by a standard application of Rabinowitz's global bifurcation result, we know that there is a global branch of positive solutions $\Gamma := \{(\lambda, u)\}$ bifurcating from the trivial solution curve $\{(\lambda, 0)\}$ at $(\lambda_1^{\Omega}, 0)$. By the maximum principle and the fact that λ_1^{Ω} is a simple eigenvalue, we conclude that the second alternative in Theorem 1.1 cannot occur and hence Γ has to be unbounded in the space $\mathbb{R}^1 \times C^1(\overline{\Omega})$.

LEMMA 2.1. *For any small $\delta > 0$, there exists $C = C_\delta > 0$ such that any positive solution of* (2.1) *with $\lambda \leqslant \lambda_1^{\Omega_0} - \delta$ satisfies $\|u\|_{C^1(\overline{\Omega})} \leqslant C$.*

PROOF. By the standard L^p-theory for elliptic equations and the Sobolev embedding theorems, we only need to show the bound in the $L^\infty(\Omega)$ norm. Indeed, if we have the bound for u in $L^\infty(\Omega)$, then the right-hand side of (2.1) has a bound in $L^\infty(\Omega)$. By the L^p-theory, this implies that u has a bound in $W^{2,q}(\Omega)$ for any $q > 1$. Then the Sobolev embedding theorem shows that u has a bound in $C^1(\overline{\Omega})$.

From the continuous dependence of λ_1^D on D, we can find a small neighborhood of Ω_0, say $\Omega_0^\sigma := \{x \in \Omega\colon d(x, \Omega_0) < \sigma\}$, so that $\lambda_1^{\Omega_0^\sigma} > \lambda_1^{\Omega_0} - \delta$. Let ϕ^σ be a positive eigenfunction corresponding to $\lambda_1^{\Omega_0^\sigma}$. Then extend $\phi^\sigma|_{\Omega_0^{\sigma/2}}$ to a smooth positive function over $\overline{\Omega}$, which we denote by ϕ. Then it is easily checked that there exists $M_0 > 0$ large such that for any $\lambda \in (\lambda_1^{\Omega}, \lambda_1^{\Omega_0} - \delta]$ and all $M \geqslant M_0$, $M\phi$ is an upper solution of (2.1). By the well-known Serrin's sweeping principle (see, e.g., Theorem 2.7.1 in [Sa]), we can conclude that any positive solution of (2.1) satisfies $u \leqslant M_0\phi$ in Ω. This proves the a priori bound in the $L^\infty(\Omega)$ norm, as required. □

By Lemma 2.1 (and its proof), we see that the global branch Γ of positive solutions of (2.1) can become unbounded only through a sequence $\{(\lambda_n, u_n)\} \subset \Gamma$ satisfying $\lambda_n \to \lambda_1^{\Omega_0}$ and $\|u_n\|_\infty \to \infty$. In particular, we have proved that (2.1) has a positive solution if and only if (2.2) holds. Moreover, $(\lambda_1^\Omega, 0)$ is a point of bifurcation from 0, and $(\lambda_1^{\Omega_0}, \infty)$ is a point of bifurcation from infinity.

Taking advantage of the special nonlinearity in (2.1), we can show that it has a unique positive solution when (2.2) holds. Indeed, for any fixed λ satisfying (2.2), we can use the family of upper solutions constructed in the proof of Lemma 2.1 and a standard iteration technique to conclude that (2.2) has a maximal positive solution u^*. If u is any other positive solution to (2.1), then applying the strong maximum principle to the equation satisfied by $u^* - u$ we find that $u^* - u > 0$ in Ω. Therefore we have

$$\lambda = \lambda_1^\Omega\big(b(x)\big(u^*\big)^{p-1}\big) > \lambda_1^\Omega\big(b(x)u^{p-1}\big) = \lambda.$$

This contradiction proves the uniqueness.

For each λ satisfying (2.2), if we denote the unique positive solution of (2.1) by u_λ, then the global bifurcation branch Γ can be expressed as $\Gamma = \{(\lambda, u_\lambda)\colon \lambda_1^\Omega < \lambda < \lambda_1^{\Omega_0}\}$. Near the end point $(\lambda_1^\Omega, 0)$ of Γ, the local bifurcation theorem, Theorem 1.2, gives a detailed description of the behavior of Γ; here one relies on a linear eigenvalue problem. To better understand Γ near its other end point $(\lambda_1^{\Omega_0}, \infty)$, instead of a linear problem, we need the following nonlinear boundary blow-up problem

$$-\Delta u = \lambda u - b(x)u^p \quad \text{in } \Omega \setminus \overline{\Omega}_0, \quad u|_{\partial\Omega_0} = \infty, u|_{\partial\Omega} = 0. \tag{2.3}$$

Here by $u|_{\partial\Omega_0} = \infty$, we mean

$$u(x) \to \infty \quad \text{when } d(x, \partial\Omega_0) \to 0.$$

We have the following result (see [DH], Theorem 2.4).

PROPOSITION 2.2. *For any $\lambda \in \mathbb{R}^1$, problem* (2.3) *has at least one positive solution. Moreover, it has a maximal positive solution $\overline{U}_\lambda$ and a minimal positive solution $\underline{U}_\lambda$ in the sense that any other positive solution u satisfies $\underline{U}_\lambda \leqslant u \leqslant \overline{U}_\lambda$.*

REMARK 2.3. If there exist constants $\alpha \geqslant 0$ and $\beta_2 > \beta_1 > 0$ such that, for all $x \in \Omega \setminus \overline{\Omega}_0$ near $\partial\Omega_0$,

$$\beta_1\big[d(x, \partial\Omega_0)\big]^\alpha \leqslant b(x) \leqslant \beta_2\big[d(x, \partial\Omega_0)\big]^\alpha,$$

then it is proved in [Du6], Theorem 3.2, that (2.3) has a unique positive solution. Whether the positive solution of (2.3) is unique without any extra condition on $b(x)$ is an open problem.

The following result gives a rather complete description of u_λ for λ close to $\lambda_1^{\Omega_0}$.

THEOREM 2.4. *Denote* $\lambda_0 := \lambda_1^{\Omega_0}$. *Then*

(i) $u_\lambda \to \infty$ *uniformly on* $\overline{\Omega}_0$ *as* λ *increases to* λ_0,

(ii) $u_\lambda \to \underline{U}_{\lambda_0}$ *uniformly in any compact subset of* $\overline{\Omega} \setminus \overline{\Omega}_0$ *as* λ *increases to* λ_0.

PROOF. We first observe that u_λ increases as λ increases. This follows from a simple upper and lower solution consideration, together with the uniqueness of u_λ. Suppose $\lambda_1^{\Omega} < \lambda < \lambda' < \lambda_1^{\Omega_0}$. Then $u_{\lambda'}$ is an upper solution of (2.1). If ϕ is a positive eigenfunction corresponding to λ_1^{Ω}, then it is easy to see that $\varepsilon\phi < u_{\lambda'}$ and is a lower solution to (2.1) for all small positive ε. Therefore (2.1) has a positive solution u satisfying $\varepsilon\phi < u < u_{\lambda'}$. We must have $u = u_\lambda$ as (2.1) has a unique positive solution. Therefore $u_\lambda < u_{\lambda'}$.

The monotonicity of u_λ in λ implies that if we can prove (i) and (ii) along a sequence $\lambda_n \to \lambda_0$, then the same conclusions hold for $\lambda \to \lambda_0$. Let $\{\delta_n\}$ be a sequence of positive numbers decreasing to 0 such that

$$\Omega_n := \left\{x \in \Omega\colon\ d(x, \Omega_0) < \delta_n\right\} \subset\subset \Omega \quad \forall n \geqslant 1.$$

Denote $\lambda_n = \lambda_1^{\Omega_n}$. Then $\lambda_1^{\Omega} < \lambda_n < \lambda_0$ and λ_n increases to λ_0 as $n \to \infty$. To simplify notation, we write $u_n = u_{\lambda_n}$. The proof below is divided into four steps.

STEP 1. $u_n(x) \to \infty$ *uniformly in any compact subset of* Ω_0.

Let ϕ_0 be the positive eigenfunction corresponding to $\lambda_0 = \lambda_1^{\Omega_0}$ normalized by $\|\phi_0\|_\infty = 1$. Let K be an arbitrarily given compact subset of Ω_0. Define

$$\alpha_0 := \inf_{x \in \Omega_0} u_1(x), \qquad \beta_0 := \min_{x \in K} \phi_0(x).$$

Clearly

$$\alpha_0 > 0, \qquad \beta_0 > 0, \qquad u_n(x) \geqslant u_1(x) \geqslant \alpha_0 \quad \forall x \in \Omega_0, \forall n \geqslant 1. \tag{2.4}$$

Given any large number $M > 0$, we can find an open connected set K^* satisfying $K \subset K^* \subset\subset \Omega_0$ such that

$$\phi_0(x) < \frac{\alpha_0 \beta_0}{2M} \quad \forall x \in \partial K^*. \tag{2.5}$$

By a standard interior regularity argument, $\phi_n \to \phi_0$ uniformly on K^*, where ϕ_n is given by

$$-\Delta \phi_n = \lambda_n \phi_n, \quad \phi_n|_{\partial \Omega_n} = 0, \phi_n \geqslant 0, \|\phi_n\|_\infty = 1.$$

Thus, by (2.5) and the definition of β_0, for all large n,

$$\frac{M}{\beta_0}\phi_n(x) < \alpha_0 \quad \forall x \in \partial K^*, \qquad \frac{M}{\beta_0}\phi_n(x) > \frac{M}{2} \quad \forall x \in K. \tag{2.6}$$

Recall that $b(x) = 0$ on K^*. Hence u_n and $(M/\beta_0)\phi_n$ satisfy the same equation $-\Delta u = \lambda_n u$. It now follows from (2.4) and (2.6) that $(M/\beta_0)\phi_n$ and u_n are, respectively, lower and upper solutions of the problem

$$-\Delta u = \lambda_n u \quad \text{in } K^*, \quad u|_{\partial K^*} = \alpha_0.$$

As $\lambda_n < \lambda_0 < \lambda_1^{K^*}$, it follows from the maximum principle that, for all large n,

$$u_n(x) \geqslant \frac{M}{\beta_0}\phi_n(x) \geqslant \frac{M}{2} \quad \forall x \in K \subset K^*.$$

Since $M > 0$ is arbitrary, this shows $\lim_{n\to\infty} u_n(x) = \infty$ uniformly in K. This proves Step 1.

Since $\partial\Omega_0$ is C^2, it satisfies a uniform interior ball condition: There exists $R > 0$ such that for any $x \in \partial\Omega_0$, there is a ball B_x of radius R such that $B_x \subset \overline{\Omega}_0$ and $B_x \cap \partial\Omega_0 = \{x\}$.

STEP 2. *Let $x_n \in \partial\Omega_0$ be such that*

$$u_n(x_n) = \min_{x\in\partial\Omega_0} u_n(x).$$

If $\{u_n(x_n)\}$ is bounded, then we can find a constant $\sigma > 0$ and a sequence $c_n \to \infty$ such that

$$u_n(x) \geqslant u_n(x_n) + c_n\psi(x), \quad \text{whenever } \frac{R}{2} \leqslant |x - y_n| \leqslant R, \tag{2.7}$$

where $\psi(x) = \mathrm{e}^{-\sigma|x-y_n|^2} - \mathrm{e}^{-\sigma R^2}$ and y_n is the center of the ball B_{x_n}.

A simple calculation gives

$$\Delta\psi + \lambda_n\psi = \big(4\sigma^2|x - y_n|^2 - 2N\sigma + \lambda_n\big)\mathrm{e}^{-\sigma|x-y_n|^2} - \lambda_n\mathrm{e}^{-\sigma R^2}.$$

We can choose a large $\sigma > 0$ such that

$$-\Delta\psi(x) \leqslant \lambda_n\psi(x) \quad \forall x \in B_{x_n} \setminus B_{R/2}(y_n),$$

where $B_{R/2}(y_n) = \{x \in \mathbb{R}^N\colon |x - y_n| < R/2\}$.

Choose a compact set $K \subset\subset \Omega_0$ such that $K \supset \bigcup_{n=1}^{\infty} B_{R/2}(y_n)$. By Step 1 and the assumption that $\{u_n(x_n)\}$ is bounded, we can find a sequence $c_n \to \infty$ such that

$$u_n(x) \geqslant u_n(x_n) + c_n\big(\mathrm{e}^{-\sigma R^2/4} - \mathrm{e}^{-\sigma R^2}\big) \quad \forall x \in B_{R/2}(y_n) \subset K.$$

On the other hand, since $\lambda_n < \lambda_0$, by the maximum principle, $u_n(x) \geqslant u_n(x_n)$ $\forall x \in \Omega_0$. In particular, $u_n(x) \geqslant u_n(x_n)$ on ∂B_{x_n}. Thus we see that u_n is an upper solution to the problem

$$\begin{cases} -\Delta u = \lambda_n u \quad \text{in } B_{x_n} \setminus \overline{B}_{R/2}(y_n), \\ u|_{\partial B_{x_n}} = u_n(x_n), \qquad u|_{\partial B_{R/2}(y_n)} = u_n(x_n) + c_n\big(\mathrm{e}^{-\sigma R^2/4} - \mathrm{e}^{-\sigma R^2}\big). \end{cases} \tag{2.8}$$

But clearly, $u_n(x_n) + c_n\psi(x)$ is a lower solution to (2.8). Hence, since $\lambda_n < \lambda_0 < \lambda_1^{(B_{x_n} \setminus \overline{B}_{R/2}(y_n))}$, by the maximum principle,

$$u_n(x) \geqslant u_n(x_n) + c_n\psi(x), \quad \text{whenever } \frac{R}{2} \leqslant |x - y_n| \leqslant R,$$

as required.

STEP 3. $\lim_{n\to\infty} u_n(x) = \infty$ *uniformly on* $\overline{\Omega}_0$.

By the maximum principle, it suffices to show that

$$u_n(x_n) = \min_{x \in \partial\Omega_0} u_n(x) \to \infty.$$

We argue indirectly. Suppose that this is not true. Then by passing to a subsequence, we may assume that $\{u_n(x_n)\}$ is bounded: $u_n(x_n) \leqslant C$ for all n.

Clearly u_n is an upper solution to

$$-\Delta u = \lambda_n u - b^* u^p \quad \text{in } \Omega \setminus \overline{\Omega}_0, \quad u|_{\partial\Omega_0} = u_n(x_n), u|_{\partial\Omega} = 0, \tag{2.9}$$

where $b^* = \|b\|_\infty$. Since 0 is a lower solution, we see that (2.9) has a positive solution $v_n \leqslant u_n$. Replacing $u_n(x_n)$ in (2.9) by its upper bound C, we similarly obtain a positive solution V of (2.9) satisfying $v_n \leqslant V$ on $\Omega \setminus \Omega_0$. In particular, $\|v_n\|_{L^\infty(\Omega\setminus\Omega_0)}$ is bounded. Then the L^p estimates and the Sobolev embedding theorems imply that $\{v_n\}$ is bounded in $C^1(\overline{\Omega} \setminus \Omega_0)$. In particular, $|\nabla v_n(x_n)|$ is bounded. Since

$$u_n(x) \geqslant v_n(x) \quad \forall x \in \Omega \setminus \Omega_0 \quad \text{and} \quad u_n(x_n) = v_n(x_n),$$

we have

$$\frac{\partial u_n(x_n)}{\partial \nu_n} \leqslant \frac{\partial v_n(x_n)}{\partial \nu_n} \leqslant C_0$$

for some $C_0 > 0$, where $\nu_n = (y_n - x_n)/|y_n - x_n|$, and y_n is as in Step 2.

On the other hand, by Step 2,

$$\frac{\partial u_n(x_n)}{\partial \nu_n} \geqslant c_n \frac{\partial \psi(x_n)}{\partial \nu_n} = c_n\big[2\sigma R\mathrm{e}^{-\sigma R^2}\big] \to \infty$$

as $n \to \infty$. This contradiction finishes the proof of Step 3.

STEP 4. *For any compact set $K \subset \overline{\Omega} \setminus \overline{\Omega}_0$, $u_n \to \overline{U}_{\lambda_0}$ in $C^1(K)$ as $n \to \infty$.*

Here we need the following comparison result.

LEMMA 2.5. *Suppose that D is a bounded domain, $\alpha(x)$ and $\beta(x)$ are continuous functions in D with $\|\alpha\|_\infty < \infty$, and $\beta(x)$ is nonnegative and not identically zero. Let $u_1, u_2 \in C^1(D)$ be positive in D and satisfy, in the weak sense,*

$$Lu_1 + \alpha(x)u_1 - \beta(x)g(u_1) \leqslant 0 \leqslant Lu_2 + \alpha(x)u_2 - \beta(x)g(u_2), \quad x \in D,$$

and

$$\overline{\lim_{x \to \partial D}}(u_2 - u_1) \leqslant 0,$$

where $Lu = \sum_{ij}[a_{ij}(x)u_{x_i}]_{x_j}$ is a uniformly elliptic operator with smooth coefficients a_{ij}, and $g(u)$ is continuous and such that $g(u)/u$ is strictly increasing for u in the range $\inf_D\{u_1, u_2\} < u < \sup_D\{u_1, u_2\}$. Then $u_2 \leqslant u_1$ in D.

Lemma 2.5 was proved in [DM], Lemma 2.1, when u_1 and u_2 are C^2; the same proof works if they are only C^1.

Applying Lemma 2.5 we find that $u_n \leqslant u_{n+1} \leqslant \underline{U}_{\lambda_0}$. Hence $\lim_{n\to\infty} u_n(x) = u_\infty(x)$ exists and $u_\infty(x) \leqslant \underline{U}_{\lambda_0}$. It follows that u_∞ satisfies (2.3) with $\lambda = \lambda_0$. Here the fact that $u_\infty = \infty$ on $\partial\Omega_0$ follows from $u_n(x) \leqslant u_{n+1}(x)$ and $u_n(x) \to \infty$ uniformly on $\partial\Omega_0$ by Step 3. Since $\underline{U}_{\lambda_0}$ is the minimal solution, we necessarily have $u_\infty = \underline{U}_{\lambda_0}$.

Using Sobolev embedding theorems and L^p estimates, we easily see that $u_n \to \underline{U}_{\lambda_0}$ in $C^1(K)$ as $n \to \infty$, for any compact set $K \subset \overline{\Omega} \setminus \overline{\Omega}_0$. This proves Step 4 and hence finishes our proof of Theorem 2.4. □

2.2. *Perturbation and patterned solutions*

If we replace $b(x)$ by $b(x) + \varepsilon$ in (2.1), we will see that its global bifurcation branch of positive solutions Γ_ε differs considerably from Γ, no matter how small is the positive constant ε. In this subsection we examine closely the evolution of Γ_ε as ε decreases to 0. We will see that some solutions on Γ_ε develop a sharp pattern as $\varepsilon \to 0$, others do not. So we consider the problem

$$-\Delta u = \lambda u - \big[b(x) + \varepsilon\big]u^p, \quad u|_{\partial\Omega} = 0. \tag{2.10}$$

As is well known, a standard global bifurcation consideration can be applied to (2.10) to yield an unbounded global branch of positive solutions $\Gamma_\varepsilon := \{(\lambda, u)\}$, bifurcating from the trivial solution curve at $(\lambda_1^\Omega, 0)$. Moreover, (2.10) can have a positive solution only if $\lambda > \lambda_1^\Omega$; this can be proved by the same trick used to prove (2.2). For any given $\Lambda > 0$ and all $\lambda \leqslant \Lambda$, we can find $M := M_\Lambda > 0$ large such that any constant $C \geqslant M$ is an upper solution of (2.10). Analogous to Lemma 2.1, this gives an a priori bound for all positive solutions of (2.10) with $\lambda < \Lambda$. Therefore Γ_ε can only become unbounded through $\lambda \to \infty$. Furthermore, (2.10) has a unique positive solution u_λ^ε for any $\lambda > \lambda_1^\Omega$, which can be proved by the same argument used for (2.1). Therefore

$$\Gamma_\varepsilon = \big\{\big(\lambda, u_\lambda^\varepsilon\big)\colon \lambda > \lambda_1^\Omega\big\}.$$

The following result describes how Γ_ε evolves as $\varepsilon \to 0$.

THEOREM 2.6. *Let u_λ and u_λ^ε be the unique positive solutions to* (2.1) *and* (2.10), *respectively. Then the following hold.*

(i) *If $\lambda_1^\Omega < \lambda < \lambda_1^{\Omega_0}$, then $u_\lambda^\varepsilon \to u_\lambda$ uniformly on $\overline{\Omega}$ as $\varepsilon \to 0$.*

(ii) *If $\lambda \geqslant \lambda_1^{\Omega_0}$, then*
   (a) *$u_\lambda^\varepsilon \to \infty$ uniformly on $\overline{\Omega}_0$ as $\varepsilon \to 0$,*
   (b) *$u_\lambda^\varepsilon \to \underline{U}_\lambda$ uniformly on compact subsets of $\overline{\Omega} \setminus \overline{\Omega}_0$ as $\varepsilon \to 0$.*

PROOF. Recall that by an upper and lower solution consideration and the uniqueness of u_λ, we deduced that $u_\lambda(x)$ is increasing in λ. The same consideration can be used to show that $u_\lambda^\varepsilon(x)$ is increasing in λ, decreasing in ε, and

$$u_\lambda^\varepsilon(x) < u_\lambda(x) \tag{2.11}$$

whenever both exist. (One can also apply Lemma 2.5 to prove these properties.)

Suppose now $\lambda_1^\Omega < \lambda < \lambda_1^{\Omega_0}$. Then by (2.11), we know that $\{u_\lambda^\varepsilon\colon \varepsilon > 0\}$ is bounded in $L^\infty(\Omega)$. By elliptic regularity and the Sobolev embedding theorem, we see that $\{u_\lambda^\varepsilon\colon \varepsilon > 0\}$ is compact in $C^1(\overline{\Omega})$. Since $\varepsilon \to u_\lambda^\varepsilon(x)$ is decreasing, (2.11) implies that $u_\lambda^0(x) := \lim_{\varepsilon \to 0} u_\lambda^\varepsilon(x)$ exists for all $x \in \Omega$. The above compactness conclusion then implies that $u_\lambda^\varepsilon \to u_\lambda^0$ in $C^1(\overline{\Omega})$ and furthermore, u_λ^0 is a positive solution of (2.1). Therefore we must have $u_\lambda^0 = u_\lambda$, due to uniqueness. This proves conclusion (i) of the theorem.

We next prove conclusion (ii). So we assume that $\lambda \geqslant \lambda_1^{\Omega_0}$.

Let

$$m_\varepsilon = \min_{x \in \overline{\Omega}_0} u_\varepsilon(x) = u_\varepsilon(x_\varepsilon), \quad x_\varepsilon \in \overline{\Omega}_0.$$

We claim that $m_\varepsilon \to \infty$ as $\varepsilon \to 0$. Clearly this implies part (a). We prove this claim by an indirect argument and divide the proof into several steps.

STEP 1. *If $m_\varepsilon \leqslant M$ for some constant M and all $\varepsilon > 0$, then $d(x_\varepsilon, \partial\Omega_0) \to 0$.*

Since $\lambda \geqslant \lambda_1^{\Omega_0}(\phi)$, we must have $\|u_\varepsilon\|_{L^\infty(\Omega)} \to \infty$ as $\varepsilon \to 0$, for otherwise u_ε increases to a positive solution of (2.1) with $\varepsilon = 0$ as ε decreases to 0, contradicting the fact that

(2.2) is a necessary condition for (2.1) to possess a positive solution. Let us now pick up a sequence $\varepsilon_n \to 0$, and define $\hat{u}_n = u_n / \|u_n\|_\infty$, where $u_n = u_\lambda^{\varepsilon_n}$. We easily see that

$$-\Delta \hat{u}_n = \lambda \hat{u}_n - \bigl[b(x) + \varepsilon_n\bigr] \|u_n\|_\infty^{p-1} \hat{u}_n^p, \quad \hat{u}_n|_{\partial\Omega} = 0.$$

It follows that

$$-\Delta \hat{u}_n \leqslant \lambda \hat{u}_n. \tag{2.12}$$

Therefore

$$\int_\Omega |\nabla \hat{u}_n|^2 \, \mathrm{d}x \leqslant \lambda \int_\Omega \hat{u}_n^2 \, \mathrm{d}x \leqslant \lambda |\Omega|,$$

and $\{\hat{u}_n\}$ is bounded in $W_0^{1,2}(\Omega)$. This implies that, subject to a subsequence, $\hat{u}_n$ converges weakly in $W_0^{1,2}(\Omega)$ and strongly in $L^q(\Omega)$ (for all $q > 1$) to some $\hat{u} \in W_0^{1,2}(\Omega)$. We claim that $\hat{u} \not\equiv 0$. Indeed, if $\hat{u} = 0$, then from $\hat{u}_n \to 0$ in $L^q(\Omega)$ for all $q > 1$ we deduce $(-\Delta)^{-1} \hat{u}_n \to 0$ in the C^1 norm; in particular, $(-\Delta)^{-1} \hat{u}_n \to 0$ uniformly in Ω. From (2.12) we deduce

$$0 \leqslant \hat{u}_n \leqslant \lambda (-\Delta)^{-1} \hat{u}_n \to 0,$$

contradicting the fact that $\|\hat{u}_n\|_\infty = 1$. This proves that $\hat{u} \not\equiv 0$.

An application of Lemma 2.5 shows that u_n is bounded from above by $\underline{U}_\lambda$ on $\Omega_+ := \Omega \setminus \overline{\Omega}_0$. From this we easily see that $\hat{u} \equiv 0$ on Ω_+. Thus, as $\partial \Omega_0$ is smooth, $\hat{u}|_{\Omega_0} \in W_0^{1,2}(\Omega_0)$.

If $\|u_n\|_\infty = u(x_n)$, $x_n \in \Omega$. Then by Bony's maximum principle (see [Bo] and [L]), there exists a sequence $\tilde{x}_k \to x_n$ such that $\lim_{k \to \infty} \Delta u_n(\tilde{x}_k) \leqslant 0$ and hence, from the equation for u_n, we obtain

$$0 \leqslant \lambda u_n(x_n) - \bigl[b(x_n) + \varepsilon_n\bigr] u_n(x_n)^p.$$

It follows that $\varepsilon_n \|u_n\|_\infty^{p-1} \leqslant \lambda$. Hence we may assume that $\varepsilon_n \|u_n\|_\infty^{p-1} \to \xi$ for some $\xi \geqslant 0$.

Now we multiply the equation for $\hat{u}_n$ by an arbitrary $\psi \in C_0^\infty(\Omega_0)$ and integrate over Ω_0, and pass to the limit $n \to \infty$, to obtain that

$$\int_{\Omega_0} \nabla \hat{u} \cdot \nabla \psi \, \mathrm{d}x = \int_{\Omega_0} \bigl(\lambda \hat{u} - \xi \hat{u}^p\bigr) \psi \, \mathrm{d}x.$$

That is to say that $\hat{u}|_{\Omega_0}$ is a weak solution to

$$-\Delta u = \bigl(\lambda - \xi u^{p-1}\bigr) u, \quad u|_{\partial \Omega_0} = 0.$$

By the weak Harnack inequality, we deduce $\hat{u} > 0$ in Ω_0.

From the equation for $\hat{u}_n$, we see that $-\Delta\hat{u}_n$ is uniformly bounded on $\overline{\Omega}_0$. By standard interior L^p-theory for elliptic equations (see [GT]), we find that $\hat{u}_n$ is bounded in $W^{2,q}(\Omega')$ for any $q > 1$ and any compact subdomain Ω' of Ω_0. By the Sobolev embedding theorem (see [GT]), we know that subject to a subsequence, $\hat{u}_n \to \hat{u}$ in $C^1(\overline{\Omega}')$. As $\hat{u} > 0$ on Ω_0, and $\|u_n\|_\infty \to \infty$, we find that $u_n(x) \to \infty$ uniformly on any compact subset of Ω_0. As $\varepsilon \to u_\lambda^\varepsilon$ is monotone, $u_\lambda^\varepsilon \to \infty$ uniformly on any compact subset of Ω_0 as $\varepsilon \to 0$. Thus we must have $d(x_\varepsilon, \partial\Omega_0) \to 0$ as $\varepsilon \to 0$.

STEP 2. *If $m_\varepsilon < M$ for some M and all $\varepsilon > 0$, then $\{\partial u_\lambda^\varepsilon(x_\varepsilon)/\partial\nu_\varepsilon\}$ is bounded from above, where ν_ε is a unit vector in $\mathbb{R}^N$ to be specified later.*

It suffices to show that, for any sequence $\varepsilon_n \to 0$, $\{\partial u_\lambda^{\varepsilon_n}(x_{\varepsilon_n})/\partial\nu_{\varepsilon_n}\}$ has a subsequence which is bounded from above. Let us denote

$$u_n = u_\lambda^{\varepsilon_n}, \qquad x_n = x_{\varepsilon_n} \quad \text{and} \quad \Omega_n = \big\{x \in \Omega_0\colon d(x, \partial\Omega_0) \geqslant d(x_n, \partial\Omega_0)\big\}.$$

Note that if $x_n \in \partial\Omega_0$, then $\Omega_n = \Omega_0$, and if Ω_n is different from Ω_0, then for large n, it is close to Ω_0 by Step 1. Thus for any $\Omega' \subset\subset \Omega_0$, $\Omega' \subset\subset \Omega_n$ for all large n. Clearly u_n is an upper solution to the problem

$$-\Delta u = \lambda u - \big[b(x) + 1\big]u^p \quad \text{in } \Omega \setminus \overline{\Omega}_n, \quad u|_{\partial\Omega} = 0, u|_{\partial\Omega_n} = u_n(x_n), \tag{2.13}$$

and 0 is a lower solution. Therefore (2.13) has a positive solution v_n satisfying $0 \leqslant v_n \leqslant u_n$ in $\Omega \setminus \Omega_n$. As $u_n(x_n) = v_n(x_n)$, it follows that

$$\frac{\partial u_n(x_n)}{\partial\nu_n} \leqslant \frac{\partial v_n(x_n)}{\partial\nu_n},$$

where ν_n is the unit normal vector of $\partial\Omega_n$ at x_n pointing inward of Ω_n. Thus it suffices to show that $\partial v_n(x_n)/\partial\nu_n$ is bounded.

Clearly $C_0 := \max\{\lambda^{1/(p-1)}, M\}$ is an upper solution to (2.13). By Lemma 2.5, we conclude that $v_n \leqslant C_0$. This implies that $-\Delta v_n$ has an L^∞ bound on $\Omega \setminus \Omega_n$ which is independent of n. Since, furthermore,

(1) $v_n|_{\partial\Omega_n}$ is a constant which has a bound independent of n, and

(2) for all large n, $\partial\Omega_n$ is as smooth as Ω_0 with the smoothness not depending on n,

by the L^p-theory of elliptic equations up to the boundary (see, e.g., [GT]), we see that, for any $q > 1$, $\|v_n\|_{W^{2,q}(\Omega\setminus\Omega_n)}$ has a bound independent of n. By Sobolev embeddings and the uniform smoothness of Ω_n, this implies that $\|v_n\|_{C^1(\overline{\Omega}\setminus\Omega_n)}$ has a bound independent of n. In particular $\{|\nabla v_n(x_n)|\}$ is bounded, and thus $\{\partial v_n(x_n)/\partial\nu_n\}$ is bounded, as required.

STEP 3. *$m_\varepsilon \to \infty$ as $\varepsilon \to 0$.*

Otherwise we can find a sequence $\varepsilon_n \to 0$ such that m_{ε_n} is bounded. By Step 2, $\{\partial u_n(x_n)/\partial\nu_n\}$ is bounded from above, where ν_n is the unit normal vector of $\partial\Omega_n$ at x_n pointing inward of Ω_n. Here we follow the notation in Step 2. We show that this is impossible, and hence proving the claim. For all large n, $\partial\Omega_n$ is as smooth as $\partial\Omega_0$ and hence it satisfies a uniform interior ball condition: There exists $R > 0$ such that for any large n and $x \in \partial\Omega_n$, one can find a closed ball B_x of radius R such that $B_x \subset \overline{\Omega}_n$ and $B_x \cap \partial\Omega_n = \{x\}$. Let y_n denote the center of B_{x_n} and define

$$\psi(x) = \mathrm{e}^{-\sigma|x-y_n|^2} - \mathrm{e}^{-\sigma R^2},$$

where σ is a positive number to be specified. We may assume that $\varepsilon_n < 1$ for all n. Then, for any constant c satisfying $1 < c < \varepsilon_n^{-1/p}$ and $x \in B_{x_n} \setminus B^n$, where $B^n = \{x\colon |x - y_n| < R/2\}$, we have

$$\begin{aligned}
&\Delta\big[u_n(x_n) + c\psi\big] + \lambda\big[u_n(x_n) + c\psi\big] - \varepsilon_n\big[u_n(x_n) + c\psi\big]^p \\
&\quad \geqslant c\mathrm{e}^{-\sigma|x-y_n|^2}\big[4\sigma^2|x - y_n|^2 - 2N\sigma\big] - \varepsilon_n c^p\left[\frac{u_n(x_n)}{c} + \psi\right]^p \\
&\quad \geqslant c\mathrm{e}^{-\sigma R^2}\big(\sigma^2 R^2 - 2N\sigma\big) - \big[u_n(x_n) + \psi\big]^p \\
&\quad > 0,
\end{aligned}$$

if σ, c and n are large enough. We fix σ at such a value.

Choose a compact set $K \subset\subset \Omega_0$ such that $K \supset \bigcup_{n=1}^{\infty} B^n$. By the proof of Step 1, $u_n \to \infty$ on K. Hence we can find a sequence $c_n \to \infty$ satisfying $c_n \leqslant \varepsilon_n^{-1/p}$ and

$$u_n(x) \geqslant M + c_n\psi|_{\partial B^n} \quad \text{for all } x \in \partial B^n \subset K.$$

Thus, u_n is an upper solution to the problem

$$\begin{cases} -\Delta u = \lambda u - \varepsilon_n u^p & \text{in } B_{x_n} \setminus B^n, \\ u|_{\partial B_{x_n}} = u_n(x_n), \qquad u|_{\partial B^n} = u_n(x_n) + c_n\psi|_{\partial B^n}. \end{cases}$$

By our choice of σ, for all large n, $u_n(x_n) + c_n\psi$ is a lower solution to this problem. Using Lemma 2.5 we deduce $u_n \geqslant u_n(x_n) + c_n\psi$ in $B_{x_n} \setminus B^n$, and it follows that

$$\frac{\partial u_n(x_n)}{\partial \nu_n} \geqslant c_n \frac{\partial\psi(x_n)}{\partial \nu_n} = c_n 2\sigma R\mathrm{e}^{-\sigma R^2} \to \infty.$$

This contradicts the conclusion in Step 2. Thus the claim and hence part (a) in conclusion (ii) of the theorem is proved.

It remains to prove part (b). By the above proved part (a), we see that $u_n|_{\partial\Omega_0} \to \infty$ uniformly as $n \to \infty$. By Lemma 2.5 we deduce $u_n \leqslant u_{n+1} \leqslant \underline{U}_\lambda$. Therefore $u_n \to u_0 \leqslant \underline{U}_\lambda$ as $n \to \infty$. It follows that u_0 is a positive solution of (2.3). Since $\underline{U}_\lambda$ is the minimal positive solution, we must have $u_0 = \underline{U}_\lambda$. The proof is complete. □

It was proved in [Du3], Part II, Proposition 2.3, that the minimal positive solution $\underline{U}_\lambda$ varies continuously with $\lambda \in \mathbb{R}^1$ in the space $X := C(\Omega_+ \cup \partial\Omega)$, where $\Omega_+ := \Omega \setminus \overline{\Omega}_0$, and X has the metric defined by

$$d(u, v) = \sum_{n=1}^{\infty} \frac{2^{-n} d_n(u, v)}{1 + d_n(u, v)}$$

with

$$d_n(u, v) = \|u - v\|_{C(\Omega_n)}, \quad \Omega_n := \left\{ x \in \overline{\Omega}_+ \colon d(x, \partial\Omega_0) \geqslant \frac{\delta}{n} \right\},$$

where $\delta > 0$ is small enough. This implies that

$$\Gamma_\infty := \left\{ (\lambda, \underline{U}_\lambda) \colon \lambda \in \mathbb{R}^1 \right\}$$

is a continuous curve in $\mathbb{R}^1 \times C(\Omega_+ \cup \partial\Omega)$. If we define

$$\widetilde{\underline{U}}_\lambda(x) = \begin{cases} +\infty, & x \in \overline{\Omega}_0, \\ \underline{U}_\lambda(x), & x \in \Omega_+, \end{cases}$$

and consider $\widetilde{\Gamma}_\infty = \{(\lambda, \widetilde{\underline{U}}_\lambda) \colon \lambda \in \mathbb{R}^1\}$ as a continuous bifurcation curve at infinity, then Theorems 2.4 and 2.6 can be interpreted as follows:

(i) *The positive solution curve* $\Gamma = \{(\lambda, u_\lambda)\}$ *of* (2.1) *bifurcates from the trivial solution curve* $\Gamma_0 = \{(\lambda, 0)\}$ *at* $\lambda = \lambda_0^{\Omega}$, *then joins the bifurcation curve from infinity* $\widetilde{\Gamma}_\infty$ *at* $\lambda = \lambda_1^{\Omega_0}$.

(ii) *As* $\varepsilon \to 0$, *the positive solution curve* $\Gamma_\varepsilon = \{(\lambda, u_\lambda^\varepsilon)\}$ *of* (2.10) *approaches* Γ *when* $\lambda \in (\lambda_1^{\Omega}, \lambda_1^{\Omega_0})$, *and it approaches* $\widetilde{\Gamma}_\infty$ *when* $\lambda \geqslant \lambda_1^{\Omega_0}$.

In order to better understand the profile of u_λ^ε (which is the unique positive solution of (2.10)), we consider $w_\lambda^\varepsilon := \varepsilon^{p-1} u_\lambda^\varepsilon$. It is easily seen that w_λ^ε is the unique positive solution of the problem

$$-\Delta w = \lambda w - \left[1 + \varepsilon^{-1} b(x)\right] w^p, \quad w|_{\partial\Omega} = 0. \tag{2.14}$$

If $\lambda \in (\lambda_1^{\Omega}, \lambda_1^{\Omega_0})$, then by Theorem 2.6(i), we see that $w_\lambda^\varepsilon \to 0$ uniformly in Ω as $\varepsilon \to 0$. We now consider the case that $\lambda > \lambda_1^{\Omega_0}$. If we denote by θ_λ the unique positive solution of

$$-\Delta w = \lambda w - w^p, \quad w|_{\partial\Omega} = 0,$$

then by Lemma 2.5 we see that $w_\lambda^\varepsilon \leqslant \theta_\lambda$. Also by Lemma 2.5, we find that w_λ^ε is nonincreasing with ε. Therefore $w_\lambda^0(x) = \lim_{\varepsilon \to 0} w_\lambda^\varepsilon(x) \in [0, \theta_\lambda(x)]$ exists. Furthermore, on any compact subset K of Ω_0, $-\Delta w_\lambda^\varepsilon = \lambda w_\lambda^\varepsilon - (w_\lambda^\varepsilon)^p$ has an L^∞ bound from above independent of ε. By the L^p estimates and Sobolev embedding theorem we find that

w_λ^ε converges to w_λ^0 in $C^1(K)$. By Theorem 2.6(ii)(b), we see that $w_\lambda^\varepsilon \to 0$ uniformly on any compact subset of $\overline{\Omega} \setminus \overline{\Omega}_0$. It follows that $w_\lambda^0 = 0$ over $\overline{\Omega} \setminus \overline{\Omega}_0$.

Let θ_λ^0 denote the unique positive solution of

$$-\Delta w = \lambda w - w^p, \quad w|_{\partial\Omega_0} = 0. \tag{2.15}$$

By Lemma 2.5 we obtain $w_\lambda^\varepsilon \geqslant \theta_\lambda^0$. Therefore $w_\lambda^0 \geqslant \theta_\lambda^0$ in Ω_0. We show that $w_\lambda^0 = \theta_\lambda^0$ in Ω_0. Indeed, from the inequality $-\Delta w_\lambda^\varepsilon \leqslant \lambda w_\lambda^\varepsilon \leqslant \lambda\theta_\lambda$ we deduce that $\{w_\lambda^\varepsilon\}$ is bounded in $W_0^{1,2}(\Omega)$ and therefore by a compactness argument, $w_\lambda^\varepsilon \to w_\lambda^0$ weakly in $W_0^{1,2}(\Omega)$ and strongly in $L^q(\Omega)$ for any $q > 1$ (we also use the fact that $\|w_\lambda^\varepsilon\|_\infty$ is bounded). Since $w_\lambda^0 = 0$ in Ω_+ and $\partial\Omega_0$ is smooth, we conclude that $w_\lambda^0|_{\Omega_0} \in W_0^{1,2}(\Omega_0)$. It follows easily that $w_\lambda^0|_{\Omega_0}$ is a week positive solution of (2.15). By standard elliptic regularity, w_λ^0 is also a classical positive solution. But θ_λ^0 is the unique such solution. Therefore $w_\lambda^0 = \theta_\lambda^0$ in Ω_0.

We now find that w_λ^0 is a continuous function in $\overline{\Omega}$, and as $\varepsilon \to 0$, $w_\lambda^\varepsilon \to w_\lambda^0$ in $L^q(\Omega)$, and the convergence is uniform on any compact subset of $\overline{\Omega} \setminus \partial\Omega_0$. We claim that this convergence is uniform over $\overline{\Omega}$. From the above discussions, it is clear that we need only prove the following conclusion: For any given $\delta > 0$, there exists $\sigma_0 > 0$ so that

$$w_\lambda^\varepsilon(x) < \delta \quad \forall\varepsilon \in (0, \sigma_0), \forall x \in S_{\sigma_0} := \bigl\{x \in \Omega\colon d(x, \partial\Omega_0) < \sigma_0\bigr\}. \tag{2.16}$$

Denote $\Omega_\sigma = \{x \in \Omega\colon d(x, \Omega_0) < \sigma\}$. Since $\lambda > \lambda_1^{\Omega_0}$, for all small $\sigma > 0$, $\lambda > \lambda_1^{\Omega_\sigma}$. Therefore the problem

$$-\Delta w = \lambda w - w^p, \quad w|_{\partial\Omega_\sigma} = 0,$$

has a unique positive solution θ_λ^σ. If we extend θ_λ^σ to be 0 outside Ω_σ, then a simple compactness argument shows that $\theta_\lambda^\sigma \to \theta_\lambda^0$ as $\sigma \to 0$ uniformly in Ω. In particular, we can find $\sigma_1 > 0$ small so that $\lambda > \lambda_1^{\Omega_{\sigma_1}}$ and

$$\theta_\lambda^{\sigma_1}(x) < \frac{\delta}{2} \quad \forall x \in S_{\sigma_1}. \tag{2.17}$$

On the other hand, let $\tilde{b}(x) \leqslant b(x)$ be a continuous function such that $\tilde{b}(x) = 0$ on Ω_{σ_1} and $\tilde{b}(x) > 0$ on $\overline{\Omega} \setminus \overline{\Omega}_{\sigma_1}$. Then by what has been proved above, we have $\tilde{w}_\lambda^\varepsilon \to \theta_\lambda^{\sigma_1}$ uniformly on $\Omega_{\sigma_1/2}$, where $\tilde{w}_\lambda^\varepsilon$ is the unique positive solution to

$$-\Delta w = \lambda w - \bigl[1 + \varepsilon^{-1}\tilde{b}(x)\bigr]w^p, \quad w|_{\partial\Omega} = 0.$$

By Lemma 2.5 we deduce $w_\lambda^\varepsilon \leqslant \tilde{w}_\lambda^\varepsilon$. Choose $\sigma_0 \leqslant \sigma_1/2$ such that for $\varepsilon < \sigma_0$,

$$\tilde{w}_\lambda^\varepsilon \leqslant \theta_\lambda^{\sigma_1} + \frac{\delta}{2} \quad \text{on } \Omega_{\sigma_1/2}.$$

Therefore, by (2.17), for $\varepsilon \in (0, \sigma_0)$ and $x \in S_{\sigma_0} \subset S_{\sigma_1/2}$,

$$w_\lambda^\varepsilon \leqslant \tilde{w}_\lambda^\varepsilon \leqslant \theta_\lambda^{\sigma_1} + \frac{\delta}{2} < \delta,$$

that is, (2.16) holds. We have thus proved the following theorem.

THEOREM 2.7. *Suppose that $\lambda > \lambda_1^{\Omega_0}$. Then the unique positive solution w_λ^ε of* (2.14) *converges uniformly to w_λ^0 on $\overline{\Omega}$ as $\varepsilon \to 0$.*

Let us observe that Theorem 2.7 gives a clear description of the pattern of w_λ^ε for small $\varepsilon > 0$: It is close to 0 over $\Omega \setminus \Omega_0$ and close to a definite positive function θ_λ^0 over Ω_0.

2.3. *Comments and related results*

Positive solutions of problem (2.1) seem first considered by Ouyang [Ou], motivated by some geometric questions. Soon after, the results in [Ou] were extended in several directions by a number of authors. For example, Fraile, Koch-Medina, López-Gómez and Merino [FKLM] used an upper and lower solution argument to obtain a priori bounds, which greatly simplified the arguments in [Ou]. The proof of our Lemma 2.1 follows the approach of [FKLM]. Theorem 2.4 was proved in [DH]. Under some extra conditions on $b(x)$ near $\partial\Omega_0$, similar results were proved in [GGLS]. More references can be found in [DH]. Theorem 2.6 is taken from [Du3], Part II, and Theorem 2.7 from [DL].

The restriction that Ω_0 is connected can be relaxed to the situation that it has finitely many components, each with smooth boundary; the techniques here can be easily adapted to deal with this case. Related results can be found, for example, in [Lop2] and [DL].

If $\overline{\Omega}_0$ is not contained in Ω, then some of the techniques here collapse, though it is expected that similar results hold. The case that $\partial\Omega_0 \cap \partial\Omega \neq \emptyset$ was discussed in [DG].

For the existence and uniqueness of positive solutions of (2.1), the smoothness condition on $b(x)$ and Ω_0 can be greatly relaxed, see [dP] and [Da8]. How to extend Theorems 2.4 and 2.6 to these situations remains to be investigated, though partial results were obtained in [dP].

We now come back to (2.1). By Theorem 2.4, we know that the branch of positive solutions bifurcating from the trivial solution at $\lambda = \lambda_1^\Omega$ blows up as λ approaches $\lambda_1^{\Omega_0}$. If λ_k^Ω denotes the kth eigenvalue of $-\Delta$ under Dirichlet boundary conditions, and we define $\lambda_k^{\Omega_0}$ similarly, then by Rabinowitz's global bifurcation theorem, a global branch of nontrivial solutions Γ_k of (2.1) bifurcates from the trivial solution branch at $\lambda = \lambda_k^\Omega$ if λ_k^Ω is of odd algebraic multiplicity. If Γ_k is unbounded, then must it blow up at $\lambda = \lambda_k^{\Omega_0}$? From Section 2.1, we know that this is the case when $k = 1$. In [DO] several special cases were considered where bifurcation branches starting from λ_k^Ω $(k > 1)$ indeed blow up at $\lambda = \lambda_k^{\Omega_0}$. Whether this is true in general remains open. Nevertheless, we have the following result (see [DO]), which shows that the set $\{\lambda_k^{\Omega_0}\colon k \geqslant 1\}$ contains all the possible λ values where bifurcation from infinity can occur.

THEOREM 2.8. *Given any large positive constant Λ and open neighborhood V of the finite set*

$$M = \{\lambda_k^{\Omega_0}\colon \lambda_k^{\Omega_0} \leqslant \Lambda, k \geqslant 1\},$$

we can find a constant C depending on Λ and V such that any solution (λ, u) of (2.1) with $\lambda \leqslant \Lambda$ and $\lambda \notin V$ satisfies

$$\|u\|_{L^\infty} \leqslant C.$$

PROOF. We argue indirectly. Suppose that we can find a sequence of solutions $\{(\lambda_n, u_n)\}$ of (2.1) such that $\lambda_n \leqslant \Lambda$, $\lambda_n \notin V$, and $\|u_n\|_{L^\infty} \to \infty$. Clearly we have

$$\int_\Omega |\nabla u_n| \,\mathrm{d}x \leqslant \lambda_n \int_\Omega u_n^2 \,\mathrm{d}x.$$

It follows that $\lambda_n \geqslant \lambda_1^\Omega$. Therefore, by passing to a subsequence, we may assume that $\lambda_n \to \hat{\lambda}$ and $\hat{\lambda} \in [\lambda_1^\Omega, \Lambda] \setminus V$.

CLAIM 1. *$\{u_n\}$ is uniformly bounded on any compact subset of $\Omega_+ := \Omega \setminus \overline{\Omega}_0$.*

To prove Claim 1, we let K be an arbitrary compact subset of Ω_+. By our assumption on b, for some small neighborhood U of K, there exists $\tau > 0$ such that $b(x) \geqslant \tau$ on U. Denote by V_λ the unique positive solution of

$$-\Delta u = \lambda u - \tau u^p, \quad u|_{\partial U} = \infty,$$

whose existence and uniqueness is well known (see [MV]). Choose λ^* such that $\lambda^* > \lambda_n$ for all $n \geqslant 1$. We want to show that $u_n \leqslant V_{\lambda^*}$ on K. Otherwise, we can find some $n \geqslant 1$ and a domain $U_0 \subset\subset U$ such that $u_n > V_{\lambda^*}$ in U_0 and $u_n = V_{\lambda^*}$ on ∂U_0. Hence on U_0, we have

$$-\Delta u_n = \lambda_n u_n - b(x) u_n^p$$

and

$$-\Delta V_{\lambda^*} \geqslant \lambda^* V_{\lambda^*} - b(x) V_{\lambda^*}^p.$$

An application of Lemma 2.5 on U_0 yields $u_n \leqslant V_{\lambda^*}$ on U_0, a contradiction.

Similarly, we find $-u_n \leqslant V_{\lambda^*}$ on U. Therefore

$$|u_n| \leqslant V_{\lambda^*} \quad \forall x \in U,$$

and Claim 1 follows.

CLAIM 2. $\hat{\lambda} \in M$.

Clearly this contradicts the fact that $\hat{\lambda} \notin V$. Therefore our proof is complete once Claim 2 is proved. Denoting $\hat{u}_n = u_n/\|u_n\|_{L^\infty}$ we find that

$$\int_\Omega |\nabla \hat{u}_n|^2 \, \mathrm{d}x \leqslant \lambda_n \int_\Omega \hat{u}_n^2 \, \mathrm{d}x \leqslant \Lambda |\Omega|.$$

Thus $\{\hat{u}_n\}$ is a bounded sequence in $W_0^{1,2}(\Omega)$. It follows that, subject to a subsequence, $\hat{u}_n$ converges to some $\hat{u}$ weakly in $W_0^{1,2}(\Omega)$ and strongly in $L^2(\Omega)$. As $\hat{u}_n$ has L^∞ norm 1, we find that $\hat{u}_n \to \hat{u}$ in $L^q(\Omega)$ for any $q > 1$. By Claim 1, we see that $\hat{u} \equiv 0$ over Ω_+. Since $\partial\Omega_0$ is smooth, it is well known that this implies $v_0 := \hat{u}|_{\Omega_0} \in W_0^{1,2}(\Omega_0)$.

We show next that $v_0 \not\equiv 0$. Otherwise, $\hat{u} \equiv 0$ over Ω and hence $\hat{u}_n \to 0$ in L^q for all $q > 1$. By Kato's inequality (see [Kato]), we have, in the weak sense,

$$-\Delta|\hat{u}_n| \leqslant -\frac{\hat{u}_n}{|\hat{u}_n|}\Delta\hat{u}_n \leqslant \lambda_n|\hat{u}_n| \leqslant \Lambda|\hat{u}_n|.$$

Therefore

$$0 \leqslant |\hat{u}_n| \leqslant \Lambda(-\Delta)^{-1}|\hat{u}_n|.$$

By standard elliptic regularity we find $(-\Delta)^{-1}|\hat{u}_n| \to 0$ uniformly in Ω. It follows that $\|\hat{u}_n\|_{L^\infty} \to 0$ as $n \to \infty$. But this contradicts the fact that $\|\hat{u}_n\|_{L^\infty} = 1$. Hence we have proved that $v_0 \not\equiv 0$.

We now multiply the equation satisfied by $\hat{u}_n$ by an arbitrary $\phi \in C_0^\infty(\Omega_0)$, integrate by parts and find

$$\int_{\Omega_0} \nabla\hat{u}_n \cdot \nabla\phi \, \mathrm{d}x = \int_{\Omega_0} \lambda_n \hat{u}_n \phi \, \mathrm{d}x.$$

Letting $n \to \infty$ we obtain

$$\int_{\Omega_0} \nabla v_0 \cdot \nabla\phi \, \mathrm{d}x = \int_{\Omega_0} \hat{\lambda} v_0 \phi \, \mathrm{d}x.$$

This implies that $v_0 \in W_0^{1,2}(\Omega_0)$ solves

$$-\Delta v = \hat{\lambda} v, \quad v|_{\partial\Omega_0} = 0, \tag{2.18}$$

in the weak sense. Standard elliptic regularity shows that v_0 is also a classical solution of (2.18). As we have already proved that $v_0 \not\equiv 0$, we must have $\hat{\lambda} = \lambda_k^{\Omega_0}$ for some $k \geqslant 1$. Thus $\hat{\lambda} \in M$. This proves Claim 2 and hence concludes the proof of Theorem 2.8. □

3. Bifurcation and monotonicity: A heterogeneous competition system

Bifurcation and monotonicity have been combined to produce many nice results in nonlinear analysis, and a collection of these techniques and results may be found in [Am]. In this section we present some new applications.

It is in general not easy to capture the influence of heterogeneous spatial environment on population models. Traditionally population models were considered in homogeneous spatial environment, and hence all the coefficients appearing in the models are chosen to be positive constants. To include spatial variations of the environment, these constant coefficients should be replaced by positive functions of the space variable x. However, the mathematical techniques developed to study these models are ironically either not sensitive to this change, in which case the effects of heterogeneous spatial environment are difficult to observe in the mathematical analysis, or the techniques are too sensitive to this change and become inapplicable when the constant coefficients are replaced by functions.

In this section we use a competition model to demonstrate that bifurcation techniques are useful in capturing these spatial effects. Here we combine the bifurcation arguments with a certain monotonicity property of the system. A key in this approach is the following observation: *The behavior of the model is very sensitive to certain coefficient functions becoming small in part of the underlying spatial region*. To make our ideas more transparent, we consider the following simplified steady-state competition system

$$\begin{cases} -\Delta u = \lambda u - \big[b(x) + \varepsilon\big]u^2 - cuv, \\ -\Delta v = \mu v - v^2 - duv, \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0. \end{cases} \tag{3.1}$$

Here Ω is a bounded smooth domain in $\mathbb{R}^N$ ($N \geqslant 2$) and λ, μ, c, d are constants, with $c > 0, d > 0$. The function $b(x)$ is as in (2.1). We will see that when ε is small, the profiles of certain *stable* positive solutions of (3.1) are determined by the behavior of $b(x)$. Our approach makes use of the facts that the global bifurcation branches for (3.1) with $\varepsilon = 0$ differs considerably to that for (3.1) with any $\varepsilon > 0$. By carefully following the changes of the global bifurcation branches of (3.1) as ε shrinks to 0, we will be able to obtain a rather detailed global picture of the solution branches and to describe the patterns of the stable solutions. The techniques here can be used to study systems much more general than (3.1). We refer to [Du3,Du4] for discussions of the background.

3.1. *Global bifurcation*

We will fix $\varepsilon > 0$ and apply a global bifurcation analysis to (3.1), using μ as a bifurcation parameter. We are only interested in nonnegative solutions. Firstly let us observe some preliminary results. Clearly $(u, v) = (0, 0)$ is always a solution to (3.1), which is called the trivial solution. If $(u, 0) \neq (0, 0)$ is a nonnegative solution of (3.1), then u is a positive solution of (2.10) with $p = 2$, namely

$$-\Delta u = \lambda u - \big[b(x) + \varepsilon\big]u^2, \quad u|_{\partial\Omega} = 0. \tag{3.2}$$

It is well known that (3.2) has a unique positive solution when $\lambda > \lambda_1^\Omega$, and there is no positive solution when $\lambda \leqslant \lambda_1^\Omega$. For $\lambda > \lambda_1^\Omega$, we denote the unique positive solution of (3.2) by ϕ_λ^ε, and when the ε dependence is not emphasized, we will simply denote it by ϕ_λ.

If $(0, v) \neq (0, 0)$ is a nonnegative solution of (3.1), then v is a positive solution of

$$-\Delta v = \mu v - v^2, \quad v|_{\partial\Omega} = 0. \tag{3.3}$$

Similarly to (3.2), there is a unique positive solution θ_μ to (3.3) when $\mu > \lambda_1^\Omega$, and no positive solution exists if $\mu \leqslant \lambda_1^\Omega$. The solutions $(\phi_\lambda, 0)$ and $(0, \theta_\mu)$ are called semitrivial solutions of (3.1).

We next discuss the positive solutions of (3.1), where both components u and v are positive in Ω. (By the strong maximum principle, if (u, v) solves (3.1) and $u \geqslant, \not\equiv 0$, $v \geqslant, \not\equiv 0$, then $u > 0, v > 0$ in Ω.) Suppose that (3.1) has a positive solution. Then from the first equation we obtain

$$\lambda = \lambda_1^\Omega\big(\big[b(x) + \varepsilon\big]u + cv\big) > \lambda_1^\Omega(0) = \lambda_1^\Omega.$$

Similarly, from the second equation we deduce $\mu > \lambda_1^\Omega$. Therefore the following is a necessary condition for (3.1) to possess a positive solution:

$$\lambda > \lambda_1^\Omega, \qquad \mu > \lambda_1^\Omega. \tag{3.4}$$

So from now on we fix $\lambda > \lambda_1^\Omega$. We also assume that ε, c and d are fixed; μ will be considered as our bifurcation parameter.

We now obtain some rough estimates for positive solutions of (3.1). Let (u, v) be a positive solution of (3.1). Then u is a lower solution of (3.2). Hence we can apply Lemma 2.5 to deduce that $u \leqslant \phi_\lambda$. Similarly, $v \leqslant \theta_\mu$. Using the above estimate for u we deduce

$$-\Delta v + d\phi_\lambda v \geqslant \mu v - v^2,$$

and by Lemma 2.5, $v \geqslant v_\mu$, where v_μ is the unique positive solution of

$$-\Delta v + d\phi_\lambda v = \mu v - v^2, \quad v|_{\partial\Omega} = 0,$$

provided that $\mu > \lambda_1^\Omega(d\phi_\lambda)$. We now show that there exists $\hat{\mu} > 0$ large enough so that (3.1) has no positive solution when $\mu > \hat{\mu}$. Indeed, we have

$$\lambda = \lambda_1^\Omega\big(\big[b(x) + \varepsilon\big]u + cv\big) \geqslant \lambda_1^\Omega(cv_\mu). \tag{3.5}$$

Since $v_\mu \to \infty$ as $\mu \to \infty$ uniformly on any compact subsets of Ω, we can show that $\lambda_1^\Omega(cv_\mu) \to \infty$ as $\mu \to \infty$. Hence (3.5) implies that $\mu \leqslant \hat{\mu}$ for some large $\hat{\mu}$.

To summarize, we have the following result.

THEOREM 3.1. *Suppose* $\lambda > \lambda_1^{\Omega}$ *and* ε, c, d *are fixed positive constants. Then there exists* $\hat{\mu} > 0$ *such that if* (3.1) *has a positive solution* (u, v), *then*

$$\lambda_1^{\Omega} < \mu < \hat{\mu}, \qquad u \leqslant \phi_\lambda, \qquad v \leqslant \theta_\mu \leqslant \theta_{\hat{\mu}}.$$

We now transform (3.1) into an abstract equation and apply bifurcation and monotonicity arguments to study its positive solution set. Choose $M > 0$ large enough such that for $\mu \in [0, 1 + \hat{\mu}]$ and $0 \leqslant u \leqslant \xi := 1 + \|\phi_\lambda\|_\infty$, $0 \leqslant v \leqslant \eta := 1 + \|\theta_{\hat{\mu}}\|_\infty$,

$$g(u, v) := Mu + \lambda u - \big[b(x) + \varepsilon\big]u^2 - cuv$$

is strictly increasing in u, and

$$h(\mu, u, v) := Mv + \mu v - v^2 - duv$$

is strictly increasing in v. Then define

$$A(\mu, u, v) = (-\Delta + M)^{-1}\big(g(u, v), h(\mu, u, v)\big), \quad (u, v) \in E_0, \mu \in \mathbb{R}^1,$$

where $E_0 := \{(u, v) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})\colon u|_{\partial\Omega} = v|_{\partial\Omega} = 0\}$.

Clearly $P_0 := \{(u, v) \in E_0\colon u \leqslant 0, v \geqslant 0\}$ is a cone in E_0. It introduces a partial ordering in E_0:

$$(u_1, v_1) \leqslant_{P_0} (u_2, v_2) \quad \text{if and only if} \quad (u_2 - u_1, v_2 - v_1) \in P_0.$$

Denote

$$\mathcal{J} := [0, 1 + \hat{\mu}], \qquad \mathcal{A}_0 := \big\{(u, v) \in E_0\colon 0 \leqslant u < \xi, 0 \leqslant v < \eta \text{ in } \Omega\big\}.$$

It is easily seen, by the positivity of $(-\Delta + M)^{-1}$, that for fixed $\mu \in \mathcal{J}$, $A(\mu, u, v)$ is increasing over $\mathcal{A}_0$ in the order $\leqslant_{P_0}$, namely

$$(u_1, v_1), (u_2, v_2) \in \mathcal{A}_0, \quad (u_1, v_1) \leqslant_{P_0} (u_2, v_2) \quad \text{implies}$$
$$A(\mu, u_1, v_1) \leqslant_{P_0} A(\mu, u_2, v_2).$$

Moreover, for fixed $(u, v) \in \mathcal{A}_0$, $A(\mu, u, v)$ is increasing in μ:

$$\mu_1 \leqslant \mu_2 \quad \text{implies} \quad A(\mu_1, u, v) \leqslant_{P_0} A(\mu_2, u, v).$$

If we use $(u_1, v_1) \ll_{P_0} (u_2, v_2)$ to mean $(u_2 - u_1, v_2 - v_1) \in \operatorname{int} P_0$ and use $(u_1, v_1) <_{P_0} (u_2, v_2)$ to mean $(u_2 - u_1, v_2 - v_1) \in P_0 \setminus \{(0, 0)\}$, then for $\mu \in \mathcal{J}$ and $(u_1, v_1), (u_2, v_2) \in \mathcal{A}_0$,

$$A(\mu, u_1, v_1) \ll_{P_0} A(\mu, u_2, v_2) \qquad \text{if } (u_1, v_1) <_{P_0} (u_2, v_2) \tag{3.6}$$

and for $(u,v) \in \mathcal{A}_0$ with $v \not\equiv 0$,

$$\mu_1 < \mu_2 \quad \text{implies} \quad A(\mu_1, u, v) <_{P_0} A(\mu_2, u, v). \tag{3.7}$$

By the L^p-theory and Sobolev embedding theorems we know that $A : \mathbb{R}^1 \times E_0 \to E_0$ is completely continuous, and is continuously Frechet differentiable (in fact, it is analytic). Clearly (u,v) solves (3.1) if and only if it is a solution to

$$(u,v) = A(\mu, u, v). \tag{3.8}$$

In the space $\mathbb{R}^1 \times E_0$, (3.8) has the trivial solution curve $\Gamma_0 := \{(\mu, 0, 0)\colon\ \mu \in \mathbb{R}^1\}$, the semitrivial solution curve $\Gamma_1 := \{(\mu, \phi_\lambda, 0)\colon\ \mu \in \mathbb{R}^1\}$ and the semitrivial solution curve $\Gamma_2 := \{(\mu, 0, \theta_\mu)\colon\ \mu > \lambda_1^\Omega\}$.

It is well known (see [BB2]) that we can apply the local bifurcation result, Theorem 1.2, along Γ_1 to find a unique value $\mu_0 := \lambda_1^\Omega(d\phi_\lambda)$ so that a branch of positive solutions of (3.8) bifurcates from Γ_1 at $(\mu_0, \psi_\lambda, 0) \in \Gamma_1$. Moreover, a variant of Rabinowitz's global bifurcation theorem (see [BB2], Theorem 3.2) can be used to show that this local branch can be continued globally, and by making use of the strong maximum principle and Theorem 3.1, it can be concluded that this global branch of positive solutions Γ must join the semitrivial solution branch Γ_2. One can apply Theorem 1.2 along Γ_2 to find that $(\mu^0, 0, \theta_{\mu^0})$ is the only point on Γ_2 where positive solutions bifurcate from Γ_2, where $\mu^0 > \lambda_1^\Omega$ is uniquely determined by

$$\lambda = \lambda_1^\Omega(c\theta_{\mu^0}). \tag{3.9}$$

Therefore Γ joins Γ_2 at the point $(\mu^0, 0, \theta_{\mu^0})$.

Clearly the above global bifurcation analysis implies the following existence result.

THEOREM 3.2. *For any μ between μ_0 and μ^0*, (3.1) *has a positive solution*.

We will demonstrate that by making use of the monotonicity property of the operator A, much more can be obtained. We will mainly follow the approach of [DB], though the presentation here is slightly different. Let us define

$$\Lambda := \big\{\mu \in \mathbb{R}^1\colon\ (3.1) \text{ has a positive solution}\big\}, \qquad \mu_* = \inf \Lambda, \qquad \mu^* = \sup \Lambda.$$

By Theorems 3.1 and 3.2, clearly

$$\lambda_1^\Omega \leqslant \mu_* \leqslant \min\big\{\mu_0, \mu^0\big\}, \qquad \max\big\{\mu_0, \mu^0\big\} \leqslant \mu^* \leqslant \hat{\mu}. \tag{3.10}$$

Though it is highly unclear whether the positive solution branch Γ contains all the possible positive solutions of (3.1), we will show that

$$\begin{aligned} \mu_* &= \inf\big\{\mu \in \mathbb{R}^1\colon\ (\mu, u, v) \in \Gamma\big\}, \\ \mu^* &= \sup\big\{\mu \in \mathbb{R}^1\colon\ (\mu, u, v) \in \Gamma\big\}. \end{aligned} \tag{3.11}$$

To prove this and related facts, we need the following lemmas.

LEMMA 3.3. *Suppose that* $\mu_* < \mu^0$. *Then for every* $\mu \in (\mu_*, \mu^0)$, (3.1) *has a maximal positive solution* (u^μ, v^μ), *in the sense that any positive solution* (u, v) *of* (3.1) *satisfies* $(u, v) \leqslant_{P_0} (u^\mu, v^\mu)$.

PROOF. Let $\mu \in (\mu_*, \mu^0)$ be fixed. Then $\lambda = \lambda_1^\Omega(c\theta_{\mu^0}) > \lambda_1^\Omega(c\theta_\mu)$, and hence the problem

$$-\Delta u = \lambda u - \big[b(x) + \varepsilon\big]u^2 - c\theta_\mu u, \quad u|_{\partial\Omega} = 0$$

has a unique positive solution u_0. If we denote $v_0 := \theta_\mu$, then it is easily checked that

$$A(\mu, u_0, v_0) <_{P_0} (u_0, v_0).$$

By the definition of μ_*, there exists $\tilde{\mu} \in [\mu_*, \mu)$ such that (3.1) with $\mu = \tilde{\mu}$ has a positive solution $(\tilde{u}, \tilde{v})$. By Theorem 3.1 we have $\tilde{v} \leqslant \theta_{\tilde{\mu}} < \theta_\mu$ in Ω, and hence

$$-\Delta \tilde{u} > \lambda \tilde{u} - \big[b(x) + \varepsilon\big]\tilde{u}^2 - c\theta_\mu \tilde{u}.$$

We now apply Lemma 2.5 and conclude that $\tilde{u} \geqslant u_0$ in Ω. Therefore $(\tilde{u}, \tilde{v}) <_{P_0} (u_0, v_0)$. Clearly we also have $A(\mu, \tilde{u}, \tilde{v}) >_{P_0} (\tilde{u}, \tilde{v})$. If we define

$$(u_n, v_n) := A(\mu, u_{n-1}, v_{n-1}), \quad n = 1, 2, \ldots,$$

we easily deduce that

$$(\tilde{u}, \tilde{v}) \leqslant_{P_0} (u_n, v_n) \leqslant_{P_0} (u_{n-1}, v_{n-1}) \leqslant_{P_0} (u_0, v_0), \quad n = 2, 3, \ldots. \tag{3.12}$$

A standard compactness argument shows that $(u_n, v_n) \to (u^\mu, v^\mu)$ in E_0 and $(u^\mu, v^\mu) = A(\mu, u^\mu, v^\mu)$. By (3.12), we know that (u^μ, v^μ) is a positive solution of (3.1). We claim that it is the maximal positive solution. Indeed, if (u, v) is any positive solution of (3.1), then by Theorem 3.1, $v \leqslant \theta_\mu$ and hence

$$-\Delta u \geqslant \lambda u - \big[b(x) + \varepsilon\big]u^2 - c\theta_\mu u.$$

By Lemma 2.5, this implies that $u \geqslant u_0$. Therefore $(u, v) \leqslant_{P_0} (u_0, v_0)$. It follows from this inequality and the monotonicity of A that $(u, v) \leqslant_{P_0} (u_n, v_n)$ for all $n \geqslant 0$. Thus $(u, v) \leqslant_{P_0} (u^\mu, v^\mu)$. □

LEMMA 3.4. *If* $\mu^* > \mu_0$, *then for every* $\mu \in (\mu_0, \mu^*)$, (3.1) *has a minimal positive solution* (u_μ, v_μ).

PROOF. Let $\mu \in (\mu_0, \mu^*)$. Then $\mu > \lambda_1^\Omega(d\phi_\lambda)$ and hence the problem

$$-\Delta v = \mu v - v^2 - d\phi_\lambda v, \quad v|_{\partial\Omega} = 0,$$

has a unique positive solution v_0. Denote $u_0 = \phi_\lambda$. Then (u_0, v_0) satisfies $A(\mu, u_0, v_0) \geqslant_{P_0} (u_0, v_0)$. Choose $\tilde{\mu} \in (\mu, \mu^*]$ such that (3.1) with $\mu = \tilde{\mu}$ has a positive solution $(\tilde{u}, \tilde{v})$. Then it is easily checked that

$$A(\mu, \tilde{u}, \tilde{v}) <_{P_0} (\tilde{u}, \tilde{v}), \quad (\tilde{u}, \tilde{v}) \geqslant_{P_0} (u_0, v_0).$$

Define

$$(u_n, v_n) := A(\mu, u_{n-1}, v_{n-1}), \quad n = 1, 2, \ldots.$$

Then we have

$$(u_0, v_0) \leqslant_{P_0} (u_1, v_1) \leqslant_{P_0} \cdots \leqslant_{P_0} (u_n, v_n) \leqslant_{P_0} \cdots \leqslant_{P_0} (\tilde{u}, \tilde{v}).$$

Therefore (u_n, v_n) converges to a positive solution (u_μ, v_μ) of (3.1). Moreover, if (u, v) is any positive solution of (3.1), then $(u, v) \geqslant_{P_0} (u_0, v_0)$ and hence $(u, v) \geqslant_{P_0} (u_n, v_n)$ for all $n \geqslant 1$. It follows that $(u, v) \geqslant_{P_0} (u_\mu, v_\mu)$. □

THEOREM 3.5. *The identities in* (3.11) *are true. Moreover,*
(i) *if* $\mu_* < \mu^0$, *then* $(\mu, u^\mu, v^\mu) \in \Gamma$ *for all* $\mu \in (\mu_*, \mu^0)$,
(ii) *if* $\mu^* > \mu_0$, *then* $(\mu, u_\mu, v_\mu) \in \Gamma$ *for all* $\mu \in (\mu_0, \mu^*)$.

PROOF. Clearly $\mu^0 \geqslant \inf\{\mu\colon (\mu, u, v) \in \Gamma\}$ and $\mu_0 \leqslant \sup\{\mu\colon (\mu, u, v) \in \Gamma\}$. Therefore we have nothing to prove if $\mu_* = \mu^0$ and $\mu^* = \mu_0$.

Suppose $\mu_* < \mu^0$. Let us define, for $\mu \in (\mu_*, \mu^0)$,

$$\Delta^\mu := [\mu, \infty) \times \big[(u^\mu, v^\mu), \infty\big),$$

where

$$\big[(u^\mu, v^\mu), \infty\big) := \big\{(u, v) \in E_0\colon (u, v) \geqslant_{P_0} \big(u^\mu, v^\mu\big)\big\}.$$

Since Γ connects $(\mu^0, 0, \theta_{\mu^0}) \in \operatorname{int} \Delta^\mu$ and $(\mu_0, \phi_\lambda, 0) \notin \Delta^\mu$, it follows from the connectedness of Γ that $\Gamma \cap \partial \Delta^\mu \neq \emptyset$. Let $(\tilde{\mu}, \tilde{u}, \tilde{v}) \in \Gamma \cap \partial \Delta^\mu$. We claim that $(\tilde{\mu}, \tilde{u}, \tilde{v}) = (\mu, u^\mu, v^\mu)$. Clearly

$$\partial \Delta^\mu = \big[\{\mu\} \times \big[\big(u^\mu, v^\mu\big), \infty\big)\big] \cup \big[[\mu, \infty) \times \partial\big[\big(u^\mu, v^\mu\big), \infty\big)\big].$$

If $\tilde{\mu} = \mu$, then since (u^μ, v^μ) is the maximal positive solution of (3.1), we necessarily have $(\tilde{u}, \tilde{v}) \leqslant_{P_0} (u^\mu, v^\mu)$. On the other hand, we have $(\tilde{u}, \tilde{v}) \in [(u^\mu, v^\mu), \infty)$. Therefore $(\tilde{u}, \tilde{v}) = (u^\mu, v^\mu)$. We show next that $\tilde{\mu} > \mu$ is impossible. Indeed, if $\tilde{\mu} > \mu$, then we must have $(\tilde{u}, \tilde{v}) \in \partial[(u^\mu, v^\mu), \infty)$, i.e.,

$$(\tilde{u}, \tilde{v}) \geqslant_{P_0} \big(u^\mu, v^\mu\big), \qquad (\tilde{u}, \tilde{v}) \not\gg_{P_0} \big(u^\mu, v^\mu\big).$$

By the monotonicity properties of the operator A, we have

$$(\tilde{u},\tilde{v}) = A(\tilde{\mu},\tilde{u},\tilde{v}) >_{P_0} A(\mu,\tilde{u},\tilde{v}) \geqslant_{P_0} A\big(\mu,u^{\mu},v^{\mu}\big) = \big(u^{\mu},v^{\mu}\big).$$

Now from $(\tilde{u},\tilde{v}) >_{P_0} (u^{\mu},v^{\mu})$, we further obtain $A(\mu,\tilde{u},\tilde{v}) \gg_{P_0} A(\mu,u^{\mu},v^{\mu})$ and hence $(\tilde{u},\tilde{v}) \gg_{P_0} (u^{\mu},v^{\mu})$. Therefore $(\tilde{\mu},\tilde{u},\tilde{v}) \in \operatorname{int} \varDelta^{\mu}$, contradicting the assumption that $(\tilde{\mu},\tilde{u},\tilde{v}) \in \partial\varDelta^{\mu}$. This proves assertion (i).

The proof for (ii) is similar, where we use

$$\varDelta_{\mu} = (-\infty,\mu] \times \big(-\infty,(u_{\mu},v_{\mu})\big]$$

with

$$\big(-\infty,(u_{\mu},v_{\mu})\big] := \big\{(u,v) \in E_0\colon\ (u,v) \leqslant_{P_0} (u_{\mu},v_{\mu})\big\},$$

and the fact that $\varGamma \cap \partial\varDelta_{\mu} = \{(\mu,u_{\mu},v_{\mu})\}$. We omit the details.

Clearly, (3.11) is a consequence of (i) and (ii). □

THEOREM 3.6. (i) *If* $\mu_* < \min\{\mu_0,\mu^0\}$, *then* (3.1) *has a maximal positive solution* (u^{μ_*},v^{μ_*}) *for* $\mu = \mu_*$ *and it has at least two positive solutions for each* $\mu \in (\mu_*, \min\{\mu_0,\mu^0\})$. *Moreover, all these solutions can be chosen from* $\varGamma$.

(ii) *If* $\mu^* > \max\{\mu_0,\mu^0\}$, *then* (3.1) *has a minimal positive solution for* $\mu = \mu^*$ *and it has at least two positive solutions for each* $\mu \in (\max\{\mu_0,\mu^0\},\mu^*)$. *Moreover, all these solutions can be chosen from* $\varGamma$.

PROOF. Suppose $\mu_* < \min\{\mu_0,\mu^0\}$. Let $\{\mu_n\} \subset (\mu_*,\mu^0)$ be a decreasing sequence converging to μ_*. By Theorem 3.5, (3.1) with $\mu = \mu_n$ has a maximal positive solution $(u_n,v_n) := (u^{\mu_n},v^{\mu_n})$. Using the estimates in Theorem 3.1 and a compactness argument we easily see that, subject to a subsequence, $(u_n,v_n) \to (u_*,v_*)$ in E_0, and (u_*,v_*) is a nonnegative solution of (3.1) with $\mu = \mu_*$. Since μ_n is decreasing, by the monotonicity property of A we easily deduce that $(u_n,v_n) \gg_{P_0} (u_{n+1},v_{n+1})$. Therefore $u_* \geqslant u_n > 0$ in $\varOmega$. We claim that $v_* \not\equiv 0$ in $\varOmega$. Indeed, if $v_* \equiv 0$, then from the equation for u_n in (3.1) we easily deduce that $u_n \to \phi_\lambda$ in E_0, and hence $u_* = \phi_\lambda$. But then from the equation for v_n we deduce

$$\mu_n = \lambda_1^{\varOmega}(v_n + du_n) \to \lambda_1^{\varOmega}(d\phi_\lambda) = \mu_0,$$

a contradiction to our assumption that $\mu_n \to \mu_*$. Therefore (u_*,v_*) is a positive solution of (3.1) with $\mu = \mu_*$.

If (u,v) is any positive solution of (3.1) with $\mu = \mu_*$, then by making use of the monotonicity property of A we deduce $(u,v) \ll_{P_0} (u_n,v_n)$ for all $n \geqslant 1$. It follows that $(u,v) \leqslant_{P_0} (u_*,v_*)$. Therefore (u_*,v_*) is the maximal solution. The proof of Theorem 3.5 shows that $(\mu_*,u_*,v_*) \in \varGamma$.

To show that (3.1) has at least two positive solutions for $\mu \in (\mu_*, \min\{\mu_0,\mu^0\})$, we use the following simple result from general point set theory, whose proof can be found in [DB].

LEMMA 3.7. *Suppose that X is a Banach space, C is a connected set in X and O an open set in X such that $\partial O \cap C$ consists of a single point. Then $C \setminus O$ is a connected set.*

Now for fixed $\mu \in (\mu_*, \min\{\mu_0, \mu^0\})$, we take $C = \Gamma$ and $O = \operatorname{int} \Delta^{\tilde{\mu}}$, where $\tilde{\mu} \in (\mu_*, \mu)$. By the proof of Theorem 3.5, we know that $\partial \Delta^{\tilde{\mu}} \cap \Gamma = \{(\tilde{\mu}, u^{\tilde{\mu}}, v^{\tilde{\mu}})\}$. Hence by Lemma 3.7, the set $\Gamma \setminus \operatorname{int} \Delta^{\tilde{\mu}}$ is connected. Since $(\mu_*, u_*, v_*) \in \Gamma \setminus \Delta^{\tilde{\mu}}$ and $(\mu_0, \phi_\lambda, 0) \in \overline{\Gamma} \setminus \Delta^{\tilde{\mu}}$, by the connectedness of $\Gamma \setminus \Delta^{\tilde{\mu}}$ and the fact that $\mu \in (\mu_*, \mu_0)$, we can find a point $(\mu, u, v) \in \Gamma \setminus \Delta^{\tilde{\mu}}$. Hence (u, v) is a positive solution of (3.1). Since $\tilde{\mu} < \mu$, we have $(\mu, u^\mu, v^\mu) \in \operatorname{int} \Delta^{\tilde{\mu}}$. Therefore (u, v) and (u^μ, v^μ) are different positive solutions of (3.1), and both (μ, u, v) and (μ, u^μ, v^μ) belong to Γ. This proves (i).

The proof of (ii) is parallel and we omit the details. □

REMARK 3.8. By the connectedness of Γ, if $\mu_* < \mu^*$, then for any $\mu \in (\mu_*, \mu^*)$, (3.1) has a positive solution on Γ. We will show in Section 3.3 that $\mu^* > \max\{\mu_0, \mu^0\}$ if $\varepsilon > 0$ is small enough.

3.2. *Stability analysis*

Suppose that (u_0, v_0) is a positive solution of (3.1). We want to know whether it is stable when considered as a steady-state of the corresponding parabolic problem

$$\begin{cases} u_t - \Delta u = \lambda u - \big[b(x) + \varepsilon\big]u^2 - cuv, & x \in \Omega,\ t > 0, \\ v_t - \Delta v = \mu v - v^2 - duv, & x \in \Omega,\ t > 0, \\ u = v = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{3.13}$$

THEOREM 3.9. *Suppose that (u_0, v_0) is a positive solution of (3.1), and there exists $(h, k) \in P_0 \setminus \{(0, 0)\}$ and $\sigma \in \mathbb{R}^1$ such that*

$$\begin{cases} -\Delta h = \lambda h - 2\big[b(x) + \varepsilon\big]u_0 h - cv_0 h - cu_0 k + \sigma h, \\ -\Delta k = \mu k - 2v_0 k - du_0 k - dv_0 h + \sigma k, \\ h|_{\partial\Omega} = k|_{\partial\Omega} = 0. \end{cases} \tag{3.14}$$

Then (u_0, v_0) is asymptotically stable in E_0 if $\sigma > 0$, and it is unstable if $\sigma < 0$.

PROOF. Since $u_0 > 0$ and $v_0 > 0$ in Ω, we easily see from (3.14) that $h \not\equiv 0$ and $k \not\equiv 0$. Moreover, from the first equation in (3.14), we obtain $-\Delta h \leqslant C(x)h$ in Ω for some $C \in C(\overline{\Omega})$. Therefore by the maximum principle and Hopf boundary lemma, we deduce $h < 0$ in Ω and $\partial_\nu h > 0$ on $\partial\Omega$. We similarly deduce $k > 0$ in Ω and $\partial_\mu k < 0$ on $\partial\Omega$. This implies that $(h, k) \in \operatorname{int} P_0$.

For $\delta \in \mathbb{R}^1$ let us denote $u_\delta := u_0 + \delta h$, $v_\delta := v_0 + \delta k$. Then a simple calculation gives

$$\begin{cases} -\Delta u_\delta = \lambda u_\delta - \big[b(x) + \varepsilon\big]u_\delta^2 - c u_\delta v_\delta \\ \qquad\qquad + \big(\sigma + \delta\big[b(x) + \delta\big] + c\delta k\big)\delta h, \\ -\Delta v_\delta = \mu v_\delta - v_\delta^2 - d u_\delta v_\delta + (\sigma + \delta + d\delta h)\delta k, \\ u_\delta|_{\partial\Omega} = v_\delta|_{\partial\Omega} = 0. \end{cases} \tag{3.15}$$

Due to (3.15) we can find $\delta_0 = \delta_0(\sigma) > 0$ small enough so that, if $\sigma > 0$, then for all $\delta \in (0, \delta_0]$,

$$\begin{cases} -\Delta u_\delta < \lambda u_\delta - \big[b(x) + \varepsilon\big]u_\delta^2 - c u_\delta v_\delta, \\ -\Delta v_\delta > \mu v_\delta - v_\delta^2 - d u_\delta v_\delta, \\ u_\delta|_{\partial\Omega} = v_\delta|_{\partial\Omega} = 0, \end{cases}$$

and for all $\delta \in [-\delta_0, 0)$,

$$\begin{cases} -\Delta u_\delta > \lambda u_\delta - \big[b(x) + \varepsilon\big]u_\delta^2 - c u_\delta v_\delta, \\ -\Delta v_\delta < \mu v_\delta - v_\delta^2 - d u_\delta v_\delta, \\ u_\delta|_{\partial\Omega} = v_\delta|_{\partial\Omega} = 0. \end{cases}$$

Hence $(u_{\delta_0}, v_{\delta_0})$ is an upper solution of (3.1) and $(u_{-\delta_0}, v_{-\delta_0})$ is a lower solution of (3.1).

It is well known (see [Sm]) that the semiflow generated by the solution of (3.13) preserves the order $\leqslant_{P_0}$, and by the theory of monotone dynamical systems (see [Ma] or [Hir]), the following hold:

(i) the unique solution $(\underline{u}(x,t), \underline{v}(x,t))$ of (3.13) with initial data $(u_{-\delta_0}, v_{-\delta_0})$ increases in the order $\leqslant_{P_0}$ as t increases;

(ii) the unique solution $(\bar{u}(x,t), \bar{v}(x,t))$ of (3.13) with initial data $(u_{\delta_0}, v_{\delta_0})$ decreases in the order $\leqslant_{P_0}$ as t increases;

(iii) $$(u_{-\delta_0}, v_{-\delta_0}) \leqslant_{P_0} \big(\underline{u}(\cdot,t), \underline{v}(\cdot,t)\big) \leqslant_{P_0} \big(\bar{u}(\cdot,t), \bar{v}(\cdot,t)\big) \leqslant_{P_0} (u_{\delta_0}, v_{\delta_0}) \quad \forall t > 0;$$

(iv) $(\underline{u}(x), \underline{v}(x)) = \lim_{t\to\infty}(\underline{u}(x,t), \underline{v}(x,t))$ and $(\bar{u}(x), \bar{v}(x)) = \lim_{t\to\infty}(\bar{u}(x,t), \bar{v}(x,t))$ exist and they are solutions of (3.1).

Therefore we have

$$(u_{-\delta_0}, v_{-\delta_0}) \leqslant_{P_0} (\underline{u}, \underline{v}) \leqslant_{P_0} (\bar{u}, \bar{v}) \leqslant_{P_0} (u_{\delta_0}, v_{\delta_0}).$$

Define

$$\delta_* = \inf\big\{\delta \in [0, \delta_0]\colon\ (\underline{u}, \underline{v}) \geqslant_{P_0} (u_{-\delta}, v_{-\delta})\big\}.$$

Then $0 \leqslant \delta_* \leqslant \delta_0$. If $\delta_* > 0$, then

$$(\underline{u}, \underline{v}) = A(\mu, \underline{u}, \underline{v}) \geqslant_{P_0} A(\mu, u_{-\delta_*}, v_{-\delta_*}) \gg_{P_0} (u_{-\delta_*}, v_{-\delta_*}).$$

Therefore for all $\delta < \delta_*$ but close to δ_*, we also have $(\underline{u}, \underline{v}) \geqslant_{P_0} (u_{-\delta}, v_{-\delta})$, contradicting the definition of δ_*. This proves that $(\underline{u}, \underline{v}) \geqslant_{P_0} (u_0, v_0)$. Similarly we can show that $(\bar{u}, \bar{v}) \leqslant_{P_0} (u_0, v_0)$. Therefore we must have

$$\big(\underline{u}(x), \underline{v}(x)\big) = \big(\bar{u}(x), \bar{v}(x)\big) = \big(u_0(x), v_0(x)\big).$$

By the order preserving property of (3.13), we deduce that any solution $(u(x,t), v(x,t))$ of (3.13) with initial data taken from

$$\begin{aligned}&\big[(u_{-\delta_0}, v_{-\delta_0}), (u_{\delta_0}, v_{\delta_0})\big]\\ &\quad := \big\{(u,v) \in E_0\colon\ (u_{-\delta_0}, v_{-\delta_0}) \leqslant_{P_0} (u,v) \leqslant_{P_0} (u_{\delta_0}, v_{\delta_0})\big\}\end{aligned}$$

satisfies

$$\big(\underline{u}(\cdot,t), \underline{v}(\cdot,t)\big) \leqslant_{P_0} \big(u(\cdot,t), v(\cdot,t)\big) \leqslant_{P_0} \big(\bar{u}(\cdot,t), \bar{v}(\cdot,t)\big) \quad \forall t > 0.$$

It follows that $(u(x,t), v(x,t)) \to (u_0(x), v_0(x))$ as $t \to \infty$. By standard regularity for parabolic equations this convergence can be taken in the norm of E_0. This proves the asymptotic stability of (u_0, v_0) in E_0, since $[(u_{-\delta_0}, v_{-\delta_0}), (u_{\delta_0}, v_{\delta_0})]$ is an open neighborhood of (u_0, v_0) due to $(h,k) \in \operatorname{int} P_0$.

If $\sigma < 0$, then for all small $\delta > 0$, (u_δ, v_δ) is a lower solution of (3.1). It follows that the unique solution (u, v) of (3.13) with initial date (u_δ, v_δ) increases in $\leqslant_{P_0}$ as t increases. By a simple comparison argument one sees that (u, v) stays bounded in the L^∞ norm for all $t > 0$. Hence by the theory of monotone dynamical systems (u, v) converges to a steady-state of (3.13) as $t \to \infty$, say (u_*, v_*). Clearly $(u_*, v_*) \geqslant_{P_0} (u_\delta, v_\delta) \gg_{P_0} (u_0, v_0)$. We can show that $(u_*, v_*) >_{P_0} (u_{\delta_0}, v_{\delta_0})$ by a sweeping argument. Indeed, if we define

$$\delta^* := \sup\big\{\eta \in [\delta, \delta_0]\colon\ (u_*, v_*) \geqslant_{P_0} (u_\eta, v_\eta)\big\},$$

then by the monotonicity property of A and the continuous dependence on δ of the (strict) lower solutions (u_δ, v_δ), we easily deduce that $\delta^* = \delta_0$ and $(u_*, v_*) >_{P_0} (u_{\delta_0}, v_{\delta_0})$. Therefore $(u_*, v_*) \notin [(u_{-\delta_0}, v_{-\delta_0}), (u_{\delta_0}, v_{\delta_0})]$. This implies that (u_0, v_0) is unstable. □

We now relate (3.14) to the spectral radius $r(L)$ of the linear operator $L\colon E_0 \to E_0$, where L denotes the Frechet derivative of $A(\mu, u, v)$ with respect to (u, v) at (u_0, v_0), namely

$$L := D_{(u,v)} A(\mu, u_0, v_0).$$

THEOREM 3.10. *Let* (u_0, v_0) *be a positive solution of* (3.1). *Then* (3.14) *has a solution* $(h,k) \in P_0 \setminus \{(0,0)\}$ *with* $\sigma > 0$ *if* $r(L) < 1$; *it has a solution* $(h,k) \in P_0 \setminus \{(0,0)\}$ *with* $\sigma < 0$ *if* $r(L) > 1$.

In the proof of Theorem 3.10, we will need the following version of the well-known Krein–Rutman theorem (see [De], Theorem 19.3, and [Am], Theorem 3.2).

THEOREM 3.11. *Suppose that X is a Banach space with a positive cone P which has nonempty interior, and B is a compact linear operator in X, which is strongly positive*: $B(P \setminus \{0\}) \subset \operatorname{int} P$. *Then $r(B) > 0$ and there exists a unique $x_0 \in \operatorname{int} P$ such that $Bx_0 = r(B)x_0$, $\|x_0\| = 1$; there exists $\phi \in X^*$ such that $\phi(x) > 0$ for $x \in P \setminus \{0\}$ and $B^*\phi = r(B)\phi$. Hence $Bx - r(B)x \notin P \setminus \{0\}$ for any $x \in X$.*

PROOF OF THEOREM 3.10. It is easily checked that L is compact and strongly positive in E_0. Therefore we can apply Theorem 3.11 to find $(h_0, k_0) \in \operatorname{int} P_0$ such that $L(h_0, k_0) = r(L)(h_0, k_0)$. Suppose $r(L) < 1$ and define, for $\sigma \geqslant 0$,

$$L_\sigma(h,k) = L(h,k) + \sigma(-\Delta + M)^{-1}(h,k).$$

Then L_σ is compact and strongly positive in E_0. Hence, by Theorem 3.11, $r(L_\sigma) > 0$ and there exists a unique $(h_\sigma, k_\sigma) \in \operatorname{int} P_0$ such that

$$L_\sigma(h_\sigma, k_\sigma) = r(L_\sigma)(h_\sigma, k_\sigma), \quad \|(h_\sigma, k_\sigma)\|_{E_0} = 1.$$

By the uniqueness of (h_σ, k_σ) and a standard compactness argument, we easily see that $r(L_\sigma)$ varies continuously with σ. We also have

$$\sigma^{-1} r(L_\sigma) = r\big(\sigma^{-1}L + (-\Delta + M)^{-1}\big) \to r\big((-\Delta + M)^{-1}\big) > 0 \quad \text{as } \sigma \to \infty.$$

Therefore $r(L_\sigma) > 1$ for all large σ. Since $r(L_0) = r(L) < 1$, there exists $\sigma_0 > 0$ such that $r(L_{\sigma_0}) = 1$, i.e., $L_{\sigma_0}(h_{\sigma_0}, k_{\sigma_0}) = (h_{\sigma_0}, k_{\sigma_0})$. It is easily seen that this implies that $(h_{\sigma_0}, k_{\sigma_0})$ solves (3.14) with $\sigma = \sigma_0$.

Suppose now $r(L) > 1$. We then consider, for $\sigma \geqslant 1$, the family of operators

$$\begin{aligned} L^\sigma(h,k) &:= (-\Delta + \sigma M)^{-1} \\ &\quad \times \big(g_u(u_0, v_0)h + g_v(u_0, v_0)k, h_u(u_0, v_0)h + h_v(u_0, v_0)k\big). \end{aligned}$$

We easily see that L^σ is a compact and strongly positive operator in E_0, and hence by Theorem 3.11, there exists a unique $(h^\sigma, k^\sigma) \in \operatorname{int} P_0$ such that

$$L^\sigma\big(h^\sigma, k^\sigma\big) = r\big(L^\sigma\big)\big(h^\sigma, k^\sigma\big), \quad \big\|\big(h^\sigma, k^\sigma\big)\big\|_{E_0} = 1.$$

As before, by the uniqueness of (h^σ, k^σ) and a standard compactness argument, we easily see that $r(L^\sigma)$ varies continuously with σ. We claim that $r(L^\sigma) \to 0$ as $\sigma \to \infty$. Indeed, if we write $L^\sigma(h^\sigma, k^\sigma) = r(L^\sigma)(h^\sigma, k^\sigma)$ in its differential equation form, we obtain

$$\begin{cases} -\Delta h^\sigma + \sigma M h^\sigma = r\big(L^\sigma\big)^{-1}\big[g_u(u_0, v_0)h^\sigma + g_v(u_0, v_0)k^\sigma\big], \\ -\Delta k^\sigma + \sigma M k^\sigma = r\big(L^\sigma\big)^{-1}\big[h_u(u_0, v_0)h^\sigma + h_v(u_0, v_0)k^\sigma\big], \\ h^\sigma\big|_{\partial\Omega} = k^\sigma\big|_{\partial\Omega} = 0. \end{cases}$$

Therefore there exists a large constant $C > 0$ such that

$$\int_\Omega \left(\left|\nabla h^\sigma\right|^2 + \sigma M\left|h^\sigma\right|^2\right) \mathrm{d}x \leqslant r\left(L^\sigma\right)^{-1} C \int_\Omega \left(\left|h^\sigma\right|^2 + \left|k^\sigma\right|^2\right) \mathrm{d}x,$$

$$\int_\Omega \left(\left|\nabla k^\sigma\right|^2 + \sigma M\left|k^\sigma\right|^2\right) \mathrm{d}x \leqslant r\left(L^\sigma\right)^{-1} C \int_\Omega \left(\left|h^\sigma\right|^2 + \left|k^\sigma\right|^2\right) \mathrm{d}x.$$

It follows that

$$r\left(L^\sigma\right) \leqslant \frac{2C}{M}\sigma^{-1} \to 0 \quad \text{as } \sigma \to \infty.$$

Since $r(L^1) = r(L) > 1$, we can find $\sigma^0 > 1$ such that $r(L^{\sigma^0}) = 1$ and hence $L^{\sigma^0}(h^{\sigma^0}, k^{\sigma^0}) = (h^{\sigma^0}, k^{\sigma^0})$. Writing this in the form of differential equations, we find that $(h^{\sigma^0}, k^{\sigma^0})$ solves (3.14) with $\sigma = (1 - \sigma^0)M < 0$. □

We now discuss the stability of the maximal positive solutions (u^μ, v^μ) and the minimal positive solutions (u_μ, v_μ) of (3.1). Suppose $\mu_* < \mu^0$, and for $\mu \in (\mu_*, \mu^0)$, denote

$$L^\mu := D_{(u,v)} A\left(\mu, u^\mu, v^\mu\right), \qquad O^* := \left\{\mu \in \left(\mu_*, \mu^0\right):\ r\left(L^\mu\right) < 1\right\}.$$

By Theorems 3.9 and 3.10, we know that (u^μ, v^μ) is asymptotically stable when $\mu \in O^*$. Furthermore, by the implicity function theorem, O^* is an open set and (u^μ, v^μ) varies continuously with μ for $\mu \in O^*$. The following result shows that the measure of O^* is the same as that of the interval (μ_*, μ^0).

THEOREM 3.12. *The set $(\mu_*, \mu^0) \setminus O^*$ has measure zero in $\mathbb{R}^1$.*

PROOF. We use ideas from [Du1], Section 3. To simplify notation, we denote $\mathbf{w}(\mu) := (u^\mu, v^\mu)$. From the monotonicity of A, we easily deduce that $\mathbf{w}(\mu_1) \ll_{P_0} \mathbf{w}(\mu_2)$ when $\mu_1 < \mu_2$. Moreover, $\mathbf{w}(\mu)$ is right-continuous in μ. Indeed, suppose μ_n decreases to μ; by a compactness consideration we find that subject to a subsequence, $\mathbf{w}(\mu_n) \to \mathbf{w}$ in E_0 and $\mathbf{w} = A(\mu, \mathbf{w})$. Since $\mathbf{w}(\mu) \ll_{P_0} \mathbf{w}(\mu_n)$, we deduce $\mathbf{w}(\mu) \leqslant_{P_0} \mathbf{w}$. But $\mathbf{w}(\mu)$ is the maximal solution and hence we necessarily have $\mathbf{w} = \mathbf{w}(\mu)$. This implies that $\lim_{\mu' \to \mu+0} \mathbf{w}(\mu') = \mathbf{w}(\mu)$. We divide the rest of the proof into several steps.

STEP 1. *If $\mathbf{w}(\mu)$ is discontinuous at $\hat\mu \in (\mu_*, \mu^0)$, then $\mathbf{w}(\hat\mu^-) := \lim_{\mu \to \hat\mu - 0} \mathbf{w}(\mu)$ exists and $A(\hat\mu, \mathbf{w}(\hat\mu^-)) = \mathbf{w}(\hat\mu^-) \ll_{P_0} \mathbf{w}(\hat\mu)$.*

Let $\{\mu_n\} \subset (\mu_*, \mu^0)$ be an arbitrary sequence increasing to $\hat\mu$. Then

$$u^{\mu_n}(x) \geqslant u^{\mu_{n+1}}(x) \geqslant u^{\hat\mu}(x), \qquad v^{\mu_n}(x) \leqslant v^{\mu_{n+1}}(x) \leqslant v^{\hat\mu}(x).$$

Therefore $(\hat u(x), \hat v(x)) := \lim_{n\to\infty}(u^{\mu_n}(x), v^{\mu_n}(x))$ exists and the limit is independent of the choice of $\{\mu_n\}$. A simple compactness argument shows that $\mathbf{w}(\mu_n) \to \hat{\mathbf{w}} := (\hat u, \hat v)$

in E_0. Clearly we have $\hat{\mathbf{w}} = A(\hat{\mu}, \hat{\mathbf{w}})$. We must have $\hat{\mathbf{w}} \neq \mathbf{w}(\hat{\mu})$ for otherwise $\mathbf{w}(\mu)$ would be continuous at $\hat{\mu}$. Since $\mathbf{w}(\hat{\mu})$ is the maximal positive solution, we must have $\hat{\mathbf{w}} <_{P_0} \mathbf{w}(\hat{\mu})$. By the monotonicity of A we deduce $\hat{\mathbf{w}} \ll_{P_0} \mathbf{w}(\hat{\mu})$. This proves Step 1.

STEP 2. *O^* has the following alternative expression*:

$$O^* = \left\{\mu \in \left(\mu_*, \mu^0\right): \varliminf_{\mu' \to \mu - 0} \frac{\|\mathbf{w}(\mu) - \mathbf{w}(\mu')\|}{\mu - \mu'} < +\infty\right\}.$$

If $\hat{\mu} \in O^*$, then by the implicity function theorem, we know that $\mu \to \mathbf{w}(\mu)$ is $\mathbb{C}^1$ near $\hat{\mu}$ and hence

$$\lim_{\mu \to \hat{\mu}} \frac{\|\mathbf{w}(\hat{\mu}) - \mathbf{w}(\mu)\|}{\hat{\mu} - \mu} = \left\| D_\mu \mathbf{w}(\hat{\mu}) \right\| < +\infty.$$

Conversely, suppose that, for some sequence μ_n increasing to $\hat{\mu}$, we have $\|\mathbf{w}(\hat{\mu}) - \mathbf{w}(\mu_n)\|/(\hat{\mu} - \mu_n) \to \beta \in [0, +\infty)$. Then by Step 1, we know that $\mathbf{w}(\mu)$ must be continuous at $\hat{\mu}$. Denote $\mathbf{z}_n := (\mathbf{w}(\hat{\mu}) - \mathbf{w}(\mu_n))/\|\mathbf{w}(\hat{\mu}) - \mathbf{w}(\mu_n)\|$. We obtain

$$\begin{aligned}
\mathbf{z}_n &= \frac{A(\hat{\mu}, \mathbf{w}(\hat{\mu})) - A(\mu_n, \mathbf{w}(\mu_n))}{\|\mathbf{w}(\hat{\mu}) - \mathbf{w}(\mu_n)\|} \\
&= \frac{\hat{\mu} - \mu_n}{\|\mathbf{w}(\hat{\mu}) - \mathbf{w}(\mu_n)\|}\left(D_\mu A\left(\hat{\mu}, \mathbf{w}(\hat{\mu})\right) + \mathrm{o}(1)\right) + L^{\hat{\mu}} \mathbf{z}_n + \mathrm{o}(1).
\end{aligned}$$

Since $\mathbf{z}_0 := D_\mu A(\hat{\mu}, \mathbf{w}(\hat{\mu})) >_{P_0} \mathbf{0}$, the above identities imply that $\beta > 0$ and

$$\mathbf{z}_n = \beta^{-1} \mathbf{z}_0 + L^{\hat{\mu}} \mathbf{z}_n + \mathrm{o}(1).$$

From this and the compactness of $L^{\hat{\mu}}$, it follows that, subject to a subsequence, $\mathbf{z}_n \to \hat{\mathbf{z}}$ in E_0, and $\hat{\mathbf{z}} = \beta^{-1} \mathbf{z}_0 + L^{\hat{\mu}} \hat{\mathbf{z}}$. Since $\mathbf{z}_n >_{P_0} \mathbf{0}$ and $\|\mathbf{z}_n\| = 1$, we have $\hat{\mathbf{z}} >_{P_0} \mathbf{0}$. If $r(L^{\hat{\mu}}) \geqslant 1$, then we deduce

$$L^{\hat{\mu}}(-\hat{\mathbf{z}}) - r\left(L^{\hat{\mu}}\right)(-\hat{\mathbf{z}}) \geqslant_{P_0} \hat{\mathbf{z}} - L^{\hat{\mu}} \hat{\mathbf{z}} = \beta^{-1} \mathbf{z}_0 >_{P_0} \mathbf{0},$$

a contradiction to the last conclusion of Theorem 3.11. Therefore we must have $r(L^{\hat{\mu}}) < 1$. This proves Step 2.

STEP 3. *If $\mathbf{w}(\mu)$ is continuous at $\hat{\mu} \in (\mu_*, \mu^0)$ and* $\lim_{\mu \to \hat{\mu} - 0} (\mathbf{w}(\hat{\mu}) - \mathbf{w}(\mu))/(\hat{\mu} - \mu) = +\infty$, *then there exists $\alpha(\mu) > 0$ such that*

$$\frac{\mathbf{w}(\hat{\mu}) - \mathbf{w}(\mu)}{\hat{\mu} - \mu} \geqslant \alpha(\mu) \mathbf{w}_0, \qquad \lim_{\mu \to \hat{\mu} - 0} \alpha(\mu) = +\infty, \tag{3.16}$$

where $\mathbf{w}_0 \in \operatorname{int} P_0$ is the unique solution to $L^{\hat{\mu}} \mathbf{w}_0 = r(L^{\hat{\mu}}) \mathbf{w}_0$, $\|\mathbf{w}_0\| = 1$.

Define $\mathbf{z}(\mu) := (\mathbf{w}(\hat{\mu}) - \mathbf{w}(\mu))/\|\mathbf{w}(\hat{\mu}) - \mathbf{w}(\mu)\|$. Then for $\mu \to \hat{\mu} - 0$, we have

$$\begin{aligned}\mathbf{z}(\mu) &= \frac{A(\hat{\mu}, \mathbf{w}(\hat{\mu})) - A(\mu, \mathbf{w}(\mu))}{\|\mathbf{w}(\hat{\mu}) - \mathbf{w}(\mu)\|} \\ &= \frac{\hat{\mu} - \mu}{\|\mathbf{w}(\hat{\mu}) - \mathbf{w}(\mu)\|}\big(D_\mu A\big(\hat{\mu}, \mathbf{w}(\hat{\mu})\big) + \mathrm{o}(1)\big) + L^{\hat{\mu}}\mathbf{z}(\mu) + \mathrm{o}(1) \\ &= L^{\hat{\mu}}\mathbf{z}(\mu) + \mathrm{o}(1).\end{aligned}$$

From this and a compactness argument and the uniqueness of $\mathbf{w}_0$ we easily deduce that $\mathbf{z}(\mu) \to \mathbf{w}_0$ as $\mu \to \hat{\mu}$. Since $\mathbf{w}_0 \gg_{P_0} \mathbf{0}$, we find that for all $\mu < \hat{\mu}$ but close to $\hat{\mu}$, $\mathbf{z}(\mu) \geqslant_{P_0} (1/2)\mathbf{w}_0$ and hence

$$\frac{\mathbf{w}(\hat{\mu}) - \mathbf{w}(\mu)}{\hat{\mu} - \mu} \geqslant \alpha(\mu)\mathbf{w}_0,$$

where

$$\alpha(\mu) = \frac{1}{2}\frac{\|\mathbf{w}(\hat{\mu}) - \mathbf{w}(\mu)\|}{\hat{\mu} - \mu} \to +\infty \quad \text{as } \mu \to \hat{\mu} - 0.$$

This proves Step 3.

STEP 4. *The set* $(\mu_*, \mu^0) \setminus O^*$ *has measure* 0 *in* $\mathbb{R}^1$.

Let l be a nontrivial positive functional on E_0, i.e., $l \in P_0^* \setminus \{0\}$, where $P_0^* := \{e \in E_0^*\colon e(\mathbf{w}) \geqslant 0 \text{ for } \mathbf{w} \in P_0\}$. Define $f : (\mu_*, \mu^0) \to \mathbb{R}^1$ by $f(\mu) = l(\mathbf{w}(\mu))$. Then $f(\mu)$ is increasing and hence has finite derivatives almost everywhere in (μ_*, μ^0).

If $\hat{\mu} \in (\mu_*, \mu^0) \setminus O^*$, then by the conclusions in Steps 1–3, $\mathbf{w}(\mu)$ is either discontinuous at $\hat{\mu}$, or it is continuous at $\hat{\mu}$ but (3.16) holds. In either case we deduce $(f(\hat{\mu}) - f(\mu))/(\hat{\mu} - \mu) \to +\infty$ as $\mu \to \hat{\mu} - 0$. Therefore $(\mu_*, \mu^0) \setminus O^*$ must have measure 0. □

Let us now consider the stability of $\mathbf{w}(\mu) := (u^\mu, v^\mu)$ for $\mu \in [\mu_*, \mu^0) \setminus O^*$. We have the following result.

THEOREM 3.13. *Let* $\hat{\mu} \in (\mu_*, \mu^0) \setminus O^*$. *Then the following hold*:

(i) *If* $\mathbf{w}(\mu)$ *is continuous at* $\hat{\mu}$, *then* $\mathbf{w}(\hat{\mu})$ *is asymptotically stable.*

(ii) *If* $\mathbf{w}(\mu)$ *is discontinuous at* $\hat{\mu}$, *then* $\mathbf{w}(\hat{\mu})$ *is unstable.*

(iii) *If* $\mathbf{w}(\mu)$ *is discontinuous at* $\hat{\mu}$, *then* $\mathbf{w}(\hat{\mu}^-) := \lim_{\mu \to \hat{\mu} - 0} \mathbf{w}(\mu)$ *is asymptotically stable.*

PROOF. We first claim that $r(L^{\hat{\mu}}) = 1$, where $L^{\hat{\mu}} := D_{\mathbf{w}} A(\hat{\mu}, \mathbf{w}(\hat{\mu}))$. Indeed, we can find $\mu_n \in O^*$ decreasing to $\hat{\mu}$. It follows by the right-continuity of $\mathbf{w}(\mu)$ on μ that $r(L^{\hat{\mu}}) = \lim_{n\to\infty} r(L^{\mu_n}) \leqslant 1$. Since $\hat{\mu} \notin O^*$, we must have $r(L^{\hat{\mu}}) = 1$. This proves our claim.

We can now easily check that all the conditions in Theorem 1.3 are satisfied by $F(\mu,\mathbf{w}) := \mathbf{w} - A(\mu,\mathbf{w})$ with $(\lambda_0, v_0) = (\hat\mu, \mathbf{w}(\hat\mu))$. For example, the condition

$$F_\mu\big(\hat\mu, \mathbf{w}(\hat\mu)\big) \notin R\big(F_{\mathbf{w}}\big(\hat\mu, \mathbf{w}(\hat\mu)\big)\big)$$

follows from the last conclusion in Theorem 3.11, since $F_\mu(\hat\mu, \mathbf{w}(\hat\mu)) = -(0, v^{\hat\mu}) \in (-P_0) \setminus \{(0,0)\}$ and $F_{\mathbf{w}}(\hat\mu, \mathbf{w}(\hat\mu)) = I - L^{\hat\mu}$. Hence the solutions of $\mathbf{w} - A(\mu,\mathbf{w}) = 0$ near $(\hat\mu, \mathbf{w}(\hat\mu))$ form a smooth curve

$$(\mu, \mathbf{w}) = \big(\mu(s), \mathbf{w}(\hat\mu) + s\mathbf{w}_0 + \tau(s)\big), \quad s \in [-s_0, s_0], s_0 > 0,$$

with $\mu(0) = \hat\mu$, $\mu'(0) = 0$, $\tau(0) = \tau'(0) = 0$, and $\mathbf{w}_0 \gg_{P_0} \mathbf{0}$ satisfies $L^{\hat\mu}\mathbf{w}_0 = \mathbf{w}_0$. Since A is analytic, so are $\mu(s)$ and $\tau(s)$. We cannot have $\mu(s) \equiv 0$ since by the right-continuity of $\mathbf{w}(\mu)$,

$$\big(\mu, \mathbf{w}(\mu)\big) \to \big(\hat\mu, \mathbf{w}(\hat\mu)\big) \quad \text{as } \mu \to \hat\mu + 0. \tag{3.17}$$

Therefore there exists some integer $k \geqslant 2$ such that $\mu'(0) = \cdots = \mu^{(k-1)}(0) = 0$ and $\mu^{(k)}(0) \neq 0$. We claim that $\mu^{(k)}(0) > 0$, for if $\mu^{(k)}(0) < 0$ and k is even, then $\mu(s) \leqslant \hat\mu$ for all small $|s|$, and hence $F(\mu,\mathbf{w}) = 0$ has no solution near $(\hat\mu, \mathbf{w}(\hat\mu))$ with $\mu > \hat\mu$, contradicting (3.17); if $\mu^{(k)}(0) < 0$ and k is odd, then for $s < 0$ close to 0, we have $\mu(s) > \hat\mu$ and $\mathbf{w}_s := \mathbf{w}(\hat\mu) + s\mathbf{w}_0 + \tau(s) \ll_{P_0} \mathbf{w}(\hat\mu)$, which implies that for $\mu > \hat\mu$ but close to $\hat\mu$, the only solution $(\mu,\mathbf{w})$ of $F(\mu,\mathbf{w}) = 0$ close to $\mathbf{w}(\hat\mu)$ satisfies $\mathbf{w} \ll_{P_0} \mathbf{w}(\hat\mu)$, contradicting (3.17) and the fact that $\mathbf{w}(\mu) \gg_{P_0} \mathbf{w}(\hat\mu)$ for $\mu > \hat\mu$. Therefore we always have $\mu^{(k)}(0) > 0$.

If k is odd, then $\mu(s)$ crosses $\hat\mu$ as s increases across 0. We show that $\mathbf{w}(\mu)$ is continuous at $\hat\mu$. Otherwise by Step 1 in the proof of Theorem 3.12, $\mathbf{w}(\mu) \to \mathbf{w}(\hat\mu^-) \ll_{P_0} \mathbf{w}(\hat\mu)$ as $\mu \to \hat\mu - 0$. This implies that $\mathbf{w}(\mu(s)) \ll_{P_0} \mathbf{w}_s$ for all $s < 0$ close to 0, contradicting the fact that $\mathbf{w}(\mu(s))$ is the maximal positive solution. This proves the continuity of $\mathbf{w}(\mu)$ at $\hat\mu$.

We may assume that s_0 has been chosen small enough such that $\mu(s) < \hat\mu$ for $s \in [-s_0, 0)$, $\mu(s) > \hat\mu$ for $s \in (0, s_0]$, and

$$\mathbf{w}_s \ll_{P_0} \mathbf{w}(\hat\mu) \quad \text{for } s \in [-s_0, 0), \qquad \mathbf{w}_s \gg_{P_0} \mathbf{w}(\hat\mu) \quad \text{for } s \in (0, s_0]. \tag{3.18}$$

By the monotonicity of A, we have

$$\mathbf{w}_s = A\big(\mu(s), \mathbf{w}_s\big) <_{P_0} A(\hat\mu, \mathbf{w}_s) \quad \text{for } s \in [-s_0, 0),$$
$$\mathbf{w}_s = A\big(\mu(s), \mathbf{w}_s\big) >_{P_0} A(\hat\mu, \mathbf{w}_s) \quad \text{for } s \in (0, s_0].$$

Therefore $\{\mathbf{w}_s\colon s \in [-s_0, 0)\}$ is a continuum of strict lower solutions of (3.1) with $\mu = \hat\mu$, and $\{\mathbf{w}_s\colon s \in (0, s_0]\}$ is a continuum of strict upper solutions of (3.1) with $\mu = \hat\mu$. Now the argument in the proof of Theorem 3.9 can be repeated, with $\mathbf{w}_s$, $s \in [-s_0, s_0]$ replacing (u_δ, v_δ), $\delta \in [-\delta_0, \delta_0]$, to conclude that $\mathbf{w}(\hat\mu)$ is asymptotically stable.

If k is even, then $\mu(s) \geqslant \hat\mu$ for $|s|$ small and hence there is no solution $(\mu, \mathbf{w})$ close to $(\hat\mu, \mathbf{w}(\hat\mu))$ with $\mu < \hat\mu$. Therefore $\mathbf{w}(\mu)$ has to be discontinuous at $\hat\mu$. As before we may assume that s_0 is small enough such that (3.18) holds and $\mu(s) > \hat{\mu}$ for all $s \in [-s_0, s_0] \setminus \{0\}$. Then

$$\mathbf{w}_s = A\big(\mu(s), \mathbf{w}_s\big) >_{P_0} A(\hat{\mu}, \mathbf{w}_s) \quad \text{for } s \in [-s_0, 0),$$

and hence $\{\mathbf{w}_s \colon s \in [-s_0, 0)\}$ is a continuum of strict upper solutions of (3.1) with $\mu = \hat{\mu}$. By an analogous consideration as in the proof of Theorem 3.9, we find that the unique solution of (3.13) with $\mu = \hat{\mu}$ and with initial data $\mathbf{w}_s$ decreases in the order $\leqslant_{P_0}$ as t increases, and it converges to a steady-state of (3.13) as $t \to \infty$, say $\mathbf{w}_*$. Since $\mathbf{w}_* \leqslant_{P_0} \mathbf{w}_s$, it follows by a sweeping argument that $\mathbf{w}_* \leqslant_{P_0} \mathbf{w}_{-s_0}$. This implies that $\mathbf{w}(\hat{\mu})$ is unstable.

It remains to show conclusion (iii). So we assume that $\mathbf{w}(\mu)$ is discontinuous at $\hat{\mu}$. Let $\mu_n \in O^*$ be a sequence increasing to $\hat{\mu}$ and denote $L := D_{\mathbf{w}} A(\hat{\mu}, \mathbf{w}(\hat{\mu}^-))$. Since $\mathbf{w}(\mu_n) \to \mathbf{w}(\hat{\mu}^-)$ and $r(L^{\mu_n}) < 1$, we deduce $r(L) = \lim_{n\to\infty} r(L^{\mu_n}) \leqslant 1$. If $r(L) < 1$, then the asymptotic stability of $\mathbf{w}(\hat{\mu}^-)$ follows from Theorem 3.10. Suppose next $r(L) = 1$. Then we can again apply Theorem 1.3 to conclude that the solutions of $\mathbf{w} = A(\mu, \mathbf{w})$ near $(\hat{\mu}, \mathbf{w}(\hat{\mu}^-))$ form an analytic curve:

$$(\mu, \mathbf{w}) = \big(\mu(s), \mathbf{w}(\hat{\mu}^-) + s\mathbf{w}_0 + \tau(s)\big), \quad s \in [-s_0, s_0], s_0 > 0,$$

with $\mu(0) = \hat{\mu}$, $\mu'(0) = 0$, $\tau(0) = \tau'(0) = 0$, and $\mathbf{w}_0 \gg_{P_0} \mathbf{0}$ satisfies $L\mathbf{w}_0 = \mathbf{w}_0$. Since

$$\mathbf{w}(\mu) \to \mathbf{w}(\hat{\mu}^-) \quad \text{as } \mu \to \hat{\mu} - 0, \tag{3.19}$$

we must have $\mu(s) \not\equiv 0$, and hence there exists $k \geqslant 2$ such that $\mu'(0) = \cdots = \mu^{(k-1)}(0) = 0$ and $\mu^{(k)}(0) \neq 0$. We claim that k is odd and $\mu^{(k)}(0) > 0$. Indeed, if k is even and $\mu^{(k)}(0) > 0$, then $\mu(s) \geqslant \hat{\mu}$ for $|s|$ small. This implies that $\mathbf{w} = A(\mu, \mathbf{w})$ has no solution close to $(\hat{\mu}, \mathbf{w}(\hat{\mu}^-))$ with $\mu < \hat{\mu}$, contradicting (3.19). If k is either odd or even, but $\mu^{(k)}(0) < 0$, then for $s > 0$ small, $\mu(s) < \hat{\mu}$ and $\mathbf{w}_s := \mathbf{w}(\hat{\mu}^-) + s\mathbf{w}_0 + \tau(s) \gg_{P_0} \mathbf{w}(\hat{\mu}^-) \gg_{P_0} \mathbf{w}(\mu(s))$, contradicting the fact that $\mathbf{w}(\mu(s))$ is the maximal solution of (3.1) with $\mu = \mu(s)$. This proves our claim. Now the asymptotic stability of $\mathbf{w}(\hat{\mu}^-)$ can be proved as before, since (3.1) with $\mu = \hat{\mu}$ has a continuum of strict lower solutions $\{\mathbf{w}_s \colon s \in [-s_0, 0)\}$, and a continuum of strict upper solutions $\{\mathbf{w}_s \colon s \in (0, s_0]\}$. □

REMARK 3.14. By standard regularity theory for parabolic equations, a steady-state (u_0, v_0) of (3.13) is asymptotically stable in E_0 implies that it is asymptotically stable in $L^\infty(\Omega)^2$. Under a slightly different definition for "asymptotically stable" solutions, the result of Theorem 3.13 was proved in [Da6] by a combination of fixed point index and local bifurcation argument. See also [Da7] for related results.

We have parallel stability results for the minimal positive solutions (u_μ, v_μ). More precisely, suppose that $\mu^* > \mu_0$ and for $\mu \in (\mu_0, \mu^*)$ denote

$$L_\mu := D_{(u,v)} A(\mu, u_\mu, v_\mu), \qquad O_* := \big\{\mu \in \big(\mu_0, \mu^*\big) \colon r(L_\mu) < 1\big\}.$$

THEOREM 3.15. *The set $(\mu_0, \mu^*) \setminus O_*$ has measure zero in $\mathbb{R}^1$.*

THEOREM 3.16. *$\mathbf{z}(\mu) := (u_\mu, v_\mu)$ is left continuous in (μ_0, μ^*). Moreover, for $\hat{\mu} \in (\mu_0, \mu^*) \setminus O^*$, the following hold.*

(i) *If $\mathbf{z}(\mu)$ is continuous at $\hat{\mu}$, then $\mathbf{z}(\hat{\mu})$ is asymptotically stable.*

(ii) *If $\mathbf{z}(\mu)$ is discontinuous at $\hat{\mu}$, then $\mathbf{z}(\hat{\mu})$ is unstable.*

(iii) *If $\mathbf{z}(\mu)$ is discontinuous at $\hat{\mu}$, then $\mathbf{z}(\hat{\mu}^+) := \lim_{\mu\to\hat{\mu}+0} \mathbf{z}(\mu)$ is an asymptotically stable positive solution of* (3.1).

Theorems 3.15 and 3.16 can be proved by analogous arguments to those used in the proofs of Theorems 3.12 and 3.13. We omit the details.

REMARK 3.17. If (u^{μ_*}, v^{μ_*}) exists, i.e., (3.1) has a maximal positive solution with $\mu = \mu_*$, then by the proof of Theorem 3.13, we easily see that it is unstable. Similarly, if (u_{μ^*}, v_{μ^*}) exists, it is unstable.

3.3. *Stable patterns*

We show that the global bifurcation branch of (3.1) discussed in Section 3.1 can be better described if ε is small. Moreover, we will show that, as $\varepsilon \to 0$, $\mu^* \to \infty$ and the minimal positive solution (u_μ, v_μ) develops a sharp pattern determined by $b(x)$. A key ingredient in our analysis here is the following degenerate model, that is, (3.1) with $\varepsilon = 0$

$$\begin{cases} -\Delta u = \lambda u - b(x)u^2 - cuv, \\ -\Delta v = \mu v - v^2 - duv, \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0. \end{cases} \tag{3.20}$$

We will regard (3.20) as a limiting problem for (3.1) with small ε. In a sense, our strategy here is similar to that of Section 2.2, where the perturbed problem (2.10) was studied by making use of the limiting problem (2.1); here we study the perturbed system (3.1) by its limiting problem (3.20). We will mainly follow [Du4].

We firstly apply a global bifurcation analysis to (3.20). The following a priori estimate is crucial to our analysis; we refer to [Du4], Theorem 2.1, for its proof, which is quite involved and uses some ideas in Section 2.

THEOREM 3.18. *Given real numbers λ and M, there exists $C = C(\lambda, M) > 0$ such that any positive solution (u, v) of* (3.20) *with $\mu \leqslant M$ satisfies*

$$\|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} \leqslant C.$$

Let us observe that (3.20) behaves similarly to (3.1) if $\lambda_1^\Omega < \lambda < \lambda_1^{\Omega_0}$. Indeed, we have a trivial solution branch $\Gamma_0^0 := \{(\mu, 0, 0)\colon \mu \in \mathbb{R}^1\}$, two semitrivial solution branches $\Gamma_1^0 := \{(\mu, u_\lambda, 0)\colon \mu \in \mathbb{R}^1\}$ and $\Gamma_2^0 := \{(\mu, 0, \theta_\mu)\colon \mu > \lambda_1^\Omega\}$, where u_λ is the unique positive solution of (2.1) with $p = 2$. Moreover, we can apply the local and global bifurcation analysis of [BB2] as in Section 3.1 to conclude that there exists a global bifurcation branch of positive solutions of (3.20), denoted by Γ^0, which bifurcates from Γ_1^0 at $(\tilde{\mu}_0, u_\lambda, 0)$ and joins Γ_2^0 at $(\mu^0, 0, \theta_{\mu^0})$, where $\tilde{\mu}_0 = \lambda_1^\Omega(du_\lambda)$ and μ^0 is given by (3.9). We now find that all our discussions in the rest of Section 3.1 and in Section 3.2 can be applied to (3.20) and yield the same results. Therefore we may conclude that when $\lambda_1^\Omega < \lambda < \lambda_1^{\Omega_0}$, (3.20) and (3.1) behave similarly. It is also easily seen that they behave similarly when $\lambda \leqslant \lambda_1^\Omega$.

We will show next that, when

$$\lambda \geqslant \lambda_1^{\Omega_0}, \tag{3.21}$$

essential differences arise between (3.1) and (3.20). *We henceforth assume that* (3.21) *holds*. The first difference is that the semitrivial solution branch Γ_1^0 disappears, though Γ_0^0 and Γ_2^0 are unchanged. We can still apply a local and global bifurcation analysis to (3.20): A local bifurcation analysis along Γ_2^0 shows that a branch of positive solutions Γ^0 bifurcates from Γ_2^0 at $(\mu^0, 0, \theta_{\mu^0}) \in \Gamma_2^0$; by Theorem 3.18 and Rabinowitz's global bifurcation theorem and the strong maximum principle, we find that Γ^0 is unbounded through μ becoming unbounded. Moreover, if (3.20) has a positive solution (u, v), then $\mu = \lambda_1^\Omega(v + du) > \lambda_1^\Omega$. Therefore Γ^0 becomes unbounded through $\mu \to +\infty$. Γ^0 can be further described by making use of the monotonicity property of (3.20) as in Sections 3.1 and 3.2, and we collect these results in the following theorem (see [Du3] and [Du4] for a detailed proof).

THEOREM 3.19. *Suppose* $\lambda \geqslant \lambda_1^{\Omega_0}(0)$. *Then*:

(i) (Existence and nonexistence.) *There exists* $\tilde{\mu}_* \leqslant \mu^0$ *such that* (3.20) *has no positive solution for* $\mu < \tilde{\mu}_*$, *and it has at least one positive solution for* $\mu > \tilde{\mu}_*$.

(ii) (Multiplicity and stability.) *If* $\tilde{\mu}_* < \mu^0$, *then* (3.20) *has at least two positive solutions for* $\mu \in (\tilde{\mu}_*, \mu^0)$, *and at least one positive solution for* $\mu = \tilde{\mu}_*$. *Moreover, at least one positive solution is asymptotically stable for* $\mu \in (\tilde{\mu}_*, \mu^0)$.

(iii) (Continuum.) *All the positive solutions of* (3.20) *stated in* (i) *and* (ii) *above can be chosen from the unbounded positive solution branch* Γ^0 *which joins the semitrivial solution* $(\mu^0, 0, \theta_{\mu^0})$ *and* ∞.

REMARK 3.20. If $d > 0$ is small, then $\tilde{\mu}_* < \mu^0$; see [Du3], Theorem 3.6, for some estimates of $\tilde{\mu}_*$.

We now come back to (3.1). To emphasize the dependence on ε, we denote the global positive solution branch of (3.1) by Γ^ε, instead of Γ used in Section 3.1. Let us observe that the trivial solution branch Γ_0 and the semitrivial solution branch $\Gamma_2 = \{(\mu, 0, \theta_\mu)\}$ are independent of ε, as is μ^0 given by (3.9); but Γ_1 is ε-dependent and we henceforth denote it by $\Gamma_1^\varepsilon = \{(\mu, \phi_\lambda^\varepsilon, 0)\}$. Similarly, we replace μ_* and μ^* by $\mu_*(\varepsilon)$ and $\mu^*(\varepsilon)$, respectively. We also replace μ_0 by $\mu_0(\varepsilon)$, which, we recall, is given by $\mu_0(\varepsilon) := \lambda_1^\Omega(d\phi_\lambda^\varepsilon)$. As $\varepsilon \to 0$, the behavior of ϕ_λ^ε is described by Theorem 2.6. We now consider the behavior of $\mu_0(\varepsilon)$ (see [Du3] and [Du4] for a proof).

PROPOSITION 3.21. *As* $\varepsilon \to 0$, $\mu_0(\varepsilon) = \lambda_1^{\Omega}(d\phi_\lambda^\varepsilon) \to \lambda_1^{\Omega_+}(d\underline{U}_\lambda)$, *which is finite. Here* $\Omega_+ := \Omega \setminus \overline{\Omega}_0$, $\underline{U}_\lambda$ *is the minimal positive solution of* (2.3) *with* $p = 2$, *and* $\lambda_1^{\Omega_+}(d\underline{U}_\lambda)$ *is defined by*

$$\lambda_1^{\Omega_+}(d\underline{U}_\lambda) := \lim_{n\to\infty} \lambda_1^{\Omega_+}\big(\min\{n, d\underline{U}_\lambda\}\big).$$

Next we study the changes in $\mu_*(\varepsilon)$ and $\mu^*(\varepsilon)$ as $\varepsilon \to 0$.

PROPOSITION 3.22. *The functions* $\varepsilon \to \mu^*(\varepsilon)$ *and* $\varepsilon \to \mu_*(\varepsilon)$ *are both nonincreasing. Moreover,* $\lim_{\varepsilon\to 0} \mu^*(\varepsilon) = \infty$ *and* $\lim_{\varepsilon\to 0} \mu_*(\varepsilon) = \hat{\mu}_* \leqslant \tilde{\mu}_*$, *where* $\tilde{\mu}_*$ *is defined in Theorem* 3.19.

PROOF. We first show that $\varepsilon \to \mu^*(\varepsilon)$ is nonincreasing. If $\mu^*(\varepsilon) \equiv \max\{\mu^0, \mu_0(\varepsilon)\}$, then, since $\mu_0(\varepsilon) = \lambda_1^{\Omega}(d\phi_\lambda^\varepsilon)$ is nonincreasing with ε, there is nothing to prove. If $\mu^*(\varepsilon) > \max\{\mu^0, \mu_0(\varepsilon)\}$ for some $\varepsilon = \varepsilon_0 > 0$, then by Theorem 3.6, (3.1) with $\varepsilon = \varepsilon_0$ and $\mu = \mu^*(\varepsilon_0)$ has a positive solution (u_0, v_0). Let $\varepsilon_1 \in (0, \varepsilon_0]$. Then

$$-\Delta u_0 \leqslant \lambda u_0 - \big(b(x) + \varepsilon_1\big)u_0^2 - cu_0v_0.$$

Hence (u_0, v_0) is an upper solution to (3.1) with $\varepsilon = \varepsilon_1$. Since $\mu = \mu^*(\varepsilon_0) > \mu_0(\varepsilon_0)$, if we choose ε_1 close enough to ε_0, then $\mu > \mu_0(\varepsilon_1)$ and hence the problem

$$-\Delta v = \mu v - v^2 - d\phi_\lambda^{\varepsilon_1} v, \quad v|_{\partial\Omega} = 0,$$

has a unique positive solution v_{ε_1}. It is easily checked that $(\phi_\lambda^{\varepsilon_1}, v_{\varepsilon_1})$ is a lower solution to (3.1) with $\varepsilon = \varepsilon_1$. Moreover, it is easily seen that $\phi_\lambda^{\varepsilon_1} \geqslant \phi_\lambda^{\varepsilon_0} \geqslant u_0$ and $v_{\varepsilon_1} \leqslant v_0$. Thus, by standard upper and lower solution argument for competition models, (3.1) with $\varepsilon = \varepsilon_1$ has a positive solution (u, v) satisfying $u_0 \leqslant u \leqslant u_{\varepsilon_1}$ and $v_{\varepsilon_1} \leqslant v \leqslant v_0$. By the definition of $\mu^*(\varepsilon)$, we must have $\mu^*(\varepsilon_1) \geqslant \mu = \mu^*(\varepsilon_0)$. Thus $\varepsilon \to \mu^*(\varepsilon)$ is always nonincreasing.

The fact that $\varepsilon \to \mu_*(\varepsilon)$ is nonincreasing can be proved by a similar argument.

Next we prove that $\mu^*(\varepsilon) \to \infty$ as $\varepsilon \to 0$. We view (3.1) as a perturbation of (3.20) and use a degree argument. Given any $\tilde{\mu} > \max\{\lambda_1^{\Omega_+}(d\underline{U}), \tilde{\mu}_*\}$, by Theorem 3.18, we can find a constant $C > 0$ such that any positive solution (u, v) of (3.20) with $\mu \in [0, \tilde{\mu}]$ satisfies $\|u\|_\infty \leqslant C$. Note also that we always have $v \leqslant \theta_\mu \leqslant \theta_{\tilde{\mu}}$. We now recall the definition of $A(\mu, u, v)$ in Section 3.1. By enlarging ξ, η and M there, and replacing $\hat{\mu}$ there by $\tilde{\mu}$ if $\tilde{\mu} > \hat{\mu}$, we find that the properties of A are retained for all small $\varepsilon \geqslant 0$. To emphasize the dependence on ε, we denote $A(\mu, u, v)$ by $A_\mu^\varepsilon(u, v)$.

To use a fixed point index argument, it is convenient to work in the space $E := C(\overline{\Omega}) \times C(\overline{\Omega})$ with the natural positive cone $P := \{(u, v) \in E\colon\ u \geqslant 0, v \geqslant 0\}$. If we define $\mathcal{A} := \{(u, v) \in E\colon\ 0 \leqslant u < \xi, 0 \leqslant v < \eta\}$, then it is easily checked that $A_\mu^\varepsilon : \mathcal{A} \to P$ and is completely continuous for all small $\varepsilon \geqslant 0$ and $\mu \in [0, \tilde{\mu}]$. Let us now consider the fixed point index, $\mathrm{index}_P(A_\mu^0, \mathcal{A})$. When $\lambda_1^{\Omega} < \mu < \tilde{\mu}_*$, the only nonnegative solutions of (3.20) are $(u, v) = (0, 0)$ and $(u, v) = (0, \theta_\mu)$, both are linearly unstable solutions of (3.20). By Dancer's fixed point index formula [Da2], for such μ,

$$\mathrm{index}_P\big(A_\mu^0, (0, 0)\big) = \mathrm{index}_P\big(A_\mu^0, (0, \theta_\mu)\big) = 0.$$

Therefore,

$$\mathrm{index}_P\big(A^0_\mu, \mathcal{A}\big) = \mathrm{index}_P\big(A^0_\mu, (0,0)\big) + \mathrm{index}_P\big(A^0_\mu, (0,\theta_\mu)\big) = 0.$$

As A^0_μ has no fixed point on $\partial_P\mathcal{A}$ (the relative boundary of $\mathcal{A}$ with respect to P) for any $\mu \in [0, \tilde\mu]$, by the continuity property of the fixed point index (see [Am]), $\mathrm{index}_P(A^0_\mu, \mathcal{A})$ is independent of $\mu \in [0, \tilde\mu]$ and is thus identically zero.

Consider now $\mu \in (\mu^0, \tilde\mu]$. For such μ, the trivial solution $(0,0)$ of (3.20) is linearly unstable and hence has fixed point index 0, but the semitrivial solution $(0,\theta_\mu)$ is linearly stable, and therefore it has fixed point index 1. It follows that we can find small neighborhoods N_0 of $(0,0)$ and N_1 of $(0,\theta_{\tilde\mu})$ such that

$$\begin{aligned}
&\mathrm{index}_P\big(A^0_{\tilde\mu}, \mathcal{A}\setminus(N_0\cup N_1)\big)\\
&\quad= \mathrm{index}_P\big(A^0_{\tilde\mu}, \mathcal{A}\big) - \mathrm{index}_P\big(A^0_{\tilde\mu}, (0,0)\big) - \mathrm{index}_P\big(A^0_{\tilde\mu}, (0,\theta_{\tilde\mu})\big)\\
&\quad= 0 - 0 - 1 = -1.
\end{aligned}$$

It follows from the continuity property of the fixed point index that, for all sufficiently small $\varepsilon > 0$, $\mathrm{index}_P(A^\varepsilon_{\tilde\mu}, \mathcal{A}\setminus(N_0\cup N_1))$ is well defined and equals $\mathrm{index}_P(A^0_{\tilde\mu}, \mathcal{A}\setminus(N_0\cup N_1)) = -1$. Thus $A^\varepsilon_{\tilde\mu}$ has a fixed point (u,v) in $\mathcal{A}\setminus(N_0\cup N_1)$, i.e., (3.20) has a positive solution with $\mu = \tilde\mu$ for all small $\varepsilon > 0$. In particular, $\mu^*(\varepsilon) \geqslant \tilde\mu$. As $\tilde\mu$ is arbitrary, this implies $\mu^*(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

Let us now prove that $\lim_{\varepsilon\to 0}\mu_*(\varepsilon) \leqslant \tilde\mu_*$. Since $\mu_*(\varepsilon)$ is nonincreasing with ε and $\mu_*(\varepsilon) \leqslant \mu^0$, $\hat\mu_* := \lim_{\varepsilon\to 0}\mu_*(\varepsilon)$ exists. If $\hat\mu_* > \tilde\mu_*$, then since $\hat\mu_* \leqslant \mu^0$, we must have $\mu_* < \mu^0$. By Theorem 3.19, (3.20) with $\mu = \tilde\mu_*$ has a positive solution (u_0, v_0). It is easily checked that (u_0, v_0) is a lower solution to (3.1) with $\mu = \tilde\mu_*$ for any $\varepsilon > 0$. Moreover, since $\tilde\mu_* < \mu^0$, $\lambda = \lambda_1^\Omega(c\theta_{\mu^0}) > \lambda_1^\Omega(c\theta_{\tilde\mu_*})$, and thus, the problem

$$-\Delta u = \lambda u - \big(b(x) + \varepsilon\big)u^2 - c\theta_{\tilde\mu_*}u, \quad u|_{\partial\Omega} = 0,$$

has a unique positive solution u^*. Clearly $(u^*, \theta_{\tilde\mu_*})$ is an upper solution of (3.1) with $\mu = \tilde\mu_*$, and $v_0 \leqslant \theta_{\tilde\mu_*}$, $u_0 \geqslant u^*$. Thus, (3.1) with $\mu = \tilde\mu_*$ has a positive solution, and hence $\mu_*(\varepsilon) \leqslant \tilde\mu_*$. But this implies $\hat\mu_* \leqslant \tilde\mu_*$, a contradiction. Therefore, we must have $\hat\mu_* \leqslant \tilde\mu_*$, as required. □

By Propositions 3.21 and 3.22, we find that, for small $\varepsilon > 0$, $\mu_0(\varepsilon) < \mu^*(\varepsilon)$, and for fixed $\mu > \lambda_1^{\Omega_+}(d\underline{U}_\lambda)$, $\mu \in (\mu_0(\varepsilon), \mu^*(\varepsilon))$ holds for all small $\varepsilon > 0$. Therefore, (3.1) has a minimal positive solution $(u^\varepsilon_\mu, v^\varepsilon_\mu)$. We are interested in the profile of $(u^\varepsilon_\mu, v^\varepsilon_\mu)$ as $\varepsilon \to 0$. To simplify notation, we will denote this minimal solution by $(u^\varepsilon, v^\varepsilon)$ when its dependence on μ is not emphasized. For technical reasons, in the following theorems we require $\mu > \lambda_1^{\Omega_+}(d\overline{U}_\lambda)$, where $\overline{U}_\lambda$ is the maximal positive solution of (2.3) with $p = 2$. Under some mild conditions on $b(x)$, $\overline{U}_\lambda = \underline{U}_\lambda$, see Remark 2.3.

THEOREM 3.23. *Suppose that* $\lambda > \lambda_1^{\Omega_0}$ *and* $\mu > \lambda_1^{\Omega_+}(d\overline{U}_\lambda)$. *Then the following conclusions hold*:

(i) $(u^\varepsilon, v^\varepsilon) \to (\infty, 0)$ *uniformly on* $\overline{\Omega}_0$,

(ii) $\overline{U}_\mu \leqslant \underline{\lim}_{\varepsilon\to 0} u^\varepsilon, \overline{\lim}_{\varepsilon\to 0} u^\varepsilon \leqslant \underline{U}_\mu, \underline{V}_\mu \leqslant \underline{\lim}_{\varepsilon\to 0} v^\varepsilon, \overline{\lim}_{\varepsilon\to 0} v^\varepsilon \leqslant \overline{V}_\mu$, *where the limits are uniform on any compact subset of* $\overline{\Omega} \setminus \overline{\Omega}_0$, $(\underline{U}_\mu, \underline{V}_\mu)$ *and* $(\overline{U}_\mu, \overline{V}_\mu)$ *are respectively the minimal and maximal positive solutions of the boundary blow-up problem*

$$\begin{cases} -\Delta u = \lambda u - b(x)u^2 - cuv, & x \in \Omega_+, \\ -\Delta v = \mu v - v^2 - duv, & x \in \Omega_+, \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \quad u|_{\partial\Omega_0} = \infty, & v|_{\partial\Omega_0} = 0. \end{cases} \tag{3.22}$$

Moreover, for any positive sequence $\{\varepsilon_n\}$ *that converges to* 0, $\{(u^{\varepsilon_n}, v^{\varepsilon_n})\}$ *has a subsequence that converges, uniformly on any compact subset of* $\overline{\Omega} \setminus \overline{\Omega}_0$, *to a positive solution of* (3.22).

We omit the rather involved proof of Theorem 3.23 here, and refer the interested reader to [Du4]. Let us note that for small $\varepsilon > 0$, the above result shows that $(u^\varepsilon, v^\varepsilon)$ exhibits a sharp pattern over the underlying domain Ω: u^ε is large over Ω_0 and is positive and of order 1 over Ω_+; v^ε is small over Ω_0, and it is positive and of order 1 over Ω_+. Our next result demonstrates that an intuitively clearer pattern is given by a rescaled version of $(u^\varepsilon, v^\varepsilon)$, namely $(\tilde{u}^\varepsilon, v^\varepsilon) := (\varepsilon u^\varepsilon, v^\varepsilon)$. It is easily checked that $(\tilde{u}^\varepsilon, v^\varepsilon)$ is a minimal positive solution of the following competition model

$$\begin{cases} -\Delta u = \lambda u - \left[\varepsilon^{-1} b(x) + 1\right]u^2 - cuv, \\ -\Delta v = \mu v - v^2 - \varepsilon^{-1} duv, \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0. \end{cases} \tag{3.23}$$

Clearly $(\tilde{u}^\varepsilon, v^\varepsilon)$ has the same stability properties as $(u^\varepsilon, v^\varepsilon)$ in (3.1) when regarded as a steady-state of the corresponding parabolic problem of (3.23).

THEOREM 3.24. *Suppose that* $\lambda > \lambda_1^{\Omega_0}$ *and* $\mu > \lambda_1^{\Omega_+}(d\overline{U}_\lambda)$. *Then the following are true*:

(i) $(\tilde{u}^\varepsilon, v^\varepsilon) \to (\tilde{\theta}_\lambda, 0)$ *uniformly on* $\overline{\Omega}_0$, *where*, $\tilde{\theta}_\lambda$ *is the unique positive solution of*

$$-\Delta u = \lambda u - u^2 \quad \text{in } \Omega_0, \quad u|_{\partial\Omega_0} = 0.$$

(ii) *For any positive sequence* $\varepsilon_n \to 0$, $\{(\tilde{u}^{\varepsilon_n}, v^{\varepsilon_n})\}$ *has a subsequence that converges to* $(0, V)$ *uniformly on* $\overline{\Omega} \setminus \overline{\Omega}_0$, *where* $V \in C(\overline{\Omega} \setminus \overline{\Omega}_0)$ *and is the second component of some positive solution* (U, V) *of* (3.22).

Again we refer the proof of Theorem 3.24 to [Du4]. The above two theorems give us a rather detailed description of the spatial pattern of the minimal positive solution $(u_\mu^\varepsilon, v_\mu^\varepsilon)$ with small $\varepsilon > 0$. By Theorem 3.16, this solution is asymptotically stable exactly when its dependence on μ is continuous at μ, which is the case for almost all μ. If its dependence on μ is discontinuous at some $\hat{\mu}$, then

$$\left(\hat{u}^\varepsilon, \hat{v}^\varepsilon\right) := \lim_{\mu\to\hat{\mu}+0} \left(u_\mu^\varepsilon, v_\mu^\varepsilon\right) \tag{3.24}$$

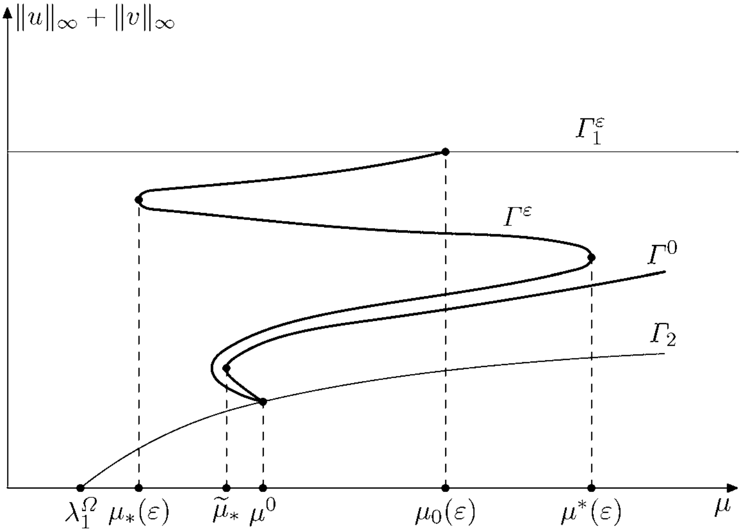

Fig. 1. Bifurcation diagram for (3.1) and (3.20).

is a positive solution of (3.1) with $\mu = \hat{\mu}$, and it is asymptotically stable. Equation (3.24) implies that, for fixed small $\varepsilon > 0$, $(\hat{u}^\varepsilon, \hat{v}^\varepsilon)$ has the spatial pattern similar to $(u_\mu^\varepsilon, v_\mu^\varepsilon)$ with $\mu > \hat{\mu}$ but close to $\hat{\mu}$.

The bifurcation diagram (Figure 1) describes a possible scenario of the global bifurcation branches for (3.1) and (3.20) with $\lambda \geqslant \lambda_1^{\Omega_0}$.

3.4. *Remarks*

1. If Ω_0 is not connected but consists of finitely many components, then our results in Sections 3.1 and 3.2 are not affected, and the results in Section 3.3 can also be extended to this case, see [Du4] for details.

2. The strategy employed here for the competition model (3.1) has been used to study certain predator–prey models; see [DD2] and [DHs] for details, and [Du5] for a survey. Since the predator–prey models do not have any kind of monotonicity property, the techniques there are very different from here.

3. It is an interesting problem to see what new features arise if we have at least two nonconstant coefficients in (3.1) that are close to zero in certain parts of the domain.

4. Our method works as well if (3.1) has the following more general form

$$\begin{cases} -\operatorname{div}\big(d_1(x)u\big) = \lambda a_1(x)u - \big[b(x)+\varepsilon\big]u^2 - c(x)uv, \\ -\operatorname{div}\big(d_2(x)v\big) = \mu a_2(x)v - e(x)v^2 - d(x)uv, \\ Bu|_{\partial\Omega} = Bv|_{\partial\Omega} = 0, \end{cases} \tag{3.25}$$

where the coefficient functions are positive except $b(x)$ which is allowed to vanish on part of the domain, and the boundary operator B is either Dirichlet, or Neumann or Robin type.

5. In a series of recent papers, Hutson–Lou–Mischaikow–Polacik studied various perturbations of the special competition model

$$\begin{cases} u_t - \mu\Delta u = \alpha(x)u - u^2 - uv, & (x,t) \in \Omega \times (0,\infty), \\ v_t - \mu\Delta v = \alpha(x)v - v^2 - uv, & (x,t) \in \Omega \times (0,\infty), \\ u_\nu = v_\nu = 0, & (x,t) \in \partial\Omega \times (0,\infty), \end{cases}$$

and obtained interesting results revealing some fundamental effects of heterogeneous environment on the competition model. We refer to [HLM,HLMP] and [HMP] for details. Further related results can be found in [AC,CC,CCH,Lop1,LS].

6. Several general approaches have been developed in the past two decades to study (3.25). For example, the method of monotone iterations and order preserving operators was developed and used by Pao [Pao], Koman and Leung [KL] and many others; the method of global bifurcation was introduced by Blat and Brown [BB1,BB2]; and the fixed point index approach was developed by Dancer [Da3,Da4]. These methods were applied to (3.25) with constant coefficients, but they work as well with nonconstant coefficients to yield similar results. Therefore it is difficult to rely on these methods alone to reveal the effects of heterogeneous environment on (3.25).

7. The spatial behavior of positive solutions of the two species competition model has received extensive studies even in the constant coefficient case, i.e., when the spatial environment is homogeneous. In [KW], it was shown that, if the spatial domain Ω is convex, then problem (3.25) with constant coefficients and Neumann boundary conditions has no stable positive steady-state that depends on x, i.e., all its stable positive steady-states are constant solutions. On the other hand, in [MM], spatially variable stable positive steady-state solutions were constructed for certain nonconvex Ω (see also [KY]). In [DD1], it was proved that in the strong competition case, positive steady-states of (3.25) with constant coefficients tend to segregate over Ω, i.e., uv is close to 0 with u close to $\max\{w,0\}$ and v close to $\max\{-w,0\}$, where w is a sign-changing solution of a scalar elliptic equation deduced from this competition system. In [LN2,LN1], the competition model with self-diffusion and cross-diffusion was closely examined and the existence and asymptotic profile of space dependent positive steady-states were obtained when certain parameters are large; see also [Mi] and [MK] for earlier result.

8. It is more realistic to assume that the coefficients in the competition model are also dependent on time, for example, they are periodic in time as well as a function of the space variable x. The general case was systematically discussed in [He]. It would be interesting to see whether the results of Sections 3.1–3.3 here can be extended to this case.

4. Bifurcation and exact multiplicity: The perturbed Gelfand equation

Exact multiplicity of solutions to nonlinear equations is in general very difficult to obtain. In most nonlinear elliptic problems, the number of solutions is not only affected by the nonlinearity in the equation, it also depends on the geometry of the underlying domain (see [Da5]). When the underlying domain is a ball, then for many classes of nonlinearities, it is possible to use a bifurcation approach to find the exact number of positive solutions (see, e.g., [OS1,OS2]). A key point is that, by the well-known result of Gidas, Ni and Nirenberg [GNN], under homogeneous Dirichlet boundary conditions, any positive solution on the ball is radially symmetric; this reduces the PDE problem to an ODE one and makes the exact multiplicity problem reachable. We will demonstrate this approach through the perturbed Gelfand equation, where a perturbation argument is also needed. We mainly follow [DLo2] and [Du2].

The perturbed Gelfand equation arises in the mathematical modelling of thermal reaction processes (see [BE], Section 1.3), which is of the following form:

$$-\Delta u = \lambda \mathrm{e}^{u/(1+\varepsilon u)} \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0, \tag{4.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, λ is a positive constant known as the Frank–Kamenetskii parameter, $\varepsilon > 0$ is a small parameter representing the reciprocal activation energy and u stands for the dimensionless temperature.

If $\varepsilon = 0$, problem (4.1) reduces to the well-known Gelfand equation

$$-\Delta u = \lambda \mathrm{e}^{u} \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0. \tag{4.2}$$

If we let $v = \varepsilon^2 u$ and $\mu = \lambda\varepsilon^2 \mathrm{e}^{1/\varepsilon}$, then (4.1) becomes

$$-\Delta v = \mu \mathrm{e}^{-1/(\varepsilon+v)} \quad \text{in } \Omega, \quad v|_{\partial\Omega} = 0, \tag{4.3}$$

which has the limiting equation

$$-\Delta v = \mu \mathrm{e}^{-1/v} \quad \text{in } \Omega, \quad v|_{\partial\Omega} = 0. \tag{4.4}$$

We will make use of both (4.2) and (4.3) to obtain a good understanding of (4.1) for small $\varepsilon > 0$.

If $\varepsilon \geqslant 1/4$, then it is easily checked that the right-hand side of (4.1) is an increasing concave function of u, and it follows easily that (4.1) has a unique positive solution for every $\lambda > 0$; see [BIS] and [CS]. For a general $\varepsilon > 0$, it is known that (4.1) has a unique positive solution for $0 < \lambda \ll 1$ and $\lambda \gg 1$; and for $0 < \varepsilon \ll 1$, it is known that there exists a nonempty bounded open interval $\Lambda \subset (0, \infty)$ such that for $\lambda \in \Lambda$, (4.1) has at least three distinct positive solutions $u_1(x) \leqslant u_2(x) \leqslant u_3(x)$; see [BIS,CS,Sh,Wie1] and [Wie2]. These results suggest that when $\varepsilon > 0$ is small, the global bifurcation branch $\{(\lambda, u)\}$ of (4.1) is roughly S-shaped. If the space dimension is one, then it is proved by the method of quadratures that this bifurcation branch is a continuous curve and is exactly S-shaped when $\varepsilon > 0$ is sufficiently small [HM], and it was further shown that this is true when $0 < \varepsilon < 1/4.4967$ in [W1], and when $0 < \varepsilon < 1/4.35$ in [KLi].

It had been conjectured that the global bifurcation curve of (4.1) is exactly S-shaped when the space dimension is two and Ω is a ball. (This kind of result is useful in understanding the profiles of the solutions to the full exothermic reaction–diffusion system; see [MS] for details.) Parter [Pa] considered this case for the equivalent problem (4.3), and gave estimates of four positive values $\underline{\mu}_1(\varepsilon) < \bar{\mu}_1(\varepsilon) < \underline{\mu}_2(\varepsilon) < \bar{\mu}_2(\varepsilon)$ such that (4.3) has a unique positive solution if $\mu \in (0, \underline{\mu}_1(\varepsilon)] \cup [\bar{\mu}_2(\varepsilon), +\infty)$ and it has at least three positive solutions if $\mu \in [\bar{\mu}_1(\varepsilon), \underline{\mu}_2(\varepsilon)]$. By using (4.2) as a limiting problem for (4.1), Dancer [Da1] proved, among other things, that for any small positive $\lambda_0 > 0$, one can find an $\varepsilon_0 > 0$ small such that if $\varepsilon \in (0, \varepsilon_0)$, then there is a constant $\lambda_2(\varepsilon) > 0$ such that (4.1) has exactly three positive solutions if $\lambda \in (\lambda_0, \lambda_2(\varepsilon))$; it has exactly two positive solutions if $\lambda = \lambda_2(\varepsilon)$; and there is a unique positive solution if $\lambda > \lambda_2(\varepsilon)$. This leaves the conjecture unsolved only for the small λ-range, $0 < \lambda < \lambda_0$. This gap was finally filled in [DLo2], where apart from (4.2), the limiting equation (4.4) was also used; we will give a proof of this result further based on (4.4) only.

As will become clear later, when the space dimension N is greater than two, the global solution curve of (4.1) needs not be S-shaped for small $\varepsilon > 0$; it is more complicated when $3 \leqslant N \leqslant 9$, and (4.2) will play an important role in understanding this.

4.1. *The limiting equations*

When Ω is the unit ball, the number of positive solutions to (4.2) was completely described in the well-known paper of Joseph and Lundgren [JL] based on a phase plane method. Their results are summarized in the following proposition.

PROPOSITION 4.1. *Suppose that Ω is the unit ball in $\mathbb{R}^N$. Then there exists a finite value λ_* depending on N, such that* (4.2) *has*

(i) *no positive solution when* $\lambda > \lambda_*$ $(1 \leqslant N \leqslant 9)$,
(ii) *exactly one positive solution when* $\lambda = \lambda_*$ $(1 \leqslant N \leqslant 9)$,
(iii) *exactly two positive solutions when* $0 < \lambda < \lambda_*$ $(N = 1, 2)$,
(iv) *an infinite number of positive solutions when* $\lambda = 2(N-2)$ $(3 \leqslant N \leqslant 9)$,
(v) *a finite but large number of solutions when* $|\lambda - 2(N-2)| \neq 0$ *is small* $(3 \leqslant N \leqslant 9)$,
(vi) *a unique positive solution when* $0 < \lambda < 2(N-2)$, *and no positive solution for* $\lambda \geqslant 2(N-2)$ $(N \geqslant 10)$.

Moreover, the solution set $\{(\lambda, u)\}$ is a smooth curve which can be illustrated in Figure 2.

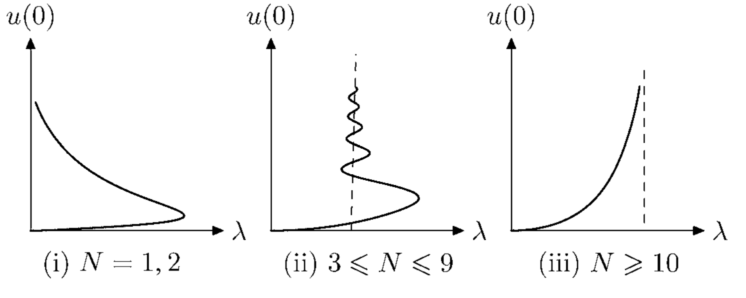

Fig. 2. Bifurcation diagrams for the Gelfand equation (4.2) over the unit ball.

We now consider the other limiting equation (4.4). We will use a bifurcation argument to show that if Ω is a ball of any dimension, the positive solution curve $\{(\mu, v)\}$ is exactly "$\subset$"-shaped. This is in sharp contrast to the Gelfand equation (4.2), whose positive solution set changes structure as the dimension of the underlying domain changes. For definiteness, we assume that Ω is the unit ball $B := \{x \in \mathbb{R}^N\colon |x| < 1\}$. For convenience of notation, we also denote

$$f(v) = f_0(v) = \mathrm{e}^{-1/v}, \qquad f_\varepsilon(v) = \mathrm{e}^{1/(\varepsilon+v)}. \tag{4.5}$$

The following lemma, with a difficult and technical proof, is crucial.

LEMMA 4.2. *Suppose $\Omega = B$. If u is a degenerate positive solution of* (4.4), *that is, the linearized problem*

$$-\Delta\phi = \mu f'(u)\phi, \quad \phi|_{\partial B} = 0,$$

has a nontrivial solution ϕ, then ϕ does not change sign in B.

PROOF. By [GNN], u is radially symmetric: $u(x) = u(r), r = |x|$; moreover, $u'(r) < 0$ on (0,1]. By Proposition 3.3 of [LN], ϕ is also radially symmetric, $\phi(x) = \phi(r)$. Hence

$$\phi'' + \frac{N-1}{r}\phi' + \mu f'(u)\phi = 0 \quad \text{in } [0,1], \quad \phi'(0) = 0, \phi(1) = 0.$$

By the Harnack inequality (or a well-known uniqueness result for the above singular second-order ordinary differential equation), $\phi(0) \neq 0$. We may assume $\phi(0) > 0$.

Direct calculations give

$$f'(u) = u^{-2}\mathrm{e}^{-1/u} > 0 \quad \forall u > 0,$$
$$f''(u) = u^{-4}\mathrm{e}^{-1/u}(1-2u).$$

Hence

$$f''(u) > 0 \quad \text{for } u \in (0, 1/2), \qquad f''(u) < 0 \quad \text{for } u \in (1/2, \infty).$$

One easily sees

$$K(u) = \frac{uf'(u)}{f(u)} = \frac{1}{u}$$

is a decreasing function of u on $(0, \infty)$. We divide our following discussion into two cases:

(i) $u(0) \leqslant 1$ and (ii) $u(0) > 1$.

Consider case (i) *first.* Let

$$v(r) = ru_r(r) + \xi u(r),$$

where ξ is a positive constant to be specified later. Then

$$-\Delta v - \mu f'(u)v = \mu\big[2f(u) - \xi\big(f'(u)u - f(u)\big)\big] \\ = \mu f(u)\big[2 - \xi\big(K(u) - 1\big)\big]. \tag{4.6}$$

Define

$$h(r) = -\frac{ru'(r)}{u(r)}, \quad r \in [0, 1).$$

Clearly $h(0) = 0$ and $h(1) = +\infty$. We show in the following that $h'(r) > 0$ in $(0, 1)$. Indeed,

$$h'(r) = \frac{ru_r^2 + (N-2)u_r u + \mu r f(u)u}{u^2} = \frac{2H(r) + \mu r Q(u(r))}{u^2}, \tag{4.7}$$

where

$$H(r) = \frac{ru_r^2 + (N-2)u_r u}{2} + \mu r F\big(u(r)\big),$$
$$F(u) = \int_0^u f(s)\,\mathrm{d}s, \qquad Q(u) = uf(u) - 2F(u).$$

Here and in what follows, u_r is sometimes used for u' to avoid notation like u'^2.

If $N = 1, 2$, then it follows from the first equality in (4.7) that $h'(r) > 0$ in $(0, 1)$. Therefore, we need only consider $N > 2$. A simple calculation gives

$$\big[r^{N-1}H(r)\big]' = \mu r^{N-1} G\big(u(r)\big) \quad \text{with } G(u) = NF(u) - \frac{N-2}{2} f(u)u.$$

Clearly, $G(0) = 0$ and

$$G'(u) = \frac{N+2}{2} f(u) - \frac{N-2}{2} f'(u)u \\ = u^{-1}\mathrm{e}^{-1/u}\frac{(N+2)u - (N-2)}{2}.$$

It follows that

$$G'(u) < 0 \quad \text{on } [0, \gamma), \qquad G'(u) > 0 \quad \text{on } (\gamma, \infty),$$

where $\gamma = (N-2)/(N+2)$. Therefore, we have either $G(u) < 0$ on $(0, \infty)$ or $G(u) < 0$ on $(0, \gamma_0)$ and $G(u) > 0$ on (γ_0, ∞) for some $\gamma_0 > \gamma$. We show that actually only the latter alternative can occur. Indeed, if $G(u(r)) \leqslant 0$ for all $r \in [0, 1]$, then

$$0 < \frac{u_r^2(1)}{2} = H(1) = \int_0^1 \mu r^{N-1} G\big(u(r)\big)\,\mathrm{d}r \leqslant 0.$$

This contradiction shows γ_0 exists, and moreover,

$$u(0) > \gamma_0 \tag{4.8}$$

whenever $N > 2$ and u is a positive solution of (4.4) with $\Omega = B$.

Let t be uniquely determined by $u(t) = \gamma_0$. We have

$$\big[r^{N-1}H(r)\big]' = \mu r^{N-1} G\big(u(r)\big) > 0 \quad \forall r \in (0, t),$$

which implies $H(r) > 0$ for $r \in (0, t]$. Moreover, for $r \in (t, 1]$, $G(u(r)) < 0$ and therefore

$$\begin{aligned} r^{N-1}H(r) &= H(1) - \int_r^1 \mu r^{N-1} G\big(u(r)\big)\,\mathrm{d}r \\ &\geqslant H(1) = \frac{u_r^2(1)}{2} > 0. \end{aligned}$$

Thus we always have $H(r) > 0$ on $(0, 1]$.

Since $Q(0) = 0$ and

$$Q'(u) = uf'(u) - f(u) = f(u)\big[K(u) - 1\big] \geqslant 0 \quad \forall u \in [0, 1),$$

we have $Q(u) \geqslant 0$ on $[0, 1)$ and hence, by (4.7), $h'(r) > 0$ on $(0, 1)$, as required.

Denote

$$\mu(r) = \frac{2}{K(u(r)) - 1}.$$

Then $\mu(r)$ is strictly decreasing for $r \in (0, 1]$, and by (4.6),

$$-\Delta v - \mu f'(u)v = g(r), \quad g(r) = \mu f(u)\big[K(u) - 1\big]\big[\mu(r) - \xi\big].$$

With these preparations, we are now ready to show that ϕ does not change sign in B in case (i). We argue indirectly. Suppose $\phi(r)$ has a zero in $(0, 1)$. Then we can find $0 < t_1 \leqslant t_2 < 1$ such that

$$\begin{aligned} &\phi(t_1) = 0, \qquad \phi(r) > 0 \quad \forall r \in [0, t_1), \\ &\phi(t_2) = 0, \qquad \phi(r) \neq 0 \quad \forall r \in (t_2, 1). \end{aligned}$$

Now we choose $\xi = h(t_1)$ in $v = ru_r + \xi u$, and have two subcases to consider

(a) $\mu(t_1) \geqslant \xi$ and (b) $\mu(t_1) < \xi$.

We have

$$v(r) = ru_r + h(t_1)u = u\big[h(t_1) - h(r)\big] > v(t_1) = 0 \quad \forall r \in [0, t_1),$$
$$v(r) < 0 \quad \forall r \in (t_1, 1). \tag{4.9}$$

In subcase (a), we easily see $g(r) > 0$ on $(0, t_1)$, and hence, using $v(t_1) = 0$, we arrive at the following contradiction

$$0 < \int_0^{t_1} g(r)\phi(r)r^{N-1}\,\mathrm{d}r = \int_{B_{t_1}} \big[-\Delta v - \mu f'(u)v\big]\phi = \int_{\partial B_{t_1}} v\phi_r = 0, \tag{4.10}$$

where we use $B_r = \{x \in \mathbb{R}^N\colon\ |x| \leqslant r\}$.

In subcase (b), we may assume $\phi(r) > 0$ on $(t_2, 1)$ for otherwise we can replace ϕ by $-\phi$. Moreover, one easily sees $g(r) < 0$ on $[t_1, 1]$. Then by (4.9) and $\phi'(t_2) > 0 > \phi'(1)$, we also arrive at a contradiction

$$\begin{aligned} 0 &> \int_{t_2}^{1} g(r)\phi(r)r^{N-1}\,\mathrm{d}r \\ &= \int_{B\setminus B_{t_2}} \big[-\Delta v - \mu f'(u)v\big]\phi \\ &= \int_{\partial B_{t_2}} -v\phi_r + \int_{\partial B} v\phi_r > 0. \end{aligned} \tag{4.11}$$

This proves the lemma for case (i).

Next we consider case (ii) *where* $u(0) > 1$. We can find $0 < r_1 < r_2 < 1$ uniquely determined by

$$u(r_1) = 1, \qquad u(r_2) = 1/4.$$

We first show $\phi(r) \neq 0$ on $(0, r_1]$. To this end, we choose $w(r) = u(r) - 1/4$ as a test function. Clearly

$$-\Delta w - \mu f'(u)w = \mu q(u), \quad q(u) = f(u) - f'(u)(u - 1/4).$$

We have

$$q'(u) = (1/4 - u)f''(u),$$

which is positive on $(0, 1/4)$, negative on $(1/4, 1/2)$ and positive on $(1/2, \infty)$. Since $q(0) = 1/4 f'(0) = 0$, it follows

$$q(u) \geqslant q(1/2) = f(1/2) - 1/4 f'(1/2) = 0 \quad \forall u > 0.$$

Hence

$$-\Delta w - \mu f'(u) w = \mu q(u) \geqslant 0 \quad \text{on } B.$$

If $\phi(r)$ has a zero in $(0, r_1]$, then we can find $t \in (0, r_1]$ such that $\phi(r) > 0$ on $[0, t)$ and $\phi(t) = 0$. Using $w(t) = u(t) - 1/4 > u(r_2) - 1/4 = 0$, we deduce

$$0 \leqslant \int_0^t \mu q\big(u(r)\big)\phi(r) r^{N-1}\,\mathrm{d}r = \int_{B_t} \big[-\Delta w - \mu f'(u) w\big]\phi = \int_{\partial B_t} w \phi_r < 0.$$

This contradiction finishes our proof for $\phi(r) \neq 0$ on $[0, r_1]$.

Next we suppose $\phi(r)$ changes sign in $(0, 1)$ and deduce a contradiction. Since $\phi(r) \neq 0$ in $[0, r_1]$, we can find $r_1 < t_1 \leqslant t_2 < 1$ such that

$$\phi(t_1) = 0, \qquad \phi(r) > 0 \quad \forall r \in [0, t_1),$$
$$\phi(t_2) = 0, \qquad \phi(r) \neq 0 \quad \forall r \in (t_2, 1).$$

We now define $v(r)$, $h(r)$, $\mu(r)$ and $g(r)$ as in case (i), and choose $\xi = h(t_1)$ in the definition of $v(r)$. Since $u(r) \leqslant 1$ on $[r_1, 1]$, our arguments in case (i) give

$$h'(r) > 0 \quad \text{on } (r_1, 1).$$

Hence

$$v(r) = u\big[h(t_1) - h(r)\big] > v(t_1) = 0 \quad \forall r \in [r_1, t_1), \qquad v(r) < 0 \quad \forall r \in (t_1, 1].$$

If $\mu(t_1) \geqslant \xi$, then since $\mu(r)$ is strictly decreasing on $(0, 1)$ and $K(u(r)) < 1$ on $[0, r_1)$, $K(u(r)) > 1$ on $(r_1, 1]$, we have

$$g(r) = \mu f(u)\big[K(u) - 1\big]\big[\mu(r) - \xi\big] > 0 \quad \forall r \in (r_1, t_1)$$

and

$$g(r) = \mu f(u)\big[2 - \xi\big(K(u) - 1\big)\big] > 0 \quad \forall r \in [0, r_1].$$

Thus $g(r) > 0$ on $[0, t_1)$. Now we can deduce the same contradiction (4.10) as in case (i).

If $\mu(t_1) < \xi$, then

$$g(r) = \mu f(u)\big[K(u) - 1\big]\big[\mu(r) - \xi\big] < 0 \quad \forall r \in (t_1, 1],$$

and we arrive at the contradiction (4.11) as in case (i). This finishes the proof of Lemma 4.2. □

Using Lemma 4.2 and the turning point theorem of Crandall and Rabinowitz [CR1,CR2], Theorem 1.3, we can prove the following result.

LEMMA 4.3. *Suppose that u_0 is a degenerate positive solution of* (4.4) *with $\Omega = B$ and $\mu = \mu_0$. Then all the positive solutions (μ, u) of* (4.4) *that are near (μ_0, u_0) in $\mathbb{R}^1 \times C(\overline{B})$ lie on a smooth curve represented by*

$$(\mu, u) = \big(\mu_0 + \tau(s), u_0 + s\phi_0 + z(s)\big) \quad \textit{with } |s| \textit{ small},$$

where $z(0) = z'(0) = 0$, $\tau(0) = \tau'(0) = 0$ and ϕ_0 is a positive eigenfunction given in Lemma 4.2. *Moreover, $\tau''(0) > 0$.*

PROOF. Set $X = C_0^{2,\alpha}(\overline{B})$, $Y = C^\alpha(\overline{B})$ and $F(\mu, u) = \Delta u + \mu f(u)$. It is easy to see that F is a smooth mapping from $\mathbb{R}^1 \times X$ to Y. The partial derivative F_u at (μ_0, u_0) is given by $F_u(\mu_0, u_0)\phi = \Delta\phi + \mu_0 f'(u_0)\phi$. By Lemma 4.2 we know that $N(F_u(\mu_0, u_0))$ is of one dimension: in fact, $N(F_u(\mu_0, u_0)) = \operatorname{span}\{\phi_0\}$. Moreover, $\operatorname{codim} F_u(\mu_0, u_0) = 1$ by the Fredholm alternative. Also

$$F_\mu(\mu_0, u_0) = f(u_0) \notin R\big(F_u(\mu_0, u_0)\big)$$

since $\int_B f(u_0)\phi_0 > 0$. Therefore we can use Theorem 1.3 to conclude the following:

Near the degenerate solution (μ_0, u_0) in $\mathbb{R}^1 \times X$, the solutions of (4.4) *form a smooth curve*

$$\big(\mu(s), u(s)\big) = \big(\mu_0 + \tau(s), u_0 + s\phi_0 + z(s)\big), \tag{4.12}$$

where $s \to (\tau(s), z(s)) \in \mathbb{R}^1 \times Z$ is a smooth function near $s = 0$ with $\tau(0) = \tau'(0) = 0$, $z(0) = z'(0) = 0$, where Z is a complement of $\operatorname{span}\{\phi_0\}$ *in X.*

By a standard elliptic regularity consideration, we know that any solution of (4.4) near (μ_0, u_0) in $\mathbb{R}^1 \times C(\overline{B})$ is also close to (μ_0, u_0) in $\mathbb{R}^1 \times X$. Therefore all the positive solutions of (4.4) near (μ_0, u_0) in $\mathbb{R}^1 \times C(\overline{B})$ are contained in the smooth curve (4.12).

We now substitute the expression (4.12) for the solutions into equation (4.4), differentiate the equation with respect to s twice at $s = 0$, multiply the resulting identity by ϕ and integrate it over B to obtain

$$\tau''(0) = -\mu_0 \frac{\int_0^1 f''(u_0)\phi_0^3 r^{N-1}\,\mathrm{d}r}{\int_0^1 f(u_0)\phi_0 r^{N-1}\,\mathrm{d}r}. \tag{4.13}$$

It remains to show that $\tau''(0) > 0$. Let use define

$$\xi(r) = \mu_0 r^{N-1}\big[f(u_0)\phi_0' - f'(u_0)\phi_0 u_0'\big].$$

One easily checks that

$$\xi'(r) = -\mu_0 r^{N-1} f''(u_0)\phi_0 \big(u_0'\big)^2, \quad \xi(0) = \xi(1) = 0. \tag{4.14}$$

It follows that

$$\int_0^1 f''(u_0)\phi_0 \big(u_0'\big)^2 r^{N-1}\,\mathrm{d}r = 0.$$

We claim that $f''(u_0(r))$ changes sign exactly once in $(0,1)$. Otherwise we necessarily have $u_0(0) \leqslant 1/2$ and $f''(u_0(r)) > 0$ in $(0,1)$. By the first part of (4.14), this implies that $\xi'(r) < 0$ in $(0,1)$, and hence $\xi(0) > \xi(1)$, contradicting the second part of (4.14). This proves our claim. Let us assume that $f''(u_0(r))$ changes sign exactly once in $(0,1)$ at $r = \bar{r}$.

Next we show that $-u_0'$ and ϕ_0 intersect exactly once in $(0,1)$. Since $-u_0'(0) = 0$, $-u_0'(1) > 0$, $\phi_0(0) > 0$ and $\phi_0(1) = 0$, $-u_0'$ and ϕ_0 intersect at least once in $(0,1)$. If they intersect at least twice, we may assume that there exist $0 < r_1 < r_2 < 1$ such that $(\phi_0 + u_0')(r_1) = (\phi_0 + u_0')(r_2) = 0$, and $(\phi_0 + u_0')(r) < 0$ for $r \in (r_1, r_2)$. This in particular implies that $(\phi_0 + u_0')'(r_1) \leqslant 0$ and $(\phi_0 + u_0')'(r_2) \geqslant 0$. By the equations of ϕ_0 and u_0', it is easy to check that the following identity holds:

$$\big[r^{N-1}u_0'\phi_0' - r^{N-1}\phi_0 u_0''\big]' = -(N-1)r^{N-2}u_0'\phi_0. \tag{4.15}$$

Integrating (4.15) from r_1 to r_2, and using $(\phi_0 + u_0')(r_1) = (\phi_0 + u_0')(r_2) = 0$, we obtain

$$\begin{aligned} &r_2^{N-1}u_0'(r_2)\big(\phi_0 + u_0'\big)'(r_2) - r_1^{N-1}u_0'(r_1)\big(\phi_0 + u_0'\big)'(r_1) \\ &\quad = -\int_{r_1}^{r_2} (N-1)r^{N-2}u_0'\phi_0\,\mathrm{d}r. \end{aligned} \tag{4.16}$$

However, the right-hand side of (4.16) is positive since $u_0' < 0$ and $\phi_0 > 0$, while the left-hand side of (4.16) is nonpositive due to the facts summarized ahead of (4.15). This contradiction shows that $-u_0'$ and ϕ_0 intersect exactly once in (0, 1). By replacing ϕ_0 by $\eta_0\phi_0$ for some suitable positive η_0 if necessary, we may assume that $-u_0'$ and ϕ_0 intersect exactly at $r = \bar{r}$, where $f''(u_0(r))$ changes sign. It now follows that

$$f''(u_0)\phi_0^2 < f''(u_0)u_0'^2 \quad \text{in } (0,\bar{r}) \cup (\bar{r},1).$$

Hence

$$\int_0^1 f''(u_0)\phi_0^3 r^{N-1}\,\mathrm{d}r < \int_0^1 f''(u_0)\phi_0\big(u_0'\big)^2 r^{N-1}\,\mathrm{d}r = 0.$$

Since $\int_0^1 f(u_0)\phi_0 r^{N-1}\,\mathrm{d}r > 0$, the fact that $\tau''(0) > 0$ now follows from (4.13). □

Before proving our main result for (4.4), we need one more preparation.

LEMMA 4.4. *Suppose $g \in C^1(\mathbb{R}^1)$ and B is the unit ball in $\mathbb{R}^N$, $N \geqslant 1$. Then for any given $c > 0$, the problem*

$$-\Delta u = \lambda g(u), \quad u|_{\partial B} = 0,$$

can have at most one solution (λ, u) satisfying $\lambda > 0$, $u \geqslant 0$ and $u(0) = c$.

PROOF. Suppose that (λ_0, u_0) is such a solution. It suffices to show that any other such solution (λ, u) must coincide with (λ_0, u_0). By Gidas, Ni and Nirenberg [GNN], both u and u_0 are radially symmetric. It is readily checked that $v(r) = u((\lambda_0/\lambda)^{1/2} r)$ satisfies

$$\left(r^{N-1} v'\right)' + \lambda_0 r^{N-1} g(v) = 0, \quad v(0) = c, v'(0) = 0.$$

Since u_0 satisfies the above equation with the same initial values, by uniqueness of solutions to the above initial value problem (see, for example, [NN], Proposition 2.35), we deduce $v = u_0$. In particular, $v(r) > 0$ for $r \in [0, 1)$ and $v(1) = 0$. This implies that $\lambda = \lambda_0$ and hence $u = v = u_0$. This finishes the proof. □

We are now ready to prove the main result of this subsection.

THEOREM 4.5. *Suppose $\Omega = B$. Then there exists $\mu_0 > 0$ such that* (4.4) *has no positive solution for $\mu < \mu_0$, has exactly one positive solution for $\mu = \mu_0$ and exactly two positive solutions for $\mu > \mu_0$. Moreover, the positive solution set $\{(\mu, v)\}$ of* (4.4) *forms a "$\subset$"-shaped smooth curve in the space $\mathbb{R}^1 \times C(\overline{B})$. Furthermore, if we denote the upper and lower branches by*

$$\left\{\left(\mu, v^{\mu}\right)\colon \mu_0 \leqslant \mu < \infty\right\} \quad \text{and} \quad \left\{(\mu, v_{\mu})\colon \mu_0 \leqslant \mu < \infty\right\},$$

respectively, then $\mu \to v^{\mu}(x)$ is strictly increasing for any fixed $|x| < 1$, $\mu \to v_{\mu}(0)$ is strictly decreasing, and

$$\lim_{\mu \to \infty} v^{\mu}(x) = \infty \quad \forall |x| < 1,$$
$$\lim_{\mu \to \infty} v_{\mu}(0) = \xi, \qquad \xi = 0 \quad \text{if } N = 1, 2 \quad \text{and} \quad \xi > 0 \quad \text{if } N > 2.$$

PROOF. We first show that for μ large, (4.4) has at least one positive solution. Let $\phi \in C_0^{\infty}(B)$ satisfy $\phi \geqslant 0$ and $\max_B \phi = 1$. Let $\underline{v}$ be the unique solution of $\Delta v + \phi = 0$, $v|_{\partial B} = 0$; let $\bar{v}$ be the unique solution of $\Delta v + \mu = 0$, $v|_{\partial B} = 0$. It is easy to check that for all large μ, $\bar{v} \geqslant \underline{v}$ and they are upper and lower solutions to (4.4), respectively. Therefore there exists $\mu_1 > 0$ such that (4.4) has at least one positive solution provided that $\mu \geqslant \mu_1$. Now we can set

$$\mu_0 = \inf\left\{\mu > 0\colon \text{(4.4) has a positive solution}\right\}.$$

We claim that $\mu_0 > 0$. If not, there exists $\mu_i \to 0$ and v_i such that $\Delta v_i + \mu_i e^{-1/v_i} = 0$. Set $\tilde{v}_i = v_i/\|v_i\|_\infty$. Then

$$\Delta\tilde{v}_i + \mu_i \frac{e^{-1/v_i}}{v_i}\tilde{v}_i = 0, \qquad \tilde{v}_i|_{\partial B} = 0.$$

As $e^{-1/v_i}/v_i$ is uniformly bounded, by standard elliptic regularity, $\|\tilde{v}_i\|_{W^{2,p}} \to 0$. The Sobolev embedding theorem implies that $\tilde{v}_i \to 0$ uniformly. However, this is impossible as $\|\tilde{v}_i\|_\infty = 1$. This contradiction implies that $\mu_0 > 0$.

Again by standard elliptic regularity, we can further show that (4.4) with $\mu = \mu_0$ has at least one positive solution, and we choose one of them and denote it as v_0. We claim that v_0 must be a degenerate solution. If not, then by the implicit function theorem we can show that for μ less than but close to μ_0, (4.4) has at least one positive solution, and this contradicts the definition of μ_0. Since v_0 is degenerate, our Lemma 4.3 implies that the solutions near (μ_0, v_0) form a smooth curve which turns to the right in the (μ, v) space. We may denote the upper and lower branches by v^μ and v_μ respectively, where $v^\mu(0) > v_\mu(0)$. As long as (μ, v^μ) and (μ, v_μ) are nondegenerate, the implicit function theorem ensures that we can continue to extend these two branches in the direction of increasing μ, and we still denote the extensions as v^μ and v_μ. This process of continuation towards larger values of μ for both branches may be stopped at some finite μ^* by one of the following three possibilities:

(i) $\|v^{\mu_n}\|_\infty$ or $\|v_{\mu_n}\|_\infty$ goes to infinity for some $\mu_n \to \mu^* - 0$;

(ii) $\|v^{\mu_n}\|_\infty$ or $\|v_{\mu_n}\|_\infty$ goes to 0 for some $\mu_n \to \mu^* - 0$ (note that by the Harnack inequality, v^μ and v_μ can only lose positivity through vanishing on the entire domain);

(iii) v^{μ^*} or v_{μ_*} is a degenerate solution.

However, (i) cannot occur since v^{μ_n} and v_{μ_n} are uniformly bounded by L^p estimates and Sobolev embedding theorem; (ii) cannot occur either as otherwise, denoting $v_n = v^{\mu_n}$ or v_{μ_n},

$$0 = \lambda_1^B\left(-\mu_n \frac{e^{-1/v_n}}{v_n}\right) \to \lambda_1^B > 0.$$

Finally, (iii) cannot occur. This is because, if, say, (μ, v^μ) becomes degenerate at $\mu = \mu^*$, then Lemma 4.3 tells us that all the solutions near (μ^*, v^{μ^*}) must lie to the right-hand side of it, which is a contradiction. Therefore we can always extend these two branches of solutions to $\mu = \infty$.

By Lemma 4.4, we see that $\mu \to v^\mu(0)$ and $\mu \to v_\mu(0)$ must be strictly monotone and $v^\mu(0) > v_0(0) > v_\mu(0)$ for any $\mu \in (\mu_0, \infty)$. Hence

$$\lim_{\mu\to\infty} v_\mu(0) = \xi \in \big[0, v_0(0)\big), \qquad \lim_{\mu\to\infty} v^\mu(0) = \eta \in \big(v_0(0), \infty\big].$$

Let us first show that $\eta = \infty$. In fact we show a little more than that. Let us denote $w_\mu = D_\mu v^\mu$. By Lemma 4.3, we find that for $\mu > \mu_0$ but close to μ_0, $w_\mu > 0$ in B.

Define $\mu^* := \sup\{\mu > \mu_0\colon\ w_{\mu'} > 0 \text{ in } B \text{ for all } \mu' \in (\mu_0, \mu)\}$. We show that $\mu^* = \infty$. Differentiate the equation for v^μ with respect to μ, we find that

$$-\Delta w_\mu = \mu f'(v^\mu) w_\mu + f(v^\mu) > \mu f'(v^\mu) w_\mu \quad \forall x \in B, \quad w_\mu|_{\partial B} = 0.$$

Therefore $w_\mu \geqslant, \not\equiv 0$ for $\mu \in (\mu_0, \mu^*]$ and by applying the strong maximum principle to the above differential inequality with $\mu = \mu^*$, we deduce $w_{\mu^*} > 0$ in B, $w'_{\mu^*}(1) < 0$. Since $\mu \to w_\mu$ is continuous in the $C^2(\overline{B})$ norm, the above conclusion for w_{μ^*} implies that $w_\mu > 0$ in B for all $\mu > \mu^*$ but close to μ^*, a contradiction to the definition of μ^*. Therefore $w_\mu > 0$ in B for all $\mu > \mu_0$. This implies that $\mu \to v^\mu(r)$ is strictly increasing and $v^\mu(r) > v_0(r)$. It follows that

$$v^\mu(r) = (-\Delta)^{-1}\left[\mu e^{-1/v^\mu}\right] \geqslant \frac{\mu}{\mu_0}(-\Delta)^{-1}\left[\mu_0 e^{-1/v_0}\right] = \frac{\mu}{\mu_0} v_0(r) \to \infty$$

as $\mu \to \infty$, for any $r \in [0, 1)$.

We next show that $\xi > 0$ when $N > 2$ and $\xi = 0$ when $N = 1, 2$. By Lemma 4.4, this would imply that all the positive solutions of (4.4) are contained in these two solution branches if we can show that there is no positive solution of (4.4) satisfying $u(0) \leqslant \xi$ when $\xi > 0$.

Let us note that when $N > 2$, $\xi > 0$ is a consequence of (4.8) in the proof of Lemma 4.2. When $N = 2$, we argue indirectly. Suppose that $\xi > 0$. Consider the initial value problem

$$(rz')' = -r e^{-1/z}, \quad z(0) = \xi, z'(0) = 0.$$

It is easily seen that $z'(r) < 0$ for $r \in (0, r_0)$ as long as z is positive on $(0, r_0)$. If z remains positive on $[0, \infty)$, then $z(x) = z(|x|) = z(r)$ satisfies $\Delta z = -e^{-1/z} < 0$ on $\mathbb{R}^2$ and hence is a bounded subharmonic function on $\mathbb{R}^2$. It is well known that in such a case, $z \equiv$ const. Clearly this is impossible. Hence z has a first zero $r_0 > 0$: $z(r) > 0$ in $[0, r_0)$ and $z(r_0) = 0$. By continuous dependence of the solutions on the initial values, for μ^* large, the unique solution z^* of the initial value problem

$$(rz')' = -r e^{-1/z}, \quad z(0) = v_{\mu^*}(0), z'(0) = 0,$$

has a first zero r^* close to r_0. But then $v^*(r) = z^*(r^* r)$ is a solution of (4.4) with $v^*(0) = v_{\mu^*}(0)$ but $\mu = (r^*)^2 \to r_0^2 \neq \mu^*$ as $\mu^* \to \infty$. This contradicts Lemma 4.4. Hence we must have $\xi = 0$.

When $N = 1$, the proof is similar but simpler. The initial value problem now is changed to

$$z'' = -z e^{-1/z}, \quad z(0) = \xi, z'(0) = 0,$$

and the existence of a first zero of z follows from $z'' < 0$ on $[0, \infty)$.

Finally we show that if $\xi > 0$, then (4.4) has no positive solution with $u(0) \leqslant \xi$. In fact, if there is such a solution, then the argument we used above can be repeated to show that there is a second smooth curve $\{(\mu, \tilde{u})\}$ of positive solutions which is "⊂"-shaped and for $(\mu, \tilde{v})$ on its upper branch, $\tilde{v}(0) \to \infty$ as $\mu \to \infty$. But this implies that for any large number $C > 0$, there are at least two solutions v^μ and $\tilde{v}$ with $v^\mu(0) = \tilde{v}(0) = C$, contradicting Lemma 4.4. The proof of Theorem 4.5 is complete. □

4.2. *The perturbed Gelfand equation in dimensions 1 and 2*

We will show that when $\Omega = B$ in dimensions 1 and 2, the positive solutions of the perturbed Gelfand equation are completely determined by (4.4). We use the equivalent form (4.3), namely

$$-\Delta u = \mu \mathrm{e}^{-1/(\varepsilon+u)} \quad \text{in } B, \quad u|_{\partial B} = 0, \tag{4.17}$$

where $B = \{x \in \mathbb{R}^N\colon |x| < 1\}$, $N = 1, 2$.

Let us first observe the following simple relationship between (4.17) and (4.4).

If (μ, v) is a positive solution of (4.4) with $\Omega = B$, and $v(0) > \varepsilon$, then we can find a unique $a \in (0, 1)$ such that $v(a) = \varepsilon$. Define

$$u(x) = v(ax) - \varepsilon, \quad x \in B.$$

Clearly

$$-\Delta u = a^2 \mu f(u + \varepsilon), \quad u|_{\partial B} = 0.$$

That is, $(a^2\mu, u)$ is a positive solution of (4.17). In dimensions 1 and 2, since $\xi = 0$ in Theorem 4.5, we find that for any $\varepsilon > 0$, we can obtain a positive solution of (4.17) from a positive solution of (4.4) through the above process. Moreover, by Lemma 4.4, we easily see that all the positive solutions of (4.17) can be obtained in this way. Therefore, the positive solutions of (4.17) are completely determined by (4.4).

The following result, for $N = 1, 2$, is the counterpart of Lemma 4.2, but we will see in Remark 4.9 that this result is not true when $3 \leqslant N \leqslant 9$. This indicates that, in this kind of conclusions, the nonlinearity and the space dimension play a very subtle role.

LEMMA 4.6. *If u is a degenerate positive solution of* (4.17) *and ϕ is a nontrivial solution to*

$$-\Delta\phi = \mu f_\varepsilon'(u)\phi, \quad \phi|_{\partial B} = 0,$$

then ϕ does not change sign in B.

PROOF. Before starting the proof, let us remark that our proof only requires $\varepsilon \geqslant 0$. Therefore, it is a simplification of the proof of Lemma 4.2 for the case $N = 1, 2$.

By Gidas, Ni and Nirenberg [GNN], u is radially symmetric: $u(x) = u(r), r = |x|$; moreover, $u'(r) < 0$ on $(0, 1]$. By Proposition 3.3 of [LN], ϕ is also radially symmetric, $\phi(x) = \phi(r)$. Hence

$$\phi'' + \frac{N-1}{r}\phi' + \mu f_\varepsilon'(u)\phi = 0 \quad \text{in } [0,1], \quad \phi'(0) = 0, \phi(1) = 0.$$

We may assume $\phi(0) > 0$.

We make use of the test function

$$v(r) = ru'(r) + \xi$$

instead of the usual $v = ru' + \xi u$ as used in the proof of Lemma 4.2, where ξ is a positive constant to be specified later. This choice of test function seems crucial. By a direct calculation,

$$\begin{aligned} &v'' + \frac{N-1}{r}v' + \mu f_\varepsilon'(u)v = \mu\big[\xi f_\varepsilon'(u) - 2f_\varepsilon(u)\big] \equiv G(r),\\ &\big[r^{N-1}\big(v'\phi - v\phi'\big)\big]' = G(r)r^{N-1}\phi, \end{aligned} \tag{4.18}$$

where

$$G(r) = \mu f_\varepsilon(u)g(r), \quad g(r) = \frac{\xi}{(u+\varepsilon)^2} - 2.$$

Clearly, $g(r)$ is strictly increasing in r.

Now we suppose $\phi(r)$ changes sign in $(0,1)$, and want to deduce a contradiction from this. Let $r_0 \in (0,1)$ be the first zero of $\phi(r)$: $\phi(r_0) = 0$ and $\phi(r) > 0$ for $r \in [0, r_0)$. We choose $\xi = -r_0 u'(r_0)$ in $v = ru' + \xi$. Since

$$v' = -r\mu f_\varepsilon(u) + (2-N)u' < 0 \quad \forall r \in (0,1],$$

we have $v(r) > v(r_0) = 0$ on $[0, r_0)$ and $v(r) < 0$ on $(r_0, 1]$.

We divide our considerations below into two cases:

$$\text{(i) } g(r_0) \leqslant 0 \quad \text{and} \quad \text{(ii) } g(r_0) > 0.$$

In case (i), using $g(r) < g(r_0) \leqslant 0$ on $[0, r_0)$, we obtain the following contradiction by integrating (4.18) from 0 to r_0:

$$0 > \int_0^{r_0} G(r)r^{N-1}\phi\,\mathrm{d}r = \big[r^{N-1}\big(v'\phi - v\phi'\big)\big]\big|_0^{r_0} = 0.$$

In case (ii), we consider the last zero of $\phi(r)$ before $r = 1$: $r_0 \leqslant r^0 < 1$, $\phi(r^0) = 0$, $\phi(r) \neq 0$ for $r \in (r^0, 1)$. We may assume that $\phi(r) > 0$ on $(r^0, 1)$ (otherwise change the sign of ϕ). Then $\phi'(r^0) > 0 > \phi'(1)$. Now using $g(r) > 0$ and $v(r) \leqslant 0$ on $[r^0, 1]$, we again deduce a contradiction:

$$0 < \int_{r^0}^{1} G(r) r^{N-1} \phi(r)\,\mathrm{d}r = \big[r^{N-1}\big(v'\phi - v\phi'\big)\big]\big|_{r^0}^{1} \leqslant 0.$$

The proof is complete. □

Using Lemma 4.6, we obtain a variant of Lemma 4.3, whose obvious proof we omit.

LEMMA 4.7. *Suppose that u_0 is a degenerate positive solution of* (4.17) *with $\mu = \mu_0$. Then all positive solutions (μ, u) of* (4.17) *that are near (μ_0, u_0) in $\mathbb{R} \times C(\overline{B})$ lie on a smooth curve represented by*

$$(\mu, u) = \big(\mu_0 + \tau(s), u_0 + s\phi + z(s)\big) \quad \textit{with } s \textit{ small},$$

where $z(0) = z'(0) = 0$, $\tau(0) = \tau'(0) = 0$ and ϕ is the positive eigenfunction given in Lemma 4.6. *Moreover,*

$$\tau''(0) = -\mu_0 \frac{\int_B f_\varepsilon''(u_0)\phi^3\,\mathrm{d}x}{\int_B f_\varepsilon(u_0)\phi\,\mathrm{d}x}. \tag{4.19}$$

We are now ready to prove the main result of this subsection.

THEOREM 4.8. *For all sufficiently small $\varepsilon > 0$, the positive solution curve $\{(\mu, u)\}$ of* (4.17) *is exactly S-shaped. There exist λ_ε^* and Λ_ε^* satisfying $\lim_{\varepsilon\to 0} \lambda_\varepsilon^* = \mu_0$, $\lim_{\varepsilon\to 0} \Lambda_\varepsilon^* = \infty$, such that* (4.17) *has*

(i) *a unique positive solution for $\mu \in (0, \lambda_\varepsilon^*) \cup (\Lambda_\varepsilon^*, \infty)$;*
(ii) *exactly two positive solutions for $\mu = \lambda_\varepsilon^*$ and $\mu = \Lambda_\varepsilon^*$;*
(iii) *exactly three positive solutions for $\mu \in (\lambda_\varepsilon^*, \Lambda_\varepsilon^*)$.*

PROOF. By Theorem 4.5, the positive solution curve of (4.4) with $\Omega = B$ is "$\subset$"-shaped with exactly one turning point at (μ_0, v_0), where $v_0 = v_{\mu_0} = v^{\mu_0}$. Denote $\xi_0 = v_0(0)$. Then for any $\varepsilon \in (0, \xi_0)$, we can find a unique $\mu_\varepsilon \in (\mu_0, \infty)$ such that

$$v_{\mu^\varepsilon}(0) = \varepsilon.$$

By Theorem 4.5, we see that μ_ε increases as ε decreases and $\mu_\varepsilon \to \infty$ as $\varepsilon \to 0$.

For any $\mu \in [\mu_0, \mu_\varepsilon)$, we can find a unique $a_\mu = a_\mu(\varepsilon) \in (0, 1)$ such that

$$v_\mu(a_\mu) = \varepsilon.$$

Clearly,

$$\begin{aligned} &\lim_{\varepsilon\to 0} a_\mu(\varepsilon) = 1 && \text{for fixed } \mu \geqslant \mu_0, \\ &\lim_{\mu\to\mu_\varepsilon - 0} a_\mu(\varepsilon) = 0 && \text{for fixed } \varepsilon \in (0, \xi_0). \end{aligned} \tag{4.20}$$

Now we define

$$\eta_\mu = (a_\mu)^2\mu, \qquad u_\mu(x) = v_\mu(a_\mu x) - \varepsilon, \quad x \in B,$$

and find that

$$\Gamma_\varepsilon = \{(\eta_\mu, u_\mu)\colon\ \mu_0 \leqslant \mu < \mu_\varepsilon\}$$

gives a piece of smooth solution curve to (4.17). Moreover, Γ_ε connects $(\eta_{\mu_0}, u_{\mu_0})$ and $(0,0)$ (when $\mu \to \mu_\varepsilon - 0$).

On the other hand, due to $\varepsilon \in (0, \xi_0)$, for any $\mu \geqslant \mu_0$, we can find a unique $a^\mu = a^\mu(\varepsilon) \in (0,1)$ satisfying $v^\mu(a^\mu) = \varepsilon$, and define

$$\eta^\mu = (a^\mu)^2\mu, \qquad u^\mu(x) = v^\mu(a^\mu x) - \varepsilon, \quad x \in B,$$

we obtain another piece of smooth solution curve of (4.17)

$$\Gamma^\varepsilon = \{(\eta^\mu, u^\mu)\colon\ \mu_0 \leqslant \mu < \infty\}.$$

By Theorem 4.5, $\mu \to a^\mu$ is strictly increasing and

$$\lim_{\mu\to\infty} a^\mu = 1. \tag{4.21}$$

Therefore $\mu \to \eta^\mu$ is strictly increasing and $\{(\mu, u(0))\colon\ (\mu, u) \in \Gamma^\varepsilon\}$ is a monotone curve in $\mathbb{R}^2$ that connects $(\eta^{\mu_0}, u^{\mu_0}(0))$ to (∞, ∞). Since

$$(\eta^{\mu_0}, u^{\mu_0}) = (\eta_{\mu_0}, u_{\mu_0}),$$

we find that

$$\Gamma(\varepsilon) = \Gamma^\varepsilon \cup \Gamma_\varepsilon$$

gives a piecewise smooth (in fact smooth) curve for (4.17) connecting $(0,0)$ and (∞, ∞). By Lemma 4.4, we know it contains all the positive solutions of (4.17). We are going to find out the shape of this curve.

Recall that

$$f''(u) > 0 \quad \text{for } u \in (0, 1/2) \quad \text{and} \quad f''(u) < 0 \quad \text{for } u \in (1/2, \infty).$$

We fix some $\xi_1 \in (0, 1/2)$ and suppose

$$\varepsilon < \varepsilon_1 \leqslant 1/2 - \xi_1.$$

Then clearly $f''(u + \varepsilon) > 0$ for $u \in (0, \xi_1)$.

Now we choose $\lambda_{\xi_1} > \mu_0$ such that

$$v_\mu(0) < \xi_1 \quad \text{when } \mu \geqslant \lambda_{\xi_1}.$$

By shrinking ε_1 we may assume that $\lambda_{\xi_1} < \mu_\varepsilon$ for all $\varepsilon \in (0, \varepsilon_1)$. We can now divide Γ_ε into two parts

$$\Gamma_\varepsilon^1 = \{(\eta_\mu, v_\mu)\colon\ \lambda_{\xi_1} \leqslant \mu < \mu_\varepsilon\} \quad \text{and} \quad \Gamma_\varepsilon^2 = \{(\eta_\mu, v_\mu)\colon\ \mu_0 \leqslant \mu \leqslant \lambda_{\xi_1}\}.$$

We first analyze the shape of Γ_ε^1. Define

$$\Lambda_\varepsilon^* = \sup_{\mu \in [\lambda_{\xi_1}, \mu_\varepsilon)} \eta_\mu.$$

By (4.20), one easily sees that there exists $\varepsilon_2 \in (0, \varepsilon_1]$ such that when $\varepsilon \in (0, \varepsilon_2)$,

$$\Lambda_\varepsilon^* \text{ is achieved at some } \mu_* \in (\lambda_{\xi_1}, \mu_\varepsilon) \quad \text{and} \quad \lim_{\varepsilon \to 0} \Lambda_\varepsilon^* = \infty.$$

By the implicit function theorem, $(\eta_{\mu_*}, u_{\mu_*})$ must be a degenerate solution of (4.17). Then by Lemma 4.7, (4.19) and our choice of ξ_1, the solutions of (4.17) near $(\eta_{\mu_*}, v_{\mu_*})$ form a smooth curve that has a turn to the left. Therefore, we have an upper branch and a lower branch of positive solutions starting from this point, and both branches can be continued towards smaller values of μ. The lower branch can be continued to reach $(0, 0)$, because (a) we cannot meet a degenerate solution in the way of continuation due to Lemma 4.7 and $u(0) < \xi_1$ on Γ_ε^1, and (b) the branch goes along Γ_ε^1. For the same reason, the upper branch can be continued till it reaches $(\eta_{\lambda_{\xi_1}}, u_{\lambda_{\xi_1}})$. This implies that Γ_ε^1 is exactly "$\supset$"-shaped.

Next we analyze the shape of Γ_ε^2. It is more convenient for our discussion if we consider a bigger piece of solution curve

$$\Gamma_\varepsilon^3 = \Gamma_\varepsilon^2 \cup \{(\eta^\mu, u^\mu)\colon\ \mu_0 \leqslant \mu \leqslant \lambda_{\xi_1}\},$$

which contains part of Γ^ε. We observe that any $(\mu, u) \in \Gamma_\varepsilon^3$ satisfies

$$\lambda_\varepsilon^* \leqslant \mu \leqslant \lambda_{\xi_1}, \qquad u_{\lambda_{\xi_1}}(0) - \varepsilon \leqslant \|u\|_\infty = u(0) \leqslant u^{\lambda_{\xi_1}}(0) - \varepsilon, \tag{4.22}$$

where

$$\lambda_\varepsilon^* = \inf\{\mu\colon\ (\mu, u) \in \Gamma_\varepsilon^3\}.$$

Since η^μ is increasing in μ, λ_ε^* is achieved at some $\eta_{\mu'}$, $\mu' \in [\mu_0, \lambda_{\xi_1})$. Therefore $(\lambda_\varepsilon^*, u_{\mu'})$ must be a degenerate solution of (4.17). Clearly

$$\lambda_\varepsilon^* \leqslant \eta_{\mu_0} = (a_{\mu_0}(\varepsilon))^2 \mu_0 < \mu_0.$$

On the other hand, it is easy to see that $a_\mu(\varepsilon) \to 1$ as $\varepsilon \to 0$ uniformly for $\mu \in [\mu_0, \lambda_{\xi_1}]$. Hence

$$\lim_{\varepsilon\to 0} \lambda_\varepsilon^* = \lim_{\varepsilon\to 0} \min\bigl\{\bigl(a_\mu(\varepsilon)\bigr)^2\mu\colon\ \mu_0 \leqslant \mu \leqslant \lambda_{\xi_1}\bigr\} = \mu_0.$$

We know from the above that Γ_ε^3 contains at least one degenerate solution $(\lambda_\varepsilon^*, u_{\mu'})$. If we can show that there exists $\varepsilon_3 \in (0, \varepsilon_2)$ such that whenever $\varepsilon \in (0, \varepsilon_3)$, any degenerate solution on Γ_ε^3 must make $\tau''(0) > 0$ in (4.19) of Lemma 4.7, then a continuation argument much as before shows Γ_ε^3 contains exactly one degenerate solution at $\mu = \lambda_\varepsilon^*$ and the curve makes a turn to the right at this point. Hence Γ_ε^3 must be smooth and "$\subset$"-shaped. This tells us that the entire solution curve $\Gamma(\varepsilon)$ is exactly S-shaped with two turning points at $\mu = \lambda_\varepsilon^*$ and $\mu = \Lambda_\varepsilon^*$, respectively. Clearly, this would finish the proof of Theorem 4.8.

It remains to show that there exists $\varepsilon_3 \in (0, \varepsilon_2)$ such that any degenerate solution on Γ_ε^3 must make $\tau''(0) > 0$ in (4.19) of Lemma 4.7 as long as $\varepsilon \in (0, \varepsilon_3)$. We argue indirectly. Suppose for some $\varepsilon_k \to 0$, we can find degenerate solutions $(\mu^k, u^k) \in \Gamma_{\varepsilon_k}^3$ such that

$$\tau_k''(0) = -\mu^k \frac{\int_B f''(u^k + \varepsilon_k)\phi_k^3 \,\mathrm{d}x}{\int_B f(u^k + \varepsilon_k)\phi_k \,\mathrm{d}x} \leqslant 0,$$

where ϕ_k is the positive eigenfunction given in Lemma 4.7 when $(\mu, u) = (\mu^k, u^k)$. We may assume that $\|\phi_k\|_\infty = 1$.

By (4.22), we may assume that $\mu^k \to \mu^0 \in [\mu_0, \lambda_{\xi_1}]$. Equation (4.22) also implies that $\|f(u^k + \varepsilon_k)\|_\infty$ is uniformly bounded. Therefore, by the equation for u^k and a standard regularity and compactness argument, $\{u^k\}$ has a convergent subsequence in C^1. We may assume $u^k \to u^0$ in C^1. Moreover, from

$$-\Delta\phi_k = \mu^k f'\bigl(u^k + \varepsilon_k\bigr)\phi_k, \quad \phi_k|_{\partial B} = 0,$$

we can use a similar regularity and compactness argument to obtain a C^1 convergent subsequence of ϕ_k. We may assume $\phi_k \to \phi^0$. Then we easily deduce

$$-\Delta u^0 = \mu^0 f\bigl(u^0\bigr), \quad u^0\big|_{\partial B} = 0, u^0 \geqslant 0, u^0 \not\equiv 0,$$

and

$$-\Delta\phi^0 = \mu^0 f'\bigl(u^0\bigr)\phi^0, \quad \phi|_{\partial B} = 0, \phi^0 \geqslant 0, \bigl\|\phi^0\bigr\|_\infty = 1.$$

This is to say (μ^0, u^0) is a degenerate positive solution of (4.4) and ϕ^0 is the corresponding positive eigenfunction. By Theorem 4.5, (4.4) has a unique degenerate positive solution which is (μ_0, u_0), and by Lemma 4.3 and (4.13),

$$\tau''(0) = -\mu_0 \frac{\int_B f''(u_0)\phi^3 \,\mathrm{d}x}{\int_B f(u_0)\phi \,\mathrm{d}x} > 0.$$

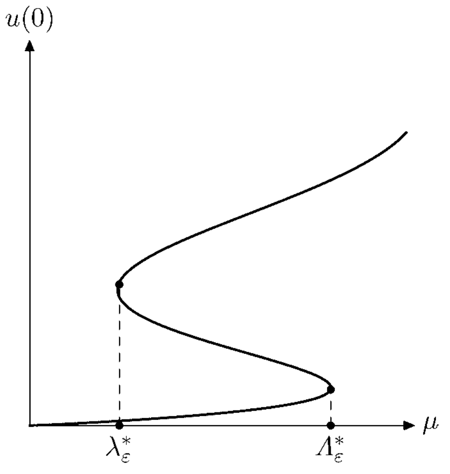

Fig. 3. Bifurcation diagram for (4.17) with small $\varepsilon > 0$.

Therefore, we must have $\mu^k \to \mu_0, u^k \to u_0$ and $\phi^0 = \phi$ (note that the positive eigenfunction is unique if it is normalized). Hence we have

$$0 \geqslant \tau_k''(0) = -\mu^k \frac{\int_B f''(u^k + \varepsilon_k)\phi_k^3 \,\mathrm{d}x}{\int_B f(u^k + \varepsilon)\phi_k \,\mathrm{d}x} \to -\mu_0 \frac{\int_B f''(u_0)\phi^3 \,\mathrm{d}x}{\int_B f(u_0)\phi \,\mathrm{d}x} > 0.$$

This contradiction finishes our proof. □

Theorem 4.8 is illustrated by the bifurcation diagram in Figure 3.

4.3. *The perturbed Gelfand equation in higher dimensions*

In this subsection, we consider the perturbed Gelfand equation with space dimension $N \geqslant 3$. For such N, by Theorem 4.5, any positive solution (μ, v) of (4.4) with $\Omega = B$ satisfies

$$v(0) > \xi > 0, \quad \text{where } \xi = \lim_{\lambda \to \infty} v_\mu(0).$$

We can still use the transformation $u(x) = v(ax) - \varepsilon$, where $v(a) = \varepsilon$, to obtain a positive solution $(a^2\mu, u)$ of (4.3) with $\Omega = B$ from a positive solution (μ, v) of (4.4). More precisely, we can define

$$\Gamma(\varepsilon) = \Gamma^\varepsilon \cup \Gamma_\varepsilon, \quad \Gamma^\varepsilon = \{(\eta^\mu, u^\mu)\colon \mu_0 \leqslant \mu < \infty\},$$
$$\Gamma_\varepsilon = \{(\eta_\mu, u_\mu)\colon \mu_0 \leqslant \mu < \infty\},$$

as in Section 4.2, except that now η_μ and u_μ can be defined for all $\mu \geqslant \mu_0$, since $\xi > 0$ (we assume that $\varepsilon < \xi$).

However, unlike in the case $N = 1$ and 2, the solution curve $\Gamma(\varepsilon)$ does not give all the positive solutions of (4.3) with $\Omega = B$, since any $(\mu, u) \in \Gamma(\varepsilon)$ satisfies $u(0) > \xi - \varepsilon$, and it is well known (and can be easily shown) that for any fixed $\varepsilon > 0$, the entire positive solution branch $\{(\mu, u)\}$ of (4.3) with $\Omega = B$ has the point $(0, 0)$ on its boundary: it approaches $(0, 0)$ as $\mu \to 0$. Let us note that this implies that η_μ remains bounded for $\mu \in [\mu_0, \infty)$. To better understand the entire positive solution branch for (4.3) with $\Omega = B$, we need the help from the Gelfand equation (4.2) and some general bifurcation results; we will follow the approach in [Da1] and [Du2].

We first recall some abstract results. Suppose that

(H_1) X and Y are real Banach spaces and $F : \mathbb{R}^2 \times X \to Y$ is a C^k-map ($k \geqslant 2$) which sends $(\varepsilon, \lambda, x) \in \mathbb{R}^2 \times X$ to $F(\varepsilon, \lambda, x) \in Y$;

(H_2) Σ is a component of the set of solutions (λ, x) of $F(0, \lambda, x) = 0$ and Σ_0 is a connected compact subset of Σ;

(H_3) For any $(\lambda, x) \in \Sigma_0$, $F_x(0, \lambda, x) : X \to Y$ is a Fredholm operator of index 0, and the mapping

$$B(\mu, h) := F_\lambda(0, \lambda, x) + F_x(0, \lambda, x) : \mathbb{R}^1 \times X \to Y$$

is onto; in other words, for any $(\lambda, x) \in \Sigma_0$, either

(i) $F_x(0, \lambda, x) : X \to Y$ has a continuous inverse, or

(ii) $\dim N(F_x(0, \lambda, x)) = \operatorname{codim} R(F_x(0, \lambda, x)) = 1$ and $F_\lambda(0, \lambda, x) \notin R(F_x(0, \lambda, x))$.

We want to know how Σ_0 is perturbed to give a solution set Σ_ε of $F(\varepsilon, \lambda, x) = 0$ when ε is small. We start with a local analysis. Suppose $(\lambda_0, x_0) \in \Sigma_0$. Then either case (i) or case (ii) in (H_3) happens. If case (i) happens, then it follows from the implicit function theorem that there exist a neighborhood V of x_0 in X and a small number $\delta > 0$ such that

$$\begin{aligned} &F^{-1}(0) \cap \big((-\delta, \delta) \times (\lambda_0 - \delta, \lambda_0 + \delta) \times V\big) \\ &\quad = \big\{\big(\varepsilon, \lambda, x(\lambda, \varepsilon)\big) : |\varepsilon|, |\lambda - \lambda_0| < \delta\big\}, \end{aligned} \tag{4.23}$$

where $(\varepsilon, \lambda) \to x(\varepsilon, \lambda)$ is C^k, and $x(0, \lambda_0) = x_0$.

If case (ii) happens, then by Theorem 1.3 and its proof (see [CR2]), there exist a small neighborhood W of (λ_0, x_0) and a small $\delta > 0$ such that

$$F^{-1}(0) \cap \big((-\delta, \delta) \times W\big) = \big\{\big(\varepsilon, \lambda(s, \varepsilon), x(s, \varepsilon)\big) : s, \varepsilon \in (-\delta, \delta)\big\}, \tag{4.24}$$

where $\lambda(s, \varepsilon)$ and $x(s, \varepsilon)$ are C^k with $\lambda(0, 0) = \lambda_0$, $\lambda_s(0, 0) = 0$, $x(s, \varepsilon) = x_0 + su_0 + z(s, \varepsilon)$, $\tau(0, 0) = \tau_s(0, 0) = 0$. Here u_0 spans $N(F_x(0, \lambda_0, x_0))$ and $z(s, \varepsilon)$ belongs to a complement Z of $\operatorname{span}\{u_0\}$ in X.

Since Σ_0 is compact, by a finite covering argument, we find from the above discussion that Σ_0 is a C^k curve and for any small ε, the solutions of $F(\varepsilon, \lambda, x) = 0$ near Σ_0 form a C^k curve Σ_ε; see [Da1], Theorem 2, and [DLo1], Proposition A.2, for more details. If we change C^k to "analytic" in (H_1), then we can do the same in all the conclusions in the above discussions.

Let us now see how the above discussions can be applied to (4.3). Set $X = C_0^{2,\alpha}(\overline{B})$, $Y = C^{\alpha}(\overline{B})$ and $F(\varepsilon, \mu, u) = \Delta u + \mu f_{\varepsilon}(u)$. Then from the proofs of Lemma 4.3 and Theorem 4.5 we know that the conditions (H_1)–(H_3) are satisfied for positive λ and ε if we choose Σ_0 to be any connected compact part of the global positive solution curve $\{(\mu, u)\}$ of (4.3) with $\Omega = B$. Moreover, if we choose Σ_0 to contain the unique turning point (μ_0, u_0), then due to Lemma 4.3, (4.23) and (4.24), we find that for all small $\varepsilon > 0$, Σ_{ε} contains exactly one turning point $(\mu_{\varepsilon}, u_{\varepsilon})$. Moreover, it is easily seen that the upper branch in Σ_{ε} contains part of Γ^{ε} defined through the transformation at the beginning of this subsection. As in Section 4.3, we easily see from Theorem 4.5 that η^{μ} is a strictly increasing function. Thus this upper branch can be parameterized by μ and gives a unique positive solution for each fixed $\mu > \mu_{\varepsilon}$. The lower branch can also be continued, and in fact, it can be continued till it reaches $(0, 0)$; however, the shape of the lower branch is much more difficult to control. Let us note that if we have chosen Σ_0 to be a rather large piece of the "$\subset$"-shaped positive solution curve of (4.4) (with $\Omega = B$), then Σ_{ε} with small $\varepsilon > 0$ is a large piece of "$\subset$"-shaped positive solution curve of (4.3) (with $\Omega = B$); hence the shape of its lower branch is controlled till a very large μ. Let us denote the union of Σ_{ε} and its well-controlled continued entire upper branch by Γ_{ε}^{*}. Then we know that Γ_{ε}^{*} is exactly "$\subset$"-shaped, its upper branch is unbounded and approaches (∞, ∞), and its lower branch contains points with large but finite μ.

In order to better understand the rest of the global positive solution curve of (4.3), we now use the equivalent form (4.1), which can be viewed as a perturbation of the Gelfand equation (4.2). We should note that the curve Γ_{ε}^{*} for (4.3) is now transformed to a positive solution curve $\widetilde{\Gamma}_{\varepsilon}^{*} := \{(\lambda, u)\colon\ \lambda = \mu/(\varepsilon^2 e^{1/\varepsilon}), u = v/\varepsilon^2, (\mu, v) \in \Gamma_{\varepsilon}^{*}\}$ for (4.1) with $\Omega = B$, its upper branch is still unbounded, but any (λ, u) belonging to its lower branch has λ small (since $\lambda = \mu/(\varepsilon^2 e^{1/\varepsilon})$ and ε is small). Let us also note that any $(\lambda, u) \in \widetilde{\Gamma}_{\varepsilon}^{*}$ satisfies $u(0) > (\xi - \varepsilon)/\varepsilon^2 \to \infty$ as $\varepsilon \to 0$.

We illustrate Γ_{ε}^{*} and $\widetilde{\Gamma}_{\varepsilon}^{*}$ in Figure 4.

Let us consider the Gelfand equation (4.2) with $\Omega = B$ in more detail. When $3 \leqslant N \leqslant 9$, we know from Proposition 4.1 and Figure 2 that its positive solution curve has infinitely

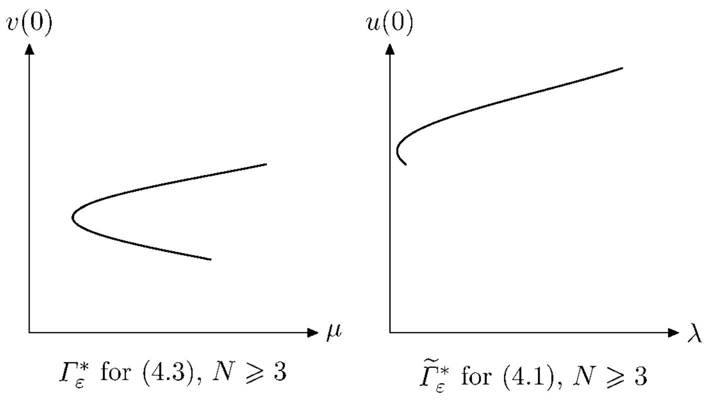

Fig. 4. Bifurcation diagrams for (4.1) and (4.3) with $\Omega = B$ and small $\varepsilon > 0$.

many turning points. Let us denote them by T_k, $k = 1, 2, \ldots,$ where the T_k's are ordered so that when we go from $T_0 := (0, 0)$ along the solution curve we meet T_1 first, then T_2, T_3, etc. It is known (see [Da1] and [NS], Section 2) that if (λ, u) lies on the open arc from T_k to T_{k+1}, then u is nondegenerate; and if $(\lambda, u) = T_k$ with $k \geqslant 1$, then u is degenerate and the nontrivial solution ϕ to

$$-\Delta\phi = \lambda e^u \phi, \quad \phi|_{\partial B} = 0,$$

is radially symmetric, $\phi(x) = \phi(r)$; moreover, $\phi(r)$ changes sign over $[0, 1)$ exactly $k - 1$ times, and is unique up to a scalar multiple, and

$$\int_B e^u \phi^3 \, dx \neq 0. \tag{4.25}$$

These facts imply that if we take $X = C_0^{2,\alpha}(\overline{B})$, $Y = C^\alpha(\overline{B})$ and $F(\varepsilon, \lambda, u) = \Delta u + \lambda e^{u/(1+\varepsilon u)}$, then the conditions (H_1)–(H_3) are satisfied with $\lambda, \varepsilon > 0$. Moreover, case (ii) in (H_3) occurs exactly when $(\lambda, u) \in \{T_k\colon\ k \geqslant 1\}$. By (4.24) and (4.25), we deduce that $\tau_{ss}(0, 0) \neq 0$ when $(\lambda, u) = T_k$, $k \geqslant 1$. Hence we can apply the above abstract results to conclude the following:

Let Σ_0 be a connected compact part of the positive solution curve $\{(\lambda, u)\}$ of (4.2) *with $\Omega = B$, and suppose that Σ_0 contains the turning points $T_1, \ldots, T_k$. Then for all small $\varepsilon > 0$, the positive solution curve of* (4.1) *with $\Omega = B$ has a part Σ_ε, which is close to Σ_0 and contains exactly k turning points $T_1^\varepsilon, \ldots, T_k^\varepsilon$, where $T_i^\varepsilon \to T_i$ as $\varepsilon \to 0$, for each $i = 1, \ldots, k$.*

Close to $T_0 = (0, 0)$, we can apply the implicity function theorem to (4.1) to conclude that there is a unique positive solution for each $\lambda > 0$ small. Therefore we can extend Σ_ε towards $(0, 0)$ so that T_0 is an end point of Σ_ε, provided that Σ_0 has been chosen properly.

REMARK 4.9. It is easy to show that, at T_i, $2 \leqslant i \leqslant k$, the linearized problem of (4.1) (with $\Omega = B$ and small $\varepsilon > 0$) has a radially symmetric nontrivial solution ϕ which changes sign exactly $i - 1$ times over $[0, 1)$. Hence Lemma 4.6 does not hold when $3 \leqslant N \leqslant 9$.

If $N \geqslant 10$, then it is known (see [Da1]) that the positive solution curve of (4.2) with $\Omega = B$ contains no turning point and all the positive solutions are nondegenerate. Therefore a similar consideration to the above yields the following conclusion:

Given any connected compact part Σ_0 of the positive solution curve $\{(\lambda, u)\}$ of (4.2) *with $\Omega = B$, for all small $\varepsilon > 0$, the positive solution curve of* (4.1) *with $\Omega = B$ has a part Σ_ε, which is close to Σ_0 and contains no turning points. Moreover, Σ_ε can be extended to reach* $(0, 0)$ *and the extended Σ_ε still does not contain any turning point.*

Combined with $\widetilde{\Gamma}_\varepsilon$, we can now illustrate the well-understood parts of the positive solution curve of (4.1) in Figure 5.

The missing part of the positive solution curve for (4.1) can be further described by using the bifurcation curve $S_0 := \{(\lambda, u'(1))\}$ of (4.2) with $\Omega = B$; this curve is completely understood in [JL], and Dancer [Da1] used S_0 to show that the missing part is roughly the

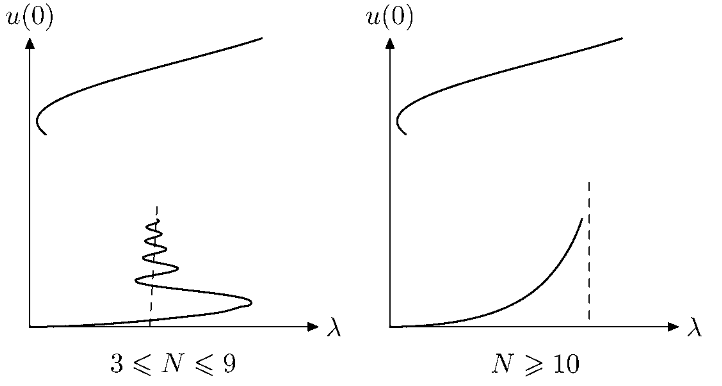

Fig. 5. Bifurcation diagrams for (4.1) with $\Omega = B$ and small $\varepsilon > 0$.

bottom part in Figure 5 reflected about the horizontal line passing through the top point of the bottom part. However, on this reflected part, the exact number of turning points is not known. We refer to [Da1], Figure 1, and [Du2], Figure 4, for bifurcation diagrams illustrating the entire positive solution curve.

4.4. *Further remarks and related results*

The above results for the perturbed Gelfand equation (4.1) with $\Omega = B$ can be extended to the following more general problem

$$-\Delta u = \lambda(1+\varepsilon u)^m e^{u/(1+\varepsilon u)}, \quad u|_{\partial B} = 0, \tag{4.26}$$

where $0 \leqslant m < 1$; see [Du2] for details. The case $m \geqslant 1$ was also discussed in [Du2], and the shape of the global positive solution curve for this case is very different. Equation (4.26) arises from combustion theory.

In catalysis theory, there is an equation closely related to (4.1); under some simple changes of variables, it reduces essentially to

$$-\Delta u = \lambda(1-\varepsilon u)^p e^{u/(1+\varepsilon u)}, \quad u|_{\partial B} = 0, \tag{4.27}$$

where p is a nonnegative integer (see, e.g., Aris [Ar], Vol. 1, Chapter 4). Let us note that when $p = 0$, (4.27) reduces to the perturbed Gelfand equation (4.1), and when $\varepsilon = 0$, (4.27) reduces to the Gelfand equation (4.2).

For (4.27), it has been conjectured that for any nonnegative integer p, the positive solution set $\{(\lambda, u)\}$ is S-shaped provided $\varepsilon > 0$ is small and the dimension $N = 1$ or 2. The conjecture was proved to be true for $N = 1$ by Hastings and McLeod [HM]. For $N = 2$ and $p > 0$, the conjecture is proved in [Da1] except for a small λ-range (with values close to 0); the higher-dimension case was also considered in [Da1].

In a series of papers, Korman, Li, Ouyang and Shi developed various useful techniques for proving exact multiplicity results for positive solutions to equations of the form

$$-\Delta u = \lambda f(u), \quad u|_{\partial B} = 0,$$

with several classes of nonlinearities; see [OS1,OS2] and the references therein. Our arguments in Section 4.1 basically follow their strategy. See also [Tang] for a more recent result along this line.

The abstract results discussed and used in Section 4.3 was further developed by Shi [S].

5. Nodal properties and global bifurcation

To a large extent, the approach in the previous section relies on the analysis of nodal properties of the solutions. In this section, we discuss some other aspects of the nodal properties and their applications in bifurcation theory.

Due to the uniqueness of initial value problems in ODEs, it is well known that the nodal properties of the solutions to a Sturm–Liouville problem for a second-order ordinary differential equation do not change when one moves along a global bifurcation branch of the solutions. This fact can be used to show that global bifurcation branches bifurcating from different eigenvalues do not meet each other and hence yielding multiple existence results; see [Ra], Theorem 2.3. This idea has been extended to elliptic partial differential equations over various two-dimensional domains with certain symmetries, in particular rectangles, by Kielhofer and his collaborators; see [K], Section III.6. In this section we look at two examples where nodal properties of solutions are used for systems of elliptic equations; here we consider radial solutions over circular domains. Let us note that, even for one-dimensional domains (i.e., intervals), the analysis of the nodal properties for systems is not trivial. It is an interesting problem to see how the techniques in [K] for equations over two-dimensional domains such as rectangles can be extended to systems (see [HK] for some efforts in this direction). Likewise, the techniques of our Section 4 may have applications to radial solutions for elliptic systems.

Our first example examines a competition system and is taken from Nakashima [N]; our second example studies a predator–prey system and follows Dancer, López-Gómez and Ortega [DLO].

5.1. *Global bifurcation of the competition system*

We consider the competition system with constant coefficients and Neumann boundary conditions

$$\begin{cases} -\Delta u = \lambda u(\alpha - u - \beta v), \\ -\Delta v = \lambda v(\gamma - v - \delta u), \\ \partial_\nu u|_{\partial\Omega} = \partial_\nu v|_{\partial\Omega} = 0, \end{cases} \tag{5.1}$$

where Ω is either a ball or an annulus: $\overline{\Omega} = \{x \in \mathbb{R}^N\colon\ R_0 \leqslant |x| \leqslant R_1\}$, $0 \leqslant R_0 < R_1$, $N \geqslant 1$. We assume throughout this subsection that the positive coefficients satisfy

$$\delta^{-1} < \frac{\alpha}{\gamma} < \beta. \tag{5.2}$$

This implies that for any $\lambda > 0$, the two semitrivial solutions $(\alpha, 0)$ and $(0, \gamma)$ are linearly stable, and (5.1) has a unique constant positive solution

$$(u^*, v^*) = \left(\frac{\beta\gamma - \alpha}{\beta\delta - 1}, \frac{\alpha\delta - \gamma}{\beta\delta - 1}\right),$$

which is linearly unstable. We are interested in the existence of other positive solutions of (5.1). We will make use of the nodal properties of radially symmetric positive solutions and global bifurcation theory to address this problem.

LEMMA 5.1. *Suppose that* $(u(x), v(x)) = (u(r), v(r))$ *is a radial, nonconstant positive solution of* (5.1), *and* $v'(r) \geqslant 0$ *for* $r \in (r_*, r^*) \subset (R_0, R_1)$, *and* $u'(r_*), u'(r^*) \leqslant 0$. *Then* $u'(r) < 0$ *in* (r_*, r^*).

Similarly, if $v'(r) \leqslant 0$ *for* $r \in (r_*, r^*) \subset (R_0, R_1)$, *and* $u'(r_*), u'(r^*) \geqslant 0$. *Then* $u'(r) > 0$ *in* (r_*, r^*).

Note that the roles of u and v in this lemma can be interchanged.

PROOF OF LEMMA 5.1. We only consider the first case; the proof for the second is parallel. Denote $f(u, v) = \lambda(\alpha - u - \beta v)$ and $g(u, v) = \lambda(\gamma - v - \delta u)$. We have

$$\begin{cases} u'' + \frac{N-1}{r}u' + uf(u, v) = 0, \\ v'' + \frac{N-1}{r}v' + vg(u, v) = 0, \\ u'(R_0) = u'(R_1) = v'(R_0) = v'(R_1) = 0. \end{cases} \tag{5.3}$$

Let $\phi(r) = u'(r)/u(r)$. We easily find

$$\phi' + \phi^2 + \frac{N-1}{r}\phi + f(u, v) = 0.$$

Differentiating this identity we obtain

$$\phi'' + \left(2\phi + \frac{N-1}{r}\right)\phi' + \left(uf_u(u, v) - \frac{N-1}{r^2}\right)\phi + f_v(u, v)v' = 0.$$

Since $f_u(u, v) < 0$, $f_v(u, v) < 0$ and $v' \geqslant 0$ in (r_*, r^*), we find that, for $r \in (r_*, r^*)$,

$$c(r) := uf_u(u, v) - \frac{N-1}{r^2} < 0$$

and

$$\phi'' + \left(2\phi + \frac{N-1}{r}\right)\phi' + c(r)\phi \geqslant 0.$$

Since $\phi(r) = u'(r)/u(r) \leqslant 0$ for $r \in \{r_*, r^*\}$, by the strong maximum principle (see [PW], p. 6, Theorem 3), we deduce either $\phi \equiv 0$ or $\phi < 0$ in (r_*, r^*). Clearly, to complete the proof of the lemma, it suffices to show that $\phi \equiv 0$ on (r_*, r^*) cannot happen. Arguing indirectly, we assume that $\phi \equiv 0$ on (r_*, r^*). Then $u' \equiv 0$ on (r_*, r^*) and by differentiating the equation for u in (5.3), we deduce $uf_v(u, v)v' \equiv 0$ on (r_*, r^*). This implies that $v' \equiv 0$ on (r_*, r^*). By the uniqueness of initial value problems for the ordinary differential equations of u' and v' we deduce that $u' = v' \equiv 0$ on (R_0, R_1), and hence (u, v) is a constant solution, contradicting our assumption. The proof is complete. □

We are now ready to give a good description of the nodal properties of the radial solutions to (5.1).

THEOREM 5.2. *Suppose that* $(u(x), v(x)) = (u(r), v(r))$ *is a radial, nonconstant positive solution of* (5.1). *Then the following hold*:

(i) $u'(r)$ *and* $v'(r)$ *have the same number of zeros in* (R_0, R_1), *and the number is finite.*

(ii) *If* $R_0 = s_0 < s_1 < \cdots < s_m = R_1$ *and* $R_0 = t_0 < t_1 < \cdots < t_m = R_1$ *are the zeros of* u' *and* v', *respectively, then for all possible* k,

$$s_{k-1} < t_k < s_{k+1}, \qquad t_{k-1} < s_k < t_{k+1}.$$

(iii) *The sign of* u' *in* (s_k, s_{k+1}) *is opposite to that of* v' *in* (t_k, t_{k+1}), *and* $u''(s_k) \neq 0$, $v''(t_k) \neq 0$.

PROOF. For the sake of clarity, we divide the proof into several steps. We assume that $f(u, v)$ and $g(u, v)$ are as defined in the proof of Lemma 5.1.

STEP 1. *The number of zeros of* u' *in* (R_0, R_1) *is finite, and the same holds for* v'.

We only prove the conclusion for u'. Suppose for contradiction that u' has infinitely many zeros in (R_0, R_1). Then we can find a limiting point of the zeros, say $r_0 \in [R_0, R_1]$. It follows that $u'(r_0) = u''(r_0) = 0$. From the equation for u we deduce that $f(u(r_0), v(r_0)) = 0$. (This is true even if $r_0 = 0$.) We now consider $v'(r_0)$. If $v'(r_0) \neq 0$, then by continuity there exists a small interval I having r_0 as an end point such that $v'(r) \neq 0$ in I and I contains a sequence of zeros of u' converging to r_0, say $\{r_k\}$. We now apply Lemma 5.1 to the interval with end points r_0 and r_1 and find that u' does not vanish and has the opposite sign to v' in this interval; this is a contradiction since infinitely many r_k's belong to this interval.

If $v'(r_0) = 0$ but $v''(r_0) \neq 0$, then we can find a small interval I as above and deduce the same contradiction. The only case left now is $v'(r_0) = v''(r_0) = 0$. In such a case, from the equation for v we obtain $g(u(r_0), v(r_0)) = 0$. Therefore, if we denote $(\xi, \eta) =$

$(u(r_0), v(r_0))$, then $(u, v) \equiv (\xi, \eta)$ is a solution to the differential equations in (5.3) with initial conditions $(u(r_0), v(r_0)) = (\xi, \eta)$, $(u'(r_0), v'(r_0)) = (0, 0)$. By uniqueness of this initial value problem (including the possible singular case $r_0 = 0$ and $N > 1$), we deduce that $(u, v) \equiv (\xi, \eta)$ in (R_0, R_1), a contradiction to our assumption. This finishes our proof of Step 1.

STEP 2. *If $r_1 < r_2 < r_3$ are three consecutive zeros of u', then $u'(r)$ changes sign as r crosses r_2; similarly, if $r_1 < r_2 < r_3$ are three consecutive zeros of v', then $v'(r)$ changes sign as r crosses r_2.*

Again we only prove the first conclusion; the proof for the second is parallel. Suppose that $u'(r)$ does not change sign at r_2. We assume for definiteness that $u'(r) > 0$ on both (r_1, r_2) and (r_2, r_3). We claim that there exists $s_1 \in (r_1, r_2)$ such that $v'(s_1) < 0$. Otherwise, we can apply Lemma 5.1 over (r_1, r_2) to deduce $u' < 0$ in (r_1, r_2). Similarly there exists $s_3 \in (r_2, r_3)$ such that $v'(s_3) < 0$. We now claim that there exists $s_2 \in (s_1, s_3)$ such that $v'(s_2) > 0$; otherwise, we can apply Lemma 5.1 over (s_1, s_3) to deduce $u' > 0$ over (s_1, s_3), contradicting the assumption that $u'(r_2) = 0$. Let (s_*, s^*) be the largest interval containing s_2 in (s_1, s_3) such that $v' > 0$ in (s_*, s^*) and $v'(s_*) = v'(s^*) = 0$. We now apply Lemma 5.1 over (s_*, s^*), where $u' \geqslant 0$, and deduce that $v' < 0$ in (s_*, s^*). This contradiction shows that $u'(r)$ changes sign at r_2. This proves Step 2.

STEP 3. *If $R_0 = s_0 < s_1 < \cdots < s_m = R_1$ and $R_0 = t_0 < t_1 < \cdots < t_n = R_1$ are the zeros of u' and v', respectively, then $m = n$ and for all possible k,*

$$s_{k-1} < t_k < s_{k+1}, \quad t_{k-1} < s_k < t_{k+1},$$

and the sign of u' in (s_k, s_{k+1}) is opposite to that of v' in (t_k, t_{k+1}).

For definiteness, we assume that $u'(r) > 0$ over (s_0, s_1). We show that $v'(r) < 0$ over (t_0, t_1). Suppose for contradiction that $v'(r) > 0$ in (t_0, t_1). If $s_1 \leqslant t_1$, then we apply Lemma 5.1 over (s_0, s_1) and deduce the contradiction that $u'(r) < 0$ in (s_0, s_1). If $s_1 > t_1$, then applying Lemma 5.1 over (t_0, t_1) we obtain $v'(r) < 0$ over (t_0, t_1), again a contradiction. Therefore we must have $v' < 0$ over (t_0, t_1). By Step 2 we now have

$$(-1)^k u'(r) > 0 \quad \forall r \in (s_k, s_{k+1}), \qquad (-1)^l v'(r) < 0 \quad \forall r \in (t_l, t_{l+1}). \tag{5.4}$$

We must have $s_1 < t_2$ for if $s_1 \geqslant t_2$ then we can apply Lemma 5.1 over (t_1, t_2) to deduce $v' < 0$ on (t_1, t_2). We can similarly prove that $s_2 < t_3$; for otherwise we can apply Lemma 5.1 over $(t_2, t_3) \subset (s_1, s_2)$ to deduce $v' > 0$ in (t_2, t_3). Continuing in this fashion, we can show that $s_k < t_{k+1}$ for all possible k, namely $k = 1, 2, \ldots, \min\{m, n-1\}$. If $n > m$, we already have a contradiction when we take $k = m$: $R_1 = s_m < t_{m+1}$. Hence we must have $n \leqslant m$.

In a similar fashion, we can show that $t_k < s_{k+1}$ for $k = 1, 2, \ldots, \min\{n, m-1\}$ and $m \leqslant n$. Therefore we must have $m = n$. This finishes the proof of Step 3.

STEP 4. *$u''(s_k) \neq 0$, $v''(t_k) \neq 0$ for $k = 0, 1, \ldots, m$.*

We will show that $u''(s_k) \neq 0$; the proof for $v''(t_k) \neq 0$ is similar. For definiteness, we assume that (5.4) holds. As in the proof of Lemma 5.1, we have, for $\phi = u'/u$,

$$\phi'' + \left(2\phi + \frac{N-1}{r}\right)\phi' + \left(uf_u(u,v) - \frac{N-1}{r^2}\right)\phi = -f_v(u,v)v'. \tag{5.5}$$

If $s_k = s_0 = R_0$, we find from (5.5) and (5.4) that

$$\phi'' + \left(2\phi + \frac{N-1}{r}\right)\phi' + \left(uf_u(u,v) - \frac{N-1}{r^2}\right)\phi < 0 \quad \forall r \in (R_0, t_1).$$

Since $\phi(r) > 0$ in (R_0, s_1) and $(2\phi + (N-1)/r) + (r - R_0)(uf_u(u,v) - (N-1)/r^2)$ is always bounded from below (even if $R_0 = 0$), we can apply Theorem 4 on page 7 of [PW] to conclude that $\phi'(R_0) > 0$, which implies $u''(R_0) > 0$. If $s_k = s_m = R_1$, the proof is similar.

Next we consider the case that $s_k \in (R_0, R_1)$. For definiteness we assume that $u' > 0$ in (s_k, s_{k+1}). If $t_k = s_k$, then the proof is similar to the case that $s_k = R_0$. Suppose now $t_k \neq s_k$. If $t_k > s_k$, then by what is proved in Steps 2 and 3 we must have $v' > 0$ in $(t_{k-1}, s_k]$, and we obtain from (5.5) that

$$\phi'' + \left(2\phi + \frac{N-1}{r}\right)\phi' + \left(uf_u(u,v) - \frac{N-1}{r^2}\right)\phi > 0 \quad \forall r \in (t_{k-1}, s_k).$$

Since $\phi < 0$ in (s_{k-1}, s_k), we can apply [PW], page 7, Theorem 4, on some $(s_k - \varepsilon, s_k)$ to obtain $\phi'(s_k) > 0$ and hence $u''(s_k) > 0$. If $t_k < s_k$, then $v' < 0$ in $[s_k, t_{k+1})$ and

$$\phi'' + \left(2\phi + \frac{N-1}{r}\right)\phi' + \left(uf_u(u,v) - \frac{N-1}{r^2}\right)\phi < 0 \quad \forall r \in [s_k, t_{k+1}).$$

Since $\phi > 0$ in (s_k, s_{k+1}), we obtain from [PW] that $\phi'(s_k) > 0$ and hence $u''(s_k) > 0$. This finishes the proof for Step 4 and hence completes the proof of Theorem 5.2. □

Following [N] we say a function pair $(u, v) \in C^2([R_0, R_1])^2$ has mode m if $u'(R_0) = u'(R_1) = 0$, $v'(R_0) = v'(R_1) = 0$ and (u', v') has the properties (i)–(iii) in Theorem 5.2; we denote

$$S_m := \left\{(u,v) \in C^2\big([R_0, R_1]\big)^2 \colon (u,v) \text{ has mode } m\right\}.$$

Clearly each S_m, $m = 1, 2, \ldots,$ is an open set in $C^2([R_0, R_1])$. We will use a global bifurcation argument to show that for any $m \geqslant 1$, (5.1) has a radial positive solution $(u, v) \in S_m$, provided that λ is large enough.

Let us now regard $\alpha, \beta, \gamma, \delta$ as fixed positive parameters satisfying (5.2) and regard λ as a bifurcation parameter. Clearly (λ, u^*, v^*) solves (5.1) for any λ. We will call this solution the *trivial solution*, and look for nontrivial positive solutions. We will restrict to the class of radial solutions and use a bifurcation approach. The linearized eigenvalue problem of (5.1) at (u^*, v^*) in the class of radial functions is given by

$$\begin{cases} \phi'' + (N-1)r^{-1}\phi' - \lambda\big(u^*\phi + \beta u^*\psi\big) = -\mu\phi, \\ \psi'' + (N-1)r^{-1}\psi' - \lambda\big(\delta v^*\phi + v^*\psi\big) = -\mu\psi, \\ \phi'(R_i) = \psi'(R_i) = 0, \quad i = 0, 1. \end{cases} \tag{5.6}$$

Denote

$$A = \begin{pmatrix} -u^* & -\beta u^* \\ -\delta v^* & -v^* \end{pmatrix}.$$

Due to (5.2), it is easily seen that A has a positive eigenvalue κ^+ and a negative eigenvalue κ^-. Therefore there exists a matrix P such that

$$P^{-1}AP = \begin{pmatrix} \kappa^- & 0 \\ 0 & \kappa^+ \end{pmatrix}, \quad \text{where } P = \begin{pmatrix} p_1^- & p_1^+ \\ p_2^- & p_2^+ \end{pmatrix}.$$

Define

$$\begin{pmatrix} \Phi \\ \Psi \end{pmatrix} = P^{-1} \begin{pmatrix} \phi \\ \psi \end{pmatrix};$$

then (5.6) becomes

$$\begin{cases} \Phi'' + (N-1)r^{-1}\Phi' + \lambda\kappa^-\Phi = -\mu\Phi, \\ \Psi'' + (N-1)r^{-1}\Psi' + \lambda\kappa^+\Psi = -\mu\Psi, \\ \Phi'(R_i) = \Psi'(R_i) = 0, \quad i = 0, 1. \end{cases} \tag{5.7}$$

The eigenvalues of (5.7) are easily found to be $\mu_n^-, \mu_n^+, n = 0, 1, \ldots,$ where

$$\mu_n^\pm = \mu_n^0 - \lambda\kappa^\pm,$$

and $0 = \mu_0^0 < \mu_1^0 < \mu_2^0 < \cdots$ are the eigenvalues of the problem

$$-\phi'' - (N-1)r^{-1}\phi' = \mu\phi, \quad \phi'(R_0) = \phi'(R_1) = 0. \tag{5.8}$$

Hence, zero is an eigenvalue of (5.7) exactly when

$$\lambda = \lambda_n^\pm := \frac{\mu_n^0}{\kappa^\pm}, \quad n = 0, 1, 2, \ldots,$$

with the corresponding eigenfunction $(\Phi_n, 0)$ for λ_n^-, and $(0, \Phi_n)$ for λ_n^+, where Φ_n is the nth eigenfunction of (5.8). It is well known that Φ_n' has exactly $n-1$ zeros in (R_0, R_1).

Returning to (5.6), we find that zero is an eigenvalue of (5.6) exactly when $\lambda = \lambda_n^{\pm}$, $n = 0, 1, 2, \ldots$, and the corresponding eigenfunctions are given by

$$\left(\phi_n^{\pm}, \psi_n^{\pm}\right) = \left(p_1^{\pm}\Phi_n, p_2^{\pm}\Phi_n\right),$$

where $p_1^{\pm}, p_2^{\pm}$ come from P and by a simple analysis one finds that $p_1^+ p_2^+ < 0$, $p_1^- p_2^- < 0$. Therefore $(\phi_n^{\pm}, \psi_n^{\pm}) \in S_{n+1}$.

The above discussions enable us to apply both Theorems 1.1 and 1.2 to a suitable abstract version of (5.1) in the space of radial functions to conclude the following theorem.

THEOREM 5.3. *Let $\mathcal{S}$ denote the closure of the radial nonconstant positive solutions* (λ, u, v) *of* (5.1) *in the space* $\mathbb{R}^1 \times C^2([R_0, R_1])^2$. *Then for each $n \geqslant 1$, $\mathcal{S}$ contains a maximal subcontinuum $\mathcal{S}_n$ such that*

(i) *$\mathcal{S}_n$ meets (λ_n^+, u^*, v^*);*

(ii) *$\mathcal{S}_n$ either is unbounded or meets $(\hat{\lambda}, u^*, v^*)$ for some $\hat{\lambda} \in \{\lambda_m^{\pm}\colon m = 0, 1, 2, \ldots\} \setminus \{\lambda_n^+\}$;*

(iii) *in a small neighborhood $O_n^{\pm}$ of $(\lambda_n^{\pm}, u^*, v^*)$ in $\mathbb{R}^1 \times C^2([R_0, R_1])^2$, all the radial solutions of* (5.1) *lie on a smooth curve $(\lambda(s), u(s), v(s))$, $|s| < s_0$, where*

$$\lambda(s) = \lambda_n^{\pm} + \mathrm{o}(s), \qquad \big(u(s), v(s)\big) = \big(u^*, v^*\big) + s\big(\phi_n^{\pm}, \psi_n^{\pm}\big) = \mathrm{o}(s).$$

Let us analyze $\mathcal{S}_n$ in more detail. By a well-known result of Conway, Hoff and Smoller [CHS], there exists $\lambda_0 > 0$ small enough such that for $\lambda \in (0, \lambda_0]$, (5.1) has only constant nonnegative solutions. We may assume that $\lambda_0 \in (0, \lambda_1^+)$.

We claim that any $(\lambda, u, v) \in \mathcal{S}_n \setminus \{(\lambda_n^+, u^*, v^*)\}$ satisfies $\lambda > \lambda_0$, $(u, v) \in S_{n+1}$ and $u(r) > 0$, $v(r) > 0$ in $[R_0, R_1]$. In other words,

$$\begin{aligned}&\big(\mathcal{S}_n \setminus \big\{\big(\lambda_n^+, u^*, v^*\big)\big\}\big)\\ &\quad\subset \Delta_n := \big\{(\lambda, u, v)\colon \lambda > \lambda_0, (u, v) \in S_{n+1}, u > 0, v > 0\big\}.\end{aligned} \tag{5.9}$$

Otherwise, by the connectedness of $\mathcal{S}_n$ and conclusions (i) and (iii) in Theorem 5.3, there exists some $(\lambda_*, u_*, v_*) \in (\mathcal{S}_n \setminus \{(\lambda_n^+, u^*, v^*)\}) \cap \partial\Delta_n$. Hence (u_*, v_*) is a nonnegative solution of (5.1) with $\lambda = \lambda_*$. If it is not a positive solution, then by the strong maximum principle, (u, v) has at least one component identically zero. This implies that $(u, v) \in \{(0, 0), (\alpha, 0), (0, \gamma)\}$. As bifurcation to positive solutions from these solutions is possible only at $\lambda = 0$ (due to (5.2)), we find this is a contradiction. Therefore (u_*, v_*) has to be a positive solution.

If $(u_*, v_*) = (u^*, v^*)$, then we can find $(\lambda_k, u_k, v_k) \in \Delta_n$ that converges to (λ_*, u^*, v^*) in $\mathbb{R}^1 \times C^2([R_0, R_1])^2$. By conclusion (iii) in Theorem 5.3, this is possible only if $\lambda_* = \lambda_n^{\pm}$. Since $\lambda_* \geqslant \lambda_0 > 0$ and $\lambda_n^- < 0$, $\lambda_* = \lambda_n^-$ is impossible. Since $(\lambda_*, u_*, v_*) \in (\mathcal{S}_n \setminus \{(\lambda_n^+, u^*, v^*)\})$, $\lambda_* = \lambda_n^+$ is also impossible. Therefore (u_*, v_*) must be a nonconstant positive solution of (5.1), and by the definition of λ_0, we know that $\lambda_* > \lambda_0$. Moreover, by Theorem 5.2, $(u_*, v_*) \in S_{m+1}$ for some $m \geqslant 0$. Since (u_*, v_*) can be approached in the C^2 norm by functions in S_{n+1}, we necessarily have $m = n$. But then (λ_*, u_*, v_*) belongs to the interior of Δ_n. This contradiction proves that (5.9) is true.

By (5.9) and conclusion (iii) in Theorem 5.3, we know that the second alternative in (ii) of Theorem 5.3 never occurs. Hence $\mathcal{S}_n$ is unbounded. A simple comparison argument shows that every nonnegative solution of (5.1) satisfies $u \leqslant \alpha$ and $v \leqslant \gamma$. Therefore by standard elliptic estimate we know that u and v have bounded C^2 norms as long as $\lambda > 0$ stays bounded. This implies that $\mathcal{S}_n$ can become unbounded only through $\lambda \to \infty$. To summarize, we have the following theorem.

THEOREM 5.4. *For each* $n \geqslant 1$, $(\lambda, u, v) \in \mathcal{S}_n \setminus \{(\lambda_n^+, u^*, v^*)\}$ *implies that* (u, v) *is a nonconstant positive radial solution of* (5.1); *moreover,* (u, v) *belongs to* S_{n+1}, *and the projection of* $\mathcal{S}_n$ *on* $\mathbb{R}^1$ *contains* (λ_n^+, ∞), *and is contained in* (λ_0, ∞) *with some* $\lambda_0 > 0$. *Hence for each* $\lambda > \lambda_m^+$, (5.1) *has at least* m *nonconstant positive radial solutions.*

REMARK 5.5. 1. The conclusion of Theorem 5.4 can be strengthened. In fact, each $\mathcal{S}_n$ can be decomposed into two unbounded subbranches $\mathcal{S}_n^+$ and $\mathcal{S}_n^-$ satisfying $\mathcal{S}_n^+ \cap \mathcal{S}_n^- = \{(\lambda_n^+, u^*, v^*)\}$. Hence (5.1) has at least $2m$ nonconstant positive solutions for $\lambda > \lambda_m^+$.

2. By considering the linearization of (5.1) at (u^*, v^*) in the space $C^2(\overline{\Omega})$, one easily sees that the linearized eigenvalue problem has eigenvalues corresponding to nonradially symmetric eigenfunctions. This fact can be used to construct examples where (5.1) has many nonradial positive solutions bifurcating from (u^*, v^*).

3. For large β and δ, one can also use the method in [DD1] to construct nonradially symmetric positive solutions for (5.1).

4. It is unclear whether $\bigcup_{n=1}^{\infty} \mathcal{S}_n$ contains "all" the radial nonconstant positive solutions of (5.1) (with $\lambda > 0$).

5. In Kan-on [Ka], it is shown that if $\alpha = \gamma$ and $\beta = \delta$ in (5.1), and if the space dimension is one, then each $\mathcal{S}_n$ is exactly "$\subset$"-shaped, and $\bigcup_{n=1}^{\infty} \mathcal{S}_n$ contains "all" the nonconstant positive solutions of (5.1) (with $\lambda > 0$).

6. If Ω is a ball (in general, if it is a convex set), then any nonconstant positive solution of (5.1) is unstable; see [KW].

5.2. *Uniqueness results for the predator–prey system*

We consider the Lotka–Volterra predator–prey system with constant coefficients and homogeneous Dirichlet boundary conditions

$$\begin{cases} -\Delta u = \lambda u - u^2 - cuv, \\ -\Delta v = \mu v - v^2 + duv, \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \end{cases} \tag{5.10}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$. It has long been conjectured that (5.10) has at most one positive solution, but the conjecture remains open except when the space dimension is one, namely when Ω is an interval.

In contrast, the existence problem for (5.10) is completely understood. In the following discussion, we will use some notations of Section 3. If $\lambda > \lambda_1^{\Omega}$, then $(\theta_\lambda, 0)$ solves (5.10).

Similarly, $(0, \theta_\mu)$ solves (5.10) when $\mu > \lambda_1^\Omega$. Suppose that (u_0, v_0) is a positive solution to (5.10). Then from the equation for u_0 we obtain

$$\lambda = \lambda_1^\Omega(u_0 + cv_0) > \lambda_1^\Omega.$$

Similarly, from the equation for v_0 it follows

$$\mu = \lambda_1^\Omega(v_0 - du_0).$$

A simple comparison argument shows that $u_0 \leqslant \theta_\lambda$ and $v_0 \geqslant \theta_\mu$; here we use the convention that $\theta_\mu = 0$ if $\mu \leqslant \lambda_1^\Omega$. Therefore

$$\mu \geqslant \lambda_1^\Omega(v_0 - d\theta_\lambda) > \lambda_1^\Omega(-d\theta_\lambda) := \mu_0.$$

Similarly,

$$\lambda \geqslant \lambda_1^\Omega(u_0 + c\theta_\mu) > \lambda_1^\Omega(c\theta_\mu).$$

Suppose $\lambda > \lambda_1^\Omega$ and let $\mu^0 > \lambda_1^\Omega$ be uniquely determined by

$$\lambda = \lambda_1^\Omega(c\theta_{\mu^0}).$$

We find from the above discussions that a necessary condition for (5.10) to possess a positive solution is that

$$\lambda > \lambda_1^\Omega, \qquad \mu_0 < \mu < \mu^0. \tag{5.11}$$

We show next that (5.11) is also sufficient for (5.10) to have a positive solution. To this end, we fix the positive coefficients λ, c, d and regard μ as a bifurcation parameter. Under (5.11), we can apply the local and global bifurcation analysis much as for the competition model in Section 3.1, to conclude that there is a global bifurcation branch $\Sigma = \{(\mu, u, v)\}$ of positive solutions to (5.10), which bifurcates from the semitrivial branch $\{(\mu, \theta_\lambda, 0)\}$ at $(\mu_0, \theta_\lambda, 0)$, and joins the other semitrivial branch $\{(\mu, 0, \theta_\mu)\}$ at $(\mu^0, 0, \theta_{\mu^0})$. Therefore, for any $\mu \in (\mu_0, \mu^0)$, (5.10) has at least one positive solution (u, v) such that $(\mu, u, v) \in \Sigma$.

In the following, we will make use of the nodal properties of certain differential systems to show that if Ω is an annulus,

$$\Omega = \{x \in \mathbb{R}^N \colon R_1 < |x| < R_2\}, \quad 0 < R_1 < R_2, \tag{5.12}$$

then the global bifurcation branch Σ consists of radially symmetric positive solutions only, and it is a smooth curve, and for each $\mu \in (\mu_0, \mu^0)$, there is exactly one positive solution (u, v) of (5.10) such that $(\mu, u, v) \in \Sigma$.

Suppose that (5.12) holds. Let us first observe that, by uniqueness, θ_λ and θ_μ are radially symmetric, and if we apply the local and global bifurcation analysis in the space of radially symmetric functions, we obtain a global branch of radially symmetric positive solutions of (5.10), which we denote by Σ_r, and Σ_r connects the semitrivial solution branches at $(\mu_0, \theta_\lambda, 0)$ and $(\mu^0, 0, \theta_{\mu^0})$, respectively. We will show that $\Sigma = \Sigma_r$. We do this by considering the linearization of (5.10) at an arbitrary radially symmetric positive solution $(u_0(x), v_0(x)) = (u_0(r), v_0(r))$

$$\begin{cases} -\Delta\phi = (\lambda - 2u_0 - cv_0)\phi - cu_0\psi, \\ -\Delta\psi = (\mu - 2v_0 + du_0)\psi + dv_0\phi, \\ \phi|_{\partial\Omega} = \psi|_{\partial\Omega} = 0. \end{cases} \tag{5.13}$$

THEOREM 5.6. *Suppose that* (5.12) *holds and* $(\phi, \psi) \in C^2(\overline{\Omega})^2$ *solves* (5.13). *Then* $(\phi, \psi) = (0, 0)$.

PROOF. We assume that $N \geqslant 2$; the case $N = 1$ can be proved by similar arguments and is simpler. Arguing indirectly, we assume that (5.13) has a solution $(\phi, \psi) \neq (0, 0)$. We are looking for a contradiction.

STEP 1. *Reduction to ODE systems by harmonic polynomials.*

We first use harmonic polynomials to reduce (5.13) to a sequence of ODE systems. (This is not needed if $N = 1$.)

Let $\mathcal{H}^n$ denote the space of homogeneous and harmonic polynomials of degree n in N variables. It is well known that the restriction of these polynomials to the unit sphere S^{N-1} are the eigenfunctions of the Laplace–Beltrami operator $\Delta_{S^{N-1}}$ corresponding to the eigenvalue $\lambda_n = -n(n + N - 2)$, $n = 0, 1, 2, \ldots$. Moreover, the following orthogonal decomposition holds

$$L^2(S^{N-1}) = \bigoplus_{n \geqslant 0} \mathcal{H}^n. \tag{5.14}$$

Given $P_n \in \mathcal{H}^n$ define

$$f(r) = \int_{S^{N-1}} \phi(r\xi) P_n(\xi)\, \mathrm{d}\sigma(\xi), \qquad g(r) = \int_{S^{N-1}} \psi(r\xi) P_n(\xi)\, \mathrm{d}\sigma(\xi).$$

Then $(f, g) \in C^2([R_1, R_2])^2$, and $f(R_1) = f(R_2) = 0$, $g(R_1) = g(R_2) = 0$. Moreover, since the Laplace operator in $\mathbb{R}^N$ can be expressed as

$$\Delta = \Delta_r + r^{-2}\Delta_{S^{N-1}},$$

where

$$r = |x|, \qquad \Delta_r = r^{1-N}\partial_r\big(r^{N-1}\partial_r\big),$$

if we multiply the differential equations in (5.13) by $P_n(\xi)$ and integrate on S^{N-1}, it results

$$\begin{cases} -\Delta_r f + n(n+N-2)r^{-2} f \\ \quad = (\lambda - 2u_0 - cv_0) f - cu_0 g, \quad r \in (R_1, R_2), \\ -\Delta_r g + n(n+N-2)r^{-2} g \\ \quad = (\mu - 2v_0 + du_0) g + dv_0 f, \quad r \in (R_1, R_2), \\ f(R_i) = g(R_i) = 0, \qquad i = 1, 2. \end{cases} \tag{5.15}$$

STEP 2. *There exists some* $P_n \in \mathcal{H}^n$, $n \geqslant 0$, *such that problem* (5.15) *has a solution* $(f, g) \neq (0, 0)$.

Otherwise, for all $P_n \in \mathcal{H}^n$, $n \geqslant 0$, (5.15) has only the solution $(0, 0)$. Hence for each $r \in (R_1, R_2)$, $\phi(r\xi)$ is orthogonal in $L^2(S^{N-1})$ to every $\mathcal{H}^n$, $n \geqslant 0$. It follows from (5.14) that $\phi(r\xi) = 0$. Hence $\phi \equiv 0$ in Ω. Similarly $\psi \equiv 0$. But this contradicts our assumption.

STEP 3. *The zeros of* f *and* g *in* $[R_1, R_2]$ *are isolated.*

We only prove the conclusion for f; that for g is similar. Suppose for contradiction that $r_0 \in [R_1, R_2]$ is an accumulating point of zeros of f in $[R_1, R_2]$. Then we easily see that $f(r_0) = f'(r_0) = 0$. If $g(r_0) = g'(r_0) = 0$, then we deduce $(f, g) \equiv (0, 0)$ by uniqueness of the initial value problem for (5.15). Hence either $g(r_0) \neq 0$ or $g'(r_0) \neq 0$. Therefore we can find an open interval $I \subset [R_1, R_2]$ with r_0 as an end point and a sequence of zeros $\{r_k\}$ of f such that $\{r_k\} \subset I$, $r_k \to r_0$ and $g(r) \neq 0$ for all $r \in I$. From the equation for u_0 we find $\lambda_1^{\Omega}(-\lambda + u_0 + cv_0) = 0$. It follows that

$$\lambda_1^{\Omega}\bigl(-\lambda + 2u_0 + cv_0 + n(n+N-2)r^{-2}\bigr) > 0.$$

If we denote $\Omega_1 = \{x \in \Omega\colon\ |x| \in I_1\}$, where I_1 denotes the interval with end points r_0 and r_1, we obtain

$$\begin{aligned} &\lambda_1^{\Omega_1}\bigl(-\lambda + 2u_0 + cv_0 + n(n+N-2)r^{-2}\bigr) \\ &\quad > \lambda_1^{\Omega}\bigl(-\lambda + 2u_0 + cv_0 + n(n+N-2)r^{-2}\bigr) > 0. \end{aligned}$$

This implies that the strong maximum principle is satisfied by the operator

$$L_1 := \bigl(-\Delta - \lambda + 2u_0 + cv_0 + n(n+N-2)r^{-2}\bigr)$$

over Ω_1: $L_1 u \geqslant 0$ in Ω_1 and $u \geqslant 0$ on $\partial\Omega_1$ imply $u > 0$ or $u \equiv 0$ in Ω_1.

If we identify $f(r)$ with $f(|x|)$, then from (5.15) we obtain $L_1 f = -cu_0 g$ and $f|_{\partial\Omega_1} = 0$. Therefore by the strong maximum principle we have either $f \equiv 0$ in Ω_1 or f does not vanish and has the opposite sign to g in Ω_1. If f is identically 0 then by (5.15) we must have g identically zero over Ω_1 and hence by uniqueness of the initial value problem for (5.15) we deduce $(f, g) = (0, 0)$ over Ω; a contradiction. If f does not vanish in Ω_1, then we arrive a contradiction with the fact that I_1 contains infinitely many zeros of $f(r)$. Therefore the zeros of f in $[R_1, R_2]$ must be isolated.

STEP 4. *Let $R_1 < \xi_1 < \cdots < \xi_p = R_2$ be the finite sequence of zeros of f where f changes sign, and assume that $f \geqslant 0$ in (R_1, ξ_1). Then*

$$(-1)^j g(\xi_j) > 0, \quad j = 1, \ldots, p. \tag{5.16}$$

Before starting the proof for Step 4, let us note that we do not lose generality by assuming $f \geqslant 0$ in (R_1, ξ_1); we may change (f, g) to $(-f, -g)$ otherwise. Note also that $(-1)^p g(\xi_p) > 0$ is a contradiction to the fact that $g(R_2) = 0$. Therefore this step finishes the proof of the theorem.

We now prove Step 4 by an induction argument. For $j = 1$, suppose for contradiction that $g(\xi_1) \geqslant 0$. Denote $\Omega_0 = \{x \in \Omega\colon\ R_1 < |x| < \xi_1\}$. From the equation for v_0 we find that $\lambda_1^{\Omega}(-\mu + v_0 + du_0) = 0$. Hence

$$\begin{aligned}&\lambda_1^{\Omega_0}\bigl(-\mu + 2v_0 + du_0 + n(n+N-2)r^{-2}\bigr)\\&\quad > \lambda_1^{\Omega}\bigl(-\mu + 2v_0 + du_0 + n(n+N-2)r^{-2}\bigr) > 0.\end{aligned}$$

It follows that the strong maximum principle holds over Ω_0 for the operator

$$L_2 := \bigl(-\Delta - \mu + 2v_0 + du_0 + n(n+N-2)r^{-2}\bigr).$$

It now follows from $L_2 g = dv_0 f \geqslant 0$ in Ω_0 and $g|_{\partial\Omega_0} \geqslant 0$ that $g > 0$ in Ω_0; note that by Step 3, g cannot be identically 0 in Ω_0.

Similarly we find that the strong maximum principle holds over Ω_0 for the operator L_1 used in the proof of Step 3. It then follows from $L_1 f = -cu_0 g < 0$ in Ω_0 and $f|_{\partial\Omega_0} = 0$ that $f < 0$ in Ω_0. This contradiction proves (5.16) for $j = 1$.

Suppose (5.16) holds for some $i \geqslant 1$. If i is odd, then $g(\xi_i) < 0$ and $f \leqslant 0$ in (ξ_i, ξ_{i+1}). Suppose for contradiction that $g(x_{i+1}) \leqslant 0$. Denote $\Omega_i = \{x \in \Omega\colon\ \xi_1 < |x| < \xi_{i+1}\}$. By the same reasoning as before, the strong maximum principle is satisfied by L_1 and L_2 over Ω_i. We have $L_2 g = dv_0 f \leqslant 0$ in Ω_i, and $g|_{\partial\Omega_i} \leqslant 0$. Hence $g < 0$ in Ω_i. It follows that $L_1 f = -cu_0 g > 0$ in Ω_i, $f|_{\partial\Omega_i} = 0$; hence $f > 0$ in Ω_i. This contradiction shows that we must have $g(\xi_{i+1}) > 0$. If i is even, the proof is similar. Therefore (5.16) holds for all $1 \leqslant j \leqslant p$. The proof is complete. □

We are now ready to prove the following result.

THEOREM 5.7. *Suppose that* (5.12) *holds. Then $\Sigma = \Sigma_r$. Moreover, Σ is a smooth curve which can be parameterized by $\mu \in (\mu_0, \mu^0)$, and Σ contains all the radially symmetric positive solutions of* (5.10). *Therefore for each $\mu \in (\mu_0, \mu^0)$,* (5.10) *has a unique radially symmetric positive solution.*

PROOF. By the local bifurcation analysis near $(\mu_0, \theta_\lambda, 0)$ and $(\mu^0, 0, \theta_{\mu^0})$, Σ and Σ_r coincide and consist of smooth curves near these points. Away from these two points, by

Theorem 5.6, we can apply the implicit function theorem to conclude that Σ_r is a smooth curve which can be parameterized by μ.

Suppose for contradiction that $\Sigma \neq \Sigma_r$. Since they agree near $(\mu_0, \theta_\lambda, 0)$ and $(\mu^0, 0, \theta_{\mu^0})$, we can find some $(\mu, u, v) \in \Sigma_r$ which can be approached by a sequence $(\mu_k, u_k, v_k) \in \Sigma \setminus \Sigma_r$. But this is impossible since near (μ, u, v), due to Theorem 5.6, we can apply the implicit function theorem to conclude that all the solutions to (5.10) form a smooth curve parameterized by μ, which necessarily agrees with Σ_r. This proves $\Sigma = \Sigma_r$.

Suppose now $(\hat{\mu}, \hat{u}, \hat{v})$ is an arbitrary radially symmetric positive solution of (5.10). By Theorem 5.6 we can apply the implicity function theorem in the space of radially symmetric functions to conclude that, near $(\hat{\mu}, \hat{u}, \hat{v})$, all the radially symmetric positive solutions of (5.10) form a smooth curve parameterized by μ. Let Σ' denotes a maximal such curve and define $\mu_* = \{\mu\colon\ (\mu, u, v) \in \Sigma'\}$. By (5.11), we necessarily have $\mu_* \in [\mu_0, \mu^0)$. If $\mu_* > \mu_0$, then we can easily deduce that there exists some $(\mu_*, u_*, v_*) \in \Sigma'$, and we can apply the implicit function theorem to continue the curve Σ' to $\mu < \mu_*$, contradicting the definition of μ_*. Hence we must have $\mu_* = \mu_0$. We now can use the equations in (5.10) to easily show that $(\mu_0, \theta_\lambda, 0)$ is an accumulating point of Σ'. By the local bifurcation analysis near $(\mu_0, \theta_\lambda, 0)$, we see that Σ' agrees with Σ_r near this point. Now $\Sigma' = \Sigma_r$ can be proved in the same way as we prove $\Sigma = \Sigma_r$. □

REMARK 5.8. 1. In Theorems 5.6 and 5.7, the restriction that the coefficients in (5.10) are constants is not necessary; they can be radially symmetric functions. Moreover, these results remain true if Ω is a ball. See [DLO] for details.

2. In the one-dimensional case, the method of using nodal properties of the solutions to (5.13) to prove Theorem 5.6 and hence the uniqueness of positive solutions to (5.10) seems first used by López-Gómez and Pardo [LP1]. This method can be extended to similar problems with general homogeneous boundary conditions, see [LP2]. In [Hs], a related method was used to discuss a problem with homogeneous Neumann and inhomogeneous Dirichlet boundary conditions.

3. Whether the unique radially symmetric positive solution of (5.10) is linearly stable is an open problem. Theorem 5.6 rules out the possibility of symmetry breaking bifurcation, but it does not rule out the possibility of Hopf bifurcation along Σ_r. In [Y], by a local bifurcation analysis, it was shown that the positive solutions are linearly stable near the semitrivial solutions.

4. If we replace the Dirichlet boundary conditions in (5.10) by Neumann boundary conditions, then it is known that the positive constant solution is a global attractor of the corresponding parabolic system with any positive initial data; this can be proved by a Lyapunov function argument and relies on the assumption that the coefficients are constants, but it works for an arbitrary Ω; see [dMR].

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CHAPTER 4

Metasolutions: Malthus versus Verhulst in Population Dynamics. A Dream of Volterra

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Contents

Abstract

This paper analyzes how the study of the interplay within the same habitat between the most classical laws of Population Dynamics is originating the *modern theory of nonlinear parabolic differential equations*, where *metasolutions* are imperative for ascertaining the dynamics in the regimes where they cannot be described with classical solutions.

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1. Introduction

Our attention in this monograph is focused into the problem of ascertaining the asymptotic behavior of the solutions of

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = \lambda u - a(x) f(x,u) u & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(\cdot,0) = u_0 > 0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geqslant 1$, with smooth boundary $\partial\Omega$, e.g., of class $\mathcal{C}^3$, $\lambda \in \mathbb{R}$, and $a \geqslant 0$, $a \neq 0$, is a function of class $\mathcal{C}^\mu(\bar{\Omega})$, for some $\mu \in (0,1]$, satisfying the following hypotheses:

(Aa) The set

$$\Omega_- := \{x \in \Omega: \ -a(x) < 0\}$$

is a subdomain of Ω with $\bar{\Omega}_- \subset \Omega$, whose boundary, $\partial\Omega_-$, is of class $\mathcal{C}^3$, and the open set

$$\Omega_0 := \Omega \setminus \bar{\Omega}_-$$

consists of two components, $\Omega_{0,i}$, $i \in \{1,2\}$, such that

$$\bar{\Omega}_{0,1} \cap \bar{\Omega}_{0,2} = \emptyset, \qquad \bar{\Omega}_{0,2} \subset \Omega,$$

and

$$\sigma_1 := \sigma[-\Delta, \Omega_{0,1}] < \sigma_2 := \sigma[-\Delta, \Omega_{0,2}]. \tag{1.2}$$

Throughout this paper, given a regular subdomain D of Ω and $V \in C(\bar{D})$, we denote by $\sigma[-\Delta + V, D]$ the principal eigenvalue of $-\Delta + V$ in D under homogeneous Dirichlet boundary conditions. Figure 1 represents a typical situation where assumption (Aa) is fulfilled. In Figure 1 we have denoted

$$\Gamma = \partial\Omega, \qquad \Gamma_1 = \partial\Omega_{0,1} \setminus \Gamma, \qquad \Gamma_2 := \partial\Omega_{0,2}, \qquad \partial\Omega_- = \Gamma_1 \cup \Gamma_2.$$

Thanks to Faber–Krahn inequality, (1.2) is reached if $\Omega_{0,2}$ has a sufficiently small Lebesgue measure (e.g., [43], Section 5 and [10], Section 10). Actually, one might think of (1.2) as a sort of hierarchical ordering size between the components $\Omega_{0,1}$ and $\Omega_{0,2}$ establishing that $\Omega_{0,1}$ is larger than $\Omega_{0,2}$, though one should take into account that $\sigma[-\Delta; D]$ can depend on certain hidden geometrical properties of D. Setting

$$\sigma_0 := \sigma[-\Delta, \Omega],$$

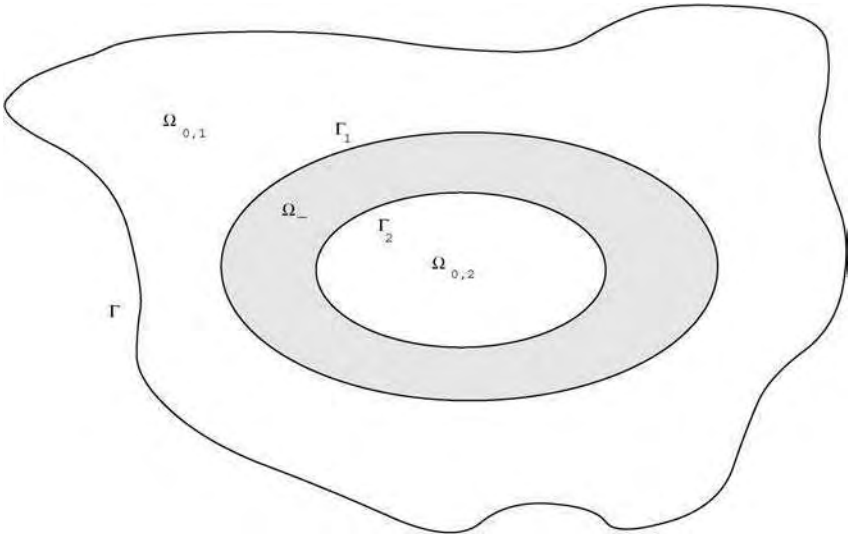

Fig. 1. Nodal configuration induced by $a(x)$.

it is apparent, from (1.2) and the monotonicity of the principal eigenvalue with respect to the domain, that

$$\sigma_0 < \sigma_1 < \sigma_2. \tag{1.3}$$

As for the function $f(x,u)$, we suppose the following:

(Af) $f \in \mathcal{C}^{\mu,1+\mu}(\overline{\Omega} \times [0,\infty))$ satisfies

$$f(x,0) = 0 \quad \text{and} \quad \partial_u f(x,u) > 0 \quad \text{for all } x \in \overline{\Omega} \text{ and } u > 0.$$

(Ag) There exists $g \in \mathcal{C}^{1+\mu}([0,\infty))$ such that

$$g(0) = 0, \qquad g(u) > 0 \quad \text{and} \quad g'(u) > 0 \quad \text{for all } u > 0, \qquad \lim_{u \uparrow \infty} g(u) = \infty,$$

where "$'$" denotes $\frac{\mathrm{d}}{\mathrm{d}u}$, and

$$f(\cdot,u) \geqslant g(u) \quad \text{if } u \geqslant 0.$$

Note that (Af), (Ag) imply

$$\lim_{u \uparrow \infty} f(x,u) = \infty \quad \text{uniformly in } x \in \overline{\Omega}.$$

Subsequently, we assume (Af), (Ag), and for every constant $\Lambda > 0$ and compact set $K \subset \Omega_-$ we consider the auxiliary function

$$h_{K,\Lambda}(u) := a_{L,K}\, g(u)u - \Lambda u, \quad u \in [0,\infty), \tag{1.4}$$

where

$$a_{L,K} = \min_K a > 0.$$

Under assumption (Ag), it is apparent that $h_{K,\Lambda}$ has a unique positive zero. Let denote it by $u_{K,\Lambda}$. The following assumption is crucial to get uniform a priori estimates for the solution of (1.1) within Ω_-:

(Ah) For every pair (K, Λ) and $u > u_{K,\Lambda}$

$$I(u) := \int_u^\infty \left[\int_u^s h_{K,\Lambda}(z)\,\mathrm{d}z \right]^{-1/2} \mathrm{d}s < \infty \tag{1.5}$$

and

$$\lim_{u \uparrow \infty} I(u) := 0. \tag{1.6}$$

Assumption (Ah) holds if

$$g(u) = \eta u^{p-1}, \quad u \geqslant 0,$$

for some $\eta > 0$ and $p > 1$. Indeed, in such case,

$$h_{K,\Lambda}(u) := a_{L,K}\, g(u) u - \Lambda u = a_{L,K}\, \eta u^p - \Lambda u, \quad u \geqslant 0,$$

and hence,

$$u_{K,\Lambda} = \left(\frac{\Lambda}{a_{L,K}\, \eta} \right)^{1/(p-1)}.$$

Thus, for each $s > u > u_{K,\Lambda}$,

$$\begin{aligned} \int_u^s h_{K,\Lambda}(z)\,\mathrm{d}z &= \int_u^s \left(a_{L,K}\, \eta z^p - \Lambda z \right) \mathrm{d}z \\ &= \frac{a_{L,K}\, \eta}{p+1} \left(s^{p+1} - u^{p+1} \right) - \frac{\Lambda}{2} \left(s^2 - u^2 \right) \end{aligned}$$

and therefore, by performing the change of variable $s = u\theta$, we find that

$$\begin{aligned} I(u) &= \int_u^\infty \left[\frac{a_{L,K}\, \eta}{p+1} \left(s^{p+1} - u^{p+1} \right) - \frac{\Lambda}{2} \left(s^2 - u^2 \right) \right]^{-1/2} \mathrm{d}s \\ &= \int_1^\infty \left[\frac{a_{L,K}\, \eta u^{p-1}}{p+1} \left(\theta^{p+1} - 1 \right) - \frac{\Lambda}{2} \left(\theta^2 - 1 \right) \right]^{-1/2} \mathrm{d}\theta \\ &< \infty, \end{aligned} \tag{1.7}$$

since the function $R(\theta)$ defined by

$$R(\theta) := \frac{a_{L,K}\eta u^{p-1}}{p+1}\left(\theta^{p+1}-1\right) - \frac{\Lambda}{2}\left(\theta^2-1\right), \quad \theta \geqslant 0,$$

satisfies

$$R(1)=0, \qquad R'(1) = a_{L,K}\eta u^{p-1} - \Lambda > a_{L,K}\eta u_{K,\Lambda}^{p-1} - \Lambda = 0$$

and

$$\lim_{\theta\uparrow\infty} \frac{R(\theta)}{\theta^{p+1}} = \frac{a_{L,K}\eta u^{p-1}}{p+1} > 0.$$

Moreover, it is easy to see that (1.7) implies (1.6). Therefore, (Ah) holds true for this choice of g.

In Ecology, (1.1) models the evolution of the distribution of a single species $u(x,t)$ randomly dispersed in the inhabiting area Ω, where λ represents the intrinsic growth rate of u and $a(x)$ measures the crowding effects of the population in Ω_-. In Ω_0, u is allowed to enjoy exponential growth according to the Malthus law. In our setting, the inhabiting area Ω is fully surrounded by completely hostile regions, because of the homogeneous Dirichlet boundary conditions on $\partial\Omega$. The function $u_0 \in \mathcal{C}(\bar{\Omega})$, $u_0 > 0$, represents the initial population distribution. Consequently, (1.1) can be viewed as a sort of intermediate prototype model linking the Malthus and the Verhulst laws of population dynamics within the same inhabiting region. Indeed, if $f(x,u)=u$, $\Omega_- = \Omega$ and $a(x)>0$ for each $x \in \partial\Omega$, then (1.1) provides us with the classical spatial logistic equation, while it provides us with the classical spatial Malthus equation if $\Omega_- = \emptyset$. Our main goal in this work is ascertaining the interplay between these two angular laws of population dynamics when they arise simultaneously in a heterogeneous environment of the type illustrated in Figure 1. Although in the classical cases when $\Omega_- \in \{\emptyset, \Omega\}$ the dynamics of (1.1) is governed by the nonnegative steady-states of (1.1), i.e., by the nonnegative solutions of

$$\begin{cases} -\Delta u = \lambda u - af(\cdot,u)u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

in our general setting a new class of *nonclassical nondistributional generalized steady-states* must be incorporated to the mathematical analysis of the problem in order to describe the asymptotic profiles of the population as time passes by. Namely, the *metasolutions*. Roughly spoken, the metasolutions of (1.1) are the extensions by infinity of the *explosive solutions*, or *large solutions*, of

$$-\Delta u = \lambda u - af(\cdot,u)u \tag{1.9}$$

in $D \in \{\Omega \setminus \bar{\Omega}_{0,1}, \Omega_-\}$. From the biological point of view, the main results of this paper can be shortly summarized as follows:

- The inhabiting region Ω cannot support the species u if $\lambda \leqslant \sigma_0$.
- The species u grows according to the Verhulst law if $\sigma_0 < \lambda < \sigma_1$.
- The species u grows according to the Malthus law in $\overline{\Omega}_{0,1}$, while it is governed by the Verhulst law in $\Omega \setminus \overline{\Omega}_{0,1}$ if $\sigma_1 \leqslant \lambda < \sigma_2$.
- The species u grows according to the Verhulst law in Ω_-, while it exhibits Malthusian growth in $\Omega \setminus \overline{\Omega}_-$ if $\lambda \geqslant \sigma_2$.

Consequently, the nature of the evolution of a single randomly distributed spatial species in a heterogeneous environment might suffer drastic changes according to the size of its intrinsic growth rate, evolving from extinction and logistic growth up to exhibit a genuine exponential growth within the most favorable regions, as in such cases most of the individuals of the population tend to abandon the most hostile areas for colonizing the *favorable regions* where natural resources are almost unlimited. A most detailed discussion, once stated the main mathematical results of this monograph, will be carried out in Section 2.

Besides the huge intrinsic interest of analyzing (1.1), as a result of the wide variety of its applications in the applied sciences and engineering, analyzing (1.1) is imperative as well from the point of view that it is a cell model for designing more sophisticated – so, more realistic – multispecies interacting models. Actually, it has been recently shown that the asymptotic profiles of the solutions of (1.1) also provide us with the dynamics of the positive solutions of large classes of superlinear indefinite problems, where $a(x)$ changes of sign (cf. Section 8). Therefore, the theory of metasolutions developed in this monograph should be a milestone to generate a great variety of new mathematical results in analyzing the effects of spatial heterogeneities in Chemistry, Biology, Ecology, Economy and Physics, so tremendously facilitating the understanding of the role of spatial heterogeneities in the formation and diversity of the Universe.

Throughout this paper, given a subdomain $D \subset \Omega$ and a function $u \in \mathcal{C}(\overline{D})$, it is said that $u > 0$ if $u \geqslant 0$ and $u \neq 0$. Accordingly, given $u, v \in \mathcal{C}(\overline{D})$, it is said that $u > v$ if $u - v > 0$. Also, given $u \in \mathcal{C}^1(\overline{D})$, it is said that $u \gg 0$ if $u(x) > 0$ for each $x \in D$ and $\frac{\partial u}{\partial n_x}(x) < 0$ for each $x \in \partial D \cap u^{-1}(0)$, where n_x stands for the outward unit normal of D at $x \in \partial D$. Accordingly, given $u, v \in \mathcal{C}^1(\overline{D})$, it is said that $u \gg v$ if $u - v \gg 0$.

Under our regularity assumptions, (1.1) possesses a unique smooth classical solution

$$u(x,t) := u_{[\lambda,\Omega]}(x,t;u_0)$$

globally defined in time since

$$-af(\cdot,u)u \leqslant 0.$$

Moreover, by the parabolic maximum principle, $u(\cdot,t) \gg 0$ for each $t > 0$, since $u_0 > 0$. The main goal of this work is ascertaining the behavior of the population distribution as time passes by, i.e., characterizing the limit

$$L := \lim_{t\uparrow\infty} u(\cdot,t), \tag{1.10}$$

if it exists, according to the several ranges of values where the intrinsic growth rate λ varies. In terms of L, the main results of this paper can be listed as follows:

- $L = 0$ if $\lambda \leqslant \sigma_0$.
- $0 < L < \infty$ if $\sigma_0 < \lambda < \sigma_1$.
- $L = \infty$ in $\bar{\Omega}_{0,1} \setminus \partial\Omega$ and $0 < L < \infty$ in $\Omega \setminus \bar{\Omega}_{0,1}$ if $\sigma_1 \leqslant \lambda < \sigma_2$.
- $L = \infty$ in $\Omega \setminus \Omega_-$ and $0 < L < \infty$ in Ω_- if $\lambda \geqslant \sigma_2$.

This monograph is distributed as follows. In Section 2 we state the main results and discuss their biological meaning. In Section 3 we collect the characterization of the strong maximum principle found by López-Gómez and Molina-Meyer in [53] and characterize the dynamics of a class of general parabolic problems related to (1.1). These results are crucial in the subsequent mathematical analysis. In Section 4 we give some preliminary results that will be necessary to prove the results of Section 2, among them count some classical problems of logistic type and some substantial improvements of the classical uniform a priori bounds of Keller [38] and Osserman [61]. In Sections 5–7 we will prove the results of Section 2. Finally, in Section 8 we describe the genesis and evolution of the mathematical theory of metasolutions and discuss some further, very recent, applications to porous media and indefinite superlinear problems.

Throughout this paper, given two real Banach spaces X and Y and a linear continuous operator between X and Y, say L, $N[L]$ and $R[L]$ will stand for the null space – kernel – and the range – image – of L.

2. The main results

Throughout this section we will assume that $a(x)$ satisfies (Aa) and that f satisfies (Af), (Ag) and (Ah), though some of the results might be valid under much weaker assumptions. Then, any weak solution of (1.9) in a smooth subdomain $D \subset \Omega$ must live in $\mathcal{C}^{2+\mu}(D)$. The solution of (1.1) satisfies

$$u \in \mathcal{C}^{2+\mu,1+\frac{\mu}{2}}\big(\bar{\Omega} \times (0,\infty)\big).$$

Throughout this paper, given a smooth subdomain $D \subset \Omega$ and $M \in [0,\infty]$, we consider the family of elliptic boundary value problems

$$\begin{cases} -\Delta u = \lambda u - a(x) f(x,u) u & \text{in } D, \\ u = M & \text{on } \partial D. \end{cases} \tag{2.1}$$

Any weak solution u of (2.1) must satisfy $u \in \mathcal{C}^{2+\mu}(\bar{D})$ if $0 \leqslant M < \infty$, while, in case $M = \infty$, a function $u \in \mathcal{C}^{2+\mu}(D)$ is said to be a solution of (2.1) if it satisfies the differential equation in D and

$$\lim_{\substack{x \in D \\ \operatorname{dist}(x,\partial D) \downarrow 0}} u(x) = \infty. \tag{2.2}$$

Throughout this paper, such solutions are called *large solutions* – or *explosive solutions* – of (1.9) in D (cf. [6,57] and the references therein).

The following result characterizes the existence of large solutions of (1.9) in Ω_-. These solutions will provide us with the limiting profiles of the solutions of (1.1) when $\lambda \geqslant \sigma_2$.

THEOREM 2.1. *Suppose $M=\infty$ and $D=\Omega_-$. Then, for each $\lambda\in\mathbb{R}$,* (2.1) *possesses a minimal and a maximal positive solution, denoted by $L^{\min}_{[\lambda,\Omega_-]}$ and $L^{\max}_{[\lambda,\Omega_-]}$, respectively – in the sense that any other positive solution L of* (2.1) *must satisfy*

$$L^{\min}_{[\lambda,\Omega_-]}\leqslant L\leqslant L^{\max}_{[\lambda,\Omega_-]}. \tag{2.3}$$

The following result characterizes the existence of large solutions of (1.9) in $\Omega\setminus\bar{\Omega}_{0,1}$. These solutions will provide us with the dynamics of (1.1) for the range $\sigma_1\leqslant\lambda<\sigma_2$.

THEOREM 2.2. *Suppose $M=\infty$ and*

$$D=\Omega\setminus\bar{\Omega}_{0,1}=\Omega_-\cup\bar{\Omega}_{0,2}.$$

Then (2.1) *possesses a positive solution if and only if $\lambda<\sigma_2$. Moreover, for each $\lambda<\sigma_2$,* (2.1) *has a minimal and a maximal solution, denoted by $L^{\min}_{[\lambda,\Omega\setminus\bar{\Omega}_{0,1}]}$ and $L^{\max}_{[\lambda,\Omega\setminus\bar{\Omega}_{0,1}]}$, respectively – in the sense that any other solution L of* (2.1) *must satisfy*

$$L^{\min}_{[\lambda,\Omega\setminus\bar{\Omega}_{0,1}]}\leqslant L\leqslant L^{\max}_{[\lambda,\Omega\setminus\bar{\Omega}_{0,1}]}. \tag{2.4}$$

Moreover, there exists a large solution of equation

$$-\Delta u=\sigma_2 u-af(\cdot,u)u$$

in Ω_-, say $L_{[\sigma_2,\Omega_-]}$, such that

$$\lim_{\lambda\uparrow\sigma_2}L^{\min}_{[\lambda,\Omega\setminus\bar{\Omega}_{0,1}]}=\begin{cases}\infty & \text{in } \bar{\Omega}_{0,2},\\ L_{[\sigma_2,\Omega_-]} & \text{in } \Omega_-.\end{cases} \tag{2.5}$$

The following result characterizes the existence of positive solutions of (1.8), which equals (2.1) if $M=0$ and $D=\Omega$. These solutions will provide us with the dynamics of (1.1) within the range $\sigma_0<\lambda<\sigma_1$.

THEOREM 2.3. *Problem* (1.8) *has a positive solution if and only if*

$$\sigma_0<\lambda<\sigma_1.$$

Moreover, it is unique, if it exists, and if we denote it by $\theta_{[\lambda,\Omega]}$, then

$$\lim_{\lambda\downarrow\sigma_0}\theta_{[\lambda,\Omega]}=0\quad \text{in } \mathcal{C}(\bar{\Omega}), \tag{2.6}$$

and

$$\lim_{\lambda\uparrow\sigma_1}\theta_{[\lambda,\Omega]}=\begin{cases}\infty & \text{in } \bar{\Omega}_{0,1}\setminus\partial\Omega,\\ L^{\min}_{[\sigma_1,\Omega\setminus\bar{\Omega}_{0,1}]} & \text{in } \Omega\setminus\bar{\Omega}_{0,1}.\end{cases} \tag{2.7}$$

Furthermore, the map

$$\begin{aligned} &(\sigma_0, \sigma_1) \xrightarrow{\theta} \mathcal{C}\big(\overline{\Omega}\,\big), \\ &\lambda \mapsto \theta(\lambda) := \theta_{[\lambda,\Omega]} \end{aligned} \tag{2.8}$$

is point-wise increasing and of class $\mathcal{C}^1$.

The following result provides us with the dynamics of (1.1).

THEOREM 2.4. *Let* $u(x,t) := u_{[\lambda,\Omega]}(x,t;u_0)$ *denote the unique solution of* (1.1). *Then*

(a) $\lim_{t\uparrow\infty} u(\cdot,t) = 0$ *in* $\mathcal{C}(\overline{\Omega})$ *if* $\lambda \leqslant \sigma_0$.

(b) $\lim_{t\uparrow\infty} u(\cdot,t) = \theta_{[\lambda,\Omega]}$ *in* $\mathcal{C}(\overline{\Omega})$ *if* $\sigma_0 < \lambda < \sigma_1$.

(c) *In case* $\sigma_1 \leqslant \lambda < \sigma_2$, *the following assertions are true*:

 (i) $\lim_{t\uparrow\infty} u(\cdot,t) = \infty$ *uniformly in compact subsets of* $\overline{\Omega}_{0,1} \setminus \partial\Omega$.

 (ii) *In* $\Omega \setminus \overline{\Omega}_{0,1}$ *the following estimate is satisfied*

$$L^{\min}_{[\lambda,\Omega\setminus\overline{\Omega}_{0,1}]} \leqslant \liminf_{t\uparrow\infty} u(\cdot,t) \leqslant \limsup_{t\uparrow\infty} u(\cdot,t) \leqslant L^{\max}_{[\lambda,\Omega\setminus\overline{\Omega}_{0,1}]}.$$

 (iii) *If, in addition,* u_0 *is a subsolution of* (1.9) *in* Ω, *then*

$$\lim_{t\uparrow\infty} u(\cdot,t) = L^{\min}_{[\lambda,\Omega\setminus\overline{\Omega}_{0,1}]} \quad \text{in } \Omega \setminus \overline{\Omega}_{0,1}.$$

(d) *In case* $\lambda \geqslant \sigma_2$, *the following assertions are true*:

 (i) $\lim_{t\uparrow\infty} u(\cdot,t) = \infty$ *uniformly in compact subsets of* $\Omega \setminus \Omega_-$.

 (ii) *In* Ω_- *the following estimate is satisfied*

$$L^{\min}_{[\lambda,\Omega_-]} \leqslant \liminf_{t\uparrow\infty} u(\cdot,t) \leqslant \limsup_{t\uparrow\infty} u(\cdot,t) \leqslant L^{\max}_{[\lambda,\Omega_-]}.$$

 (iii) *If, in addition,* u_0 *is a subsolution of* (1.9) *in* Ω, *then*

$$\lim_{t\uparrow\infty} u(\cdot,t) = L^{\min}_{[\lambda,\Omega_-]} \quad \text{in } \Omega_-.$$

The statements of Theorem 2.4(c) and (d) can be substantially shortened by introducing the following concept.

DEFINITION 2.5. Suppose $M = \infty$ and $D \in \{\Omega \setminus \overline{\Omega}_{0,1}, \Omega_-\}$. Then, a function $\mathfrak{M} : \Omega \to [0,\infty]$ is said to be a metasolution of (1.9) supported in D if there exists a large solution L of (1.9) in D for which

$$\mathfrak{M} = \begin{cases} \infty & \text{in } \Omega \setminus D, \\ L & \text{in } D. \end{cases}$$

The metasolution $\mathfrak{M}$ is said to be the minimal (resp. maximal) metasolution of (1.9) in D if L is the minimal (resp. maximal) large solution in D. The minimal and the maximal metasolution of (1.9) in D are throughout denoted by $\mathfrak{M}^{\min}_{[\lambda,D]}$ and $\mathfrak{M}^{\max}_{[\lambda,D]}$, respectively.

Using Definition 2.5, Theorem 2.4(c) shows that, if $\sigma_1 \leqslant \lambda < \sigma_2$,

$$\mathfrak{M}^{\min}_{[\lambda,\Omega\setminus\bar{\Omega}_{0,1}]} \leqslant \liminf_{t\uparrow\infty} u(\cdot,t) \leqslant \limsup_{t\uparrow\infty} u(\cdot,t) \leqslant \mathfrak{M}^{\max}_{[\lambda,\Omega\setminus\bar{\Omega}_{0,1}]} \quad \text{in } \Omega,$$

and actually,

$$\lim_{t\uparrow\infty} u(\cdot,t) = \mathfrak{M}^{\min}_{[\lambda,\Omega\setminus\bar{\Omega}_{0,1}]} \quad \text{in } \Omega$$

if, in addition, u_0 is a subsolution of (1.9) in Ω.

Similarly, thanks to Theorem 2.4(d), for any $\lambda \geqslant \sigma_2$ we have that

$$\mathfrak{M}^{\min}_{[\lambda,\Omega_-]} \leqslant \liminf_{t\uparrow\infty} u(\cdot,t) \leqslant \limsup_{t\uparrow\infty} u(\cdot,t) \leqslant \mathfrak{M}^{\max}_{[\lambda,\Omega_-]} \quad \text{in } \Omega.$$

Moreover,

$$\lim_{t\uparrow\infty} u(\cdot,t) = \mathfrak{M}^{\min}_{[\lambda,\Omega_-]} \quad \text{in } \Omega$$

if, in addition, u_0 is a subsolution of (1.9) in Ω.

As a result from these features, the dynamics of (1.1) is governed by the maximal classical nonnegative solution of (1.9) if $\lambda < \sigma_1$, by the metasolutions of (1.9) supported in $\Omega \setminus \bar{\Omega}_{0,1}$ if $\sigma_1 \leqslant \lambda < \sigma_2$, and by the metasolutions of (1.9) supported in Ω_- if $\lambda \geqslant \sigma_2$.

The following results provide us with some sufficient conditions for the uniqueness of the metasolution of (1.9) supported in $D \in \{\Omega \setminus \bar{\Omega}_{0,1}, \Omega_-\}$.

THEOREM 2.6. *Suppose*

$$f(x,u) = u^{p-1}, \quad (x,u) \in \bar{\Omega} \times [0,\infty), \tag{2.9}$$

for some $p > 1$, *and there exist*

$$\beta \in \mathcal{C}\big(\Gamma_1;(0,\infty)\big) \quad \textit{and} \quad \gamma \in \mathcal{C}\big(\Gamma_1;(0,\infty)\big)$$

such that

$$\lim_{\substack{x\to x_1\\ x\in\Omega_-}} \frac{a(x)}{\beta(x_1)[\operatorname{dist}(x,\Gamma_1)]^{\gamma(x_1)}} = 1 \quad \textit{uniformly in } x_1 \in \Gamma_1. \tag{2.10}$$

Then, for each $\lambda < \sigma_2$,

$$L^{\min}_{[\lambda,\Omega\setminus\bar{\Omega}_{0,1}]} = L^{\max}_{[\lambda,\Omega\setminus\bar{\Omega}_{0,1}]}, \tag{2.11}$$

i.e., (1.9) *possesses a unique large solution in* $\Omega \setminus \bar{\Omega}_{0,1}$. *Therefore,* (1.9) *possesses a unique metasolution supported in* $\Omega \setminus \bar{\Omega}_{0,1}$, *subsequently denoted by* $\mathfrak{M}_{[\lambda,\Omega\setminus\bar{\Omega}_{0,1}]}$, *and, due to Theorem* 2.4(c), $\mathfrak{M}_{[\lambda,\Omega\setminus\bar{\Omega}_{0,1}]}$ *is a global attractor for the solutions of* (1.1) *if* $\sigma_1 \leqslant \lambda < \sigma_2$.

THEOREM 2.7. *Suppose* (2.9) *and there exist*

$$\beta \in \mathcal{C}\big(\partial\Omega_-;(0,\infty)\big) \quad \textit{and} \quad \gamma \in \mathcal{C}\big(\partial\Omega_-;(0,\infty)\big)$$

such that

$$\lim_{\substack{x\to x_1\\ x\in\Omega_-}} \frac{a(x)}{\beta(x_1)[\operatorname{dist}(x,\Gamma_1)]^{\gamma(x_1)}} = 1 \quad \textit{uniformly in } x_1 \in \partial\Omega_-. \tag{2.12}$$

Then, for each $\lambda \in \mathbb{R}$,

$$L^{\min}_{[\lambda,\Omega_-]} = L^{\max}_{[\lambda,\Omega_-]}, \tag{2.13}$$

i.e., (1.9) *possesses a unique large solution in* Ω_-. *Consequently,* (1.9) *possesses a unique metasolution supported in* Ω_-, *subsequently denoted by* $\mathfrak{M}_{[\lambda,\Omega_-]}$, *and, due to Theorem* 2.4(d), $\mathfrak{M}_{[\lambda,\Omega_-]}$ *is a global attractor for the solutions of* (1.1) *if* $\lambda \geqslant \sigma_2$.

The next result shows the continuity and monotonicity in λ of the metasolutions of (1.9) supported in $D \in \{\Omega \setminus \bar{\Omega}_{0,1}, \Omega_-\}$ under the assumptions of Theorem 2.7, though the result remains valid assuming the uniqueness of the metasolution for every value of the parameter where it exists.

THEOREM 2.8. *Under the assumptions of Theorem* 2.7, *each of the maps*

$$\begin{aligned} &(-\infty,\sigma_2) \xrightarrow{\mathfrak{M}_{\Omega\setminus\bar{\Omega}_{0,1}}} \mathcal{C}\big(\Omega;(0,\infty]\big),\\ &\lambda \mapsto \mathfrak{M}_{\Omega\setminus\bar{\Omega}_{0,1}}(\lambda) := \mathfrak{M}_{[\lambda,\Omega\setminus\bar{\Omega}_{0,1}]} \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} &\mathbb{R} \xrightarrow{\mathfrak{M}_{\Omega_-}} \mathcal{C}\big(\Omega;(0,\infty]\big),\\ &\lambda \mapsto \mathfrak{M}_{\Omega_-}(\lambda) := \mathfrak{M}_{[\lambda,\Omega_-]} \end{aligned} \tag{2.15}$$

is continuous and point-wise increasing. Moreover,

$$\lim_{\lambda\uparrow\sigma_1} \theta(\lambda) = \mathfrak{M}_{\Omega\setminus\bar{\Omega}_{0,1}}(\sigma_1) \quad \textit{and} \quad \lim_{\lambda\uparrow\sigma_2} \mathfrak{M}_{\Omega\setminus\bar{\Omega}_{0,1}}(\lambda) = \mathfrak{M}_{\Omega_-}(\sigma_2), \tag{2.16}$$

where θ *is the solution map defined in* (2.8).

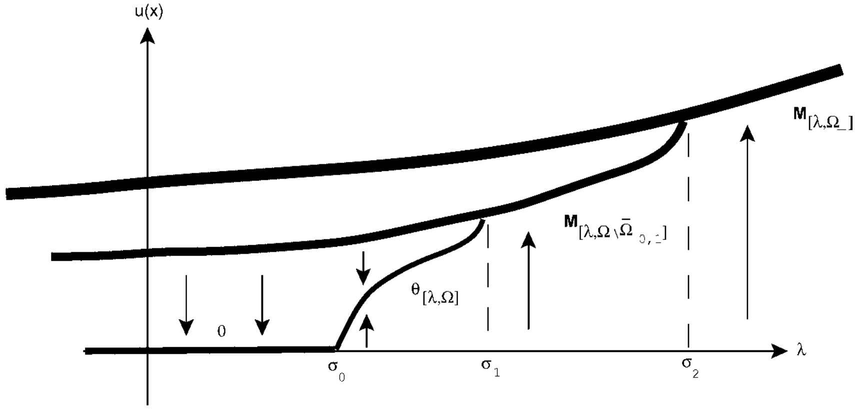

Fig. 2. The dynamics of (1.1). Classical solutions and metasolutions.

In Figure 2 we have represented the dynamics of (1.1) under the assumptions of Theorem 2.7. Most precisely, we have drown the diagram of classical solutions and significant metasolutions of (1.9) together with their respective attracting properties. We are plotting the parameter λ versus the value of the generalized solution, $u(x)$, at some point $x \in \Omega_-$, where all these generalized solutions are finite, so that the diagram cannot exhibit any *bifurcation from infinity*. The diagram shows four different kind of solutions. The λ-axis represents $u = 0$, which, according to Theorem 2.4(a), is a global attractor if $\lambda \leqslant \sigma_0$, while it is linearly unstable for any $\lambda > \sigma_0$. Then, we have represented the positive solution $\theta_{[\lambda,\Omega]}$, which bifurcates from $u = 0$ at $\lambda = \sigma_0$ and it is point-wise increasing for $\lambda \in (\sigma_0, \sigma_1)$ until it reaches the metasolution $\mathfrak{M}_{[\sigma_1,\Omega\setminus\bar{\Omega}_{0,1}]}$ at $\lambda = \sigma_1$. Due to Theorem 2.4(b), $\theta_{[\lambda,\Omega]}$ is a global attractor of (1.1) for each $\lambda \in (\sigma_0, \sigma_1)$. Then, we have represented the curve

$$\lambda \to \mathfrak{M}_{[\lambda,\Omega\setminus\bar{\Omega}_{0,1}]}(x),$$

which, according to Theorem 2.8, is continuous and increasing in its definition interval $(-\infty, \sigma_2)$. Thanks to Theorem 2.4(c), $\mathfrak{M}_{[\lambda,\Omega\setminus\bar{\Omega}_{0,1}]}$ is a global attractor for the solutions of (1.1) if $\lambda \in [\sigma_1, \sigma_2)$, and, due to Theorem 2.2, it approximates the metasolution $\mathfrak{M}_{[\sigma_2,\Omega_-]}$ as $\lambda \uparrow \sigma_2$. As for any $\lambda < \sigma_1$ the dynamics of (1.1) is described by the classical nonnegative steady states of (1.1), in such range the metasolution $\mathfrak{M}_{[\lambda,\Omega\setminus\bar{\Omega}_{0,1}]}$ must be unstable from below. Consequently, the value $\lambda = \sigma_1$ provides us with the critical value of the parameter where the attractive character of the metasolution supported in $\Omega \setminus \bar{\Omega}_{0,1}$ changes. Finally, Figure 2 shows the curve

$$\lambda \to \mathfrak{M}_{[\lambda,\Omega_-]}(x),$$

which, according to Theorem 2.8, is continuous and increasing in $\mathbb{R}$. Thanks to Theorem 2.4, $\mathfrak{M}_{[\lambda,\Omega_-]}$ is a global attractor for the solutions of (1.1) if $\lambda \geqslant \sigma_2$, while it is unstable from below if $\lambda < \sigma_2$. So, $\lambda = \sigma_2$ is the critical value of the parameter where the attractive character of $\mathfrak{M}_{[\lambda,\Omega_-]}$ changes. Figure 3 shows the corresponding asymptotic profiles of the

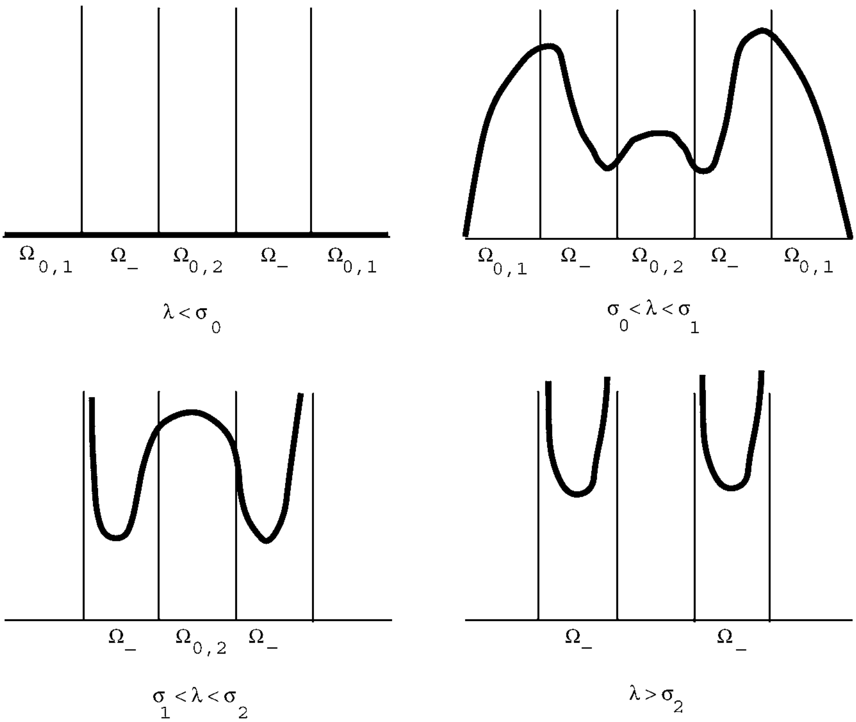

Fig. 3. The limiting profiles of the solutions of (1.1).

solutions of (1.1) according to the range of variation of the parameter λ. Although for a general nonlinearity $f(x,u)$ satisfying (Af), (Ag) and (Ah) our theory does not guarantees the uniqueness of the metasolution, due to Theorem 2.4, Figure 2 still provides us with the dynamics of (1.1) when u_0 is a subsolution of (1.9), though in this case one should think of the minimal metasolution curves, which might exhibit a number of jumps as a result of the eventual nonuniqueness of the metasolutions. Nevertheless, though we could not prove it yet, we conjecture that the metasolutions supported in $\Omega \setminus \bar{\Omega}_{0,1}$ and Ω_- must be unique for a general $a(x)$ satisfying (Aa) and $f(x,u)$, not necessarily of the special form (2.9).

Theorem 2.4 is of fundamental interest from the point of view of the applications of the abstract mathematical theory developed here to population dynamics as it provides us with the simultaneous effects of incorporating both Malthus and Verhulst laws within the same natural environment, which should be extremely realistic from the point of view of applications to real world models. Rather naturally, the density of the species should be severely limited in the regions where natural resources are drastically limited, though it might be certainly unlimited within the regions where natural resources are sufficiently abundant to maintain a huge population. A mechanism explaining why agriculture facilitated the emergence of human groups whose size gradually increased during the last 10 thousands years – nothing at the human evolution scale – until originating the extremely densely populated areas that we inhabit today. Simultaneously, in unfavorable areas, where agriculture was not possible, or simply extremely difficult, like in the rain forest areas, the level of human population has been controlled almost at the original levels.

As a consequence from Theorem 2.4, if the intrinsic growth rate of the species, measured by λ, is below the threshold σ_0, then the inhabiting area cannot support the species u, which is driven to extinction. When $\lambda \in (\sigma_0, \sigma_1)$, then the inhabiting area is able to maintain the species u around the critical level $\theta_{[\lambda,\Omega]}$, independently of the size of the initial population u_0. So, Ω cannot maintain an arbitrarily large population. Therefore, (1.1) exhibits a genuine logistic behavior if $\lambda < \sigma_1$.

Quite surprisingly from the mathematical point of view – for a reason to be explained in Section 8 – in the interval $\lambda \in [\sigma_1, \sigma_2)$ the population must be limited in the region $\Omega \setminus \bar{\Omega}_{0,1}$, while, as an effect from dispersion to the less hostile region $\Omega_{0,1}$, the population can grow arbitrarily within $\Omega_{0,1}$. Thus, (1.1) exhibits a sort of logistic growth in $\Omega \setminus \bar{\Omega}_{0,1}$ and a genuine exponential growth in $\Omega_{0,1}$ if $\lambda \in [\sigma_1, \sigma_2)$. Indeed, thanks to the parabolic maximum principle,

$$u_1 := u_{[\lambda,\Omega]}(\cdot, 1; u_0) \gg 0$$

and, for each $t \geqslant 0$,

$$u_{[\lambda,\Omega]}(\cdot, t+1; u_0) = u_{[\lambda,\Omega]}(\cdot, t; u_1) > u_{[\lambda,\Omega_{0,1}]}(\cdot, t; u_1), \tag{2.17}$$

where $u_{[\lambda,\Omega_{0,1}]}(x, t; u_1)$ stands for the solution of the linear problem

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = \lambda u & \text{in } \Omega_{0,1} \times (0, \infty), \\ u = 0 & \text{on } \partial\Omega_{0,1} \times (0, \infty), \\ u(\cdot, 0) = u_1 & \text{in } \Omega_{0,1}, \end{cases} \tag{2.18}$$

which is given by

$$u_{[\lambda,\Omega_{0,1}]}(x, t; u_1) = \mathrm{e}^{t(\lambda+\Delta)} u_1. \tag{2.19}$$

Now, suppose $\lambda > \sigma_1$ and denote by $\varphi_{0,1}$ the principal eigenfunction associated with $\sigma[-\Delta, \Omega_{0,1}]$. Then, since $u_1 \gg 0$, there exists $\alpha > 0$ such that

$$u_1 > \alpha \varphi_{0,1}$$

and hence, it follows from (2.17) and (2.19) that

$$u_{[\lambda,\Omega]}(\cdot, t+1; u_0) > \alpha \mathrm{e}^{t(\lambda+\Delta)} \varphi_{0,1} = \alpha \mathrm{e}^{(\lambda-\sigma_1)t} \varphi_{0,1},$$

which shows the exponential growth of the population within $\Omega_{0,1}$.

Similarly, when $\lambda > \sigma_2$ the population must be controlled within Ω_-, as a result of the limitation of the natural resources available there in, though the individual of the species can disperse to the most favorable areas $\Omega_{0,1}$ and $\Omega_{0,2}$, where the population density can be arbitrarily large; actually growing at the respective rates $e^{t(\lambda-\sigma_1)}$ and $e^{t(\lambda-\sigma_2)}$. Consequently, the principal eigenvalues σ_1 and σ_2 might be though as sort of measuring parameters of the quality of each of the *refuge patches* of the environment.

3. Some general pivotal results

The following characterization of the strong maximum principle goes back to [53] and [43] (cf. [48] for an extremely sharp version of the theorem).

THEOREM 3.1. *Suppose D is an open subdomain of $\mathbb{R}^N$, $N \geqslant 1$, with smooth boundary and $V \in \mathcal{C}^\mu(\overline{D})$. Then, the following assertions are equivalent.*

(a) $\sigma[-\Delta + V, D] > 0$.

(b) *There exists a function $h \in \mathcal{C}^2(D) \cap \mathcal{C}(\overline{D})$ such that $h > 0$ in D,*

$$(-\Delta + V)h \geqslant 0 \quad \text{in } D,$$

and either $h|_{\partial D} > 0$, or $(-\Delta + V)h > 0$ in D – such a function h is called a positive strict supersolution of $-\Delta + V$ in D under Dirichlet boundary conditions.

(c) *The operator $-\Delta + V$ satisfies the strong maximum principle in D, i.e., for every $f \in \mathcal{C}^\mu(\overline{D})$, $g \in \mathcal{C}^{2+\mu}(\partial D)$, such that $f \geqslant 0$, $g \geqslant 0$, $(f, g) \neq (0, 0)$, and any $u \in \mathcal{C}^{2+\mu}(\overline{D})$ satisfying*

$$\begin{cases} (-\Delta + V)u = f & \text{in } D, \\ u = g & \text{on } \partial D, \end{cases}$$

one has that $u \gg 0$ in D.

Throughout the remaining part of this section we suppose $f \in \mathcal{C}^{\mu,\mu}(\overline{\Omega} \times [0, \infty))$ and consider a smooth subdomain D of Ω such that

$$D \cap \Omega_- \neq \emptyset. \tag{3.1}$$

As an immediate consequence from the abstract theory developed by Amann in [1] and [2], the following result holds.

THEOREM 3.2. *Suppose $g \in \mathcal{C}^{2+\mu}(\partial D)$ and*

$$\begin{cases} -\Delta u = \lambda u - af(\cdot, u)u & \text{in } D, \\ u = g & \text{on } \partial D \end{cases} \tag{3.2}$$

possesses a subsolution $\underline{u} \in \mathcal{C}^{2+\mu}(\overline{D})$ and a supersolution $\bar{u} \in \mathcal{C}^{2+\mu}(\overline{D})$ such that $\underline{u} \leqslant \bar{u}$. Then (3.2) possesses a solution $u \in \mathcal{C}^{2+\mu}(\overline{D})$ such that

$$\underline{u} \leqslant u \leqslant \bar{u}.$$

Moreover, (3.2) has a minimal and a maximal solution in $[\underline{u}, \bar{u}]$.

Note that if $\underline{u}$ (resp. $\bar{u}$) is a strict subsolution (resp. supersolution) of (3.2), then any solution u of the problem in $[\underline{u}, \bar{u}]$ must satisfy $\underline{u} < u \leqslant \bar{u}$ (resp. $\underline{u} \leqslant u < \bar{u}$).

As an easy consequence from Theorem 3.2, the following result holds.

THEOREM 3.3. *Suppose $\underline{u}$, $\bar{u} \in \mathcal{C}^{2+\mu}(D)$, satisfy*

$$-\Delta \underline{u} \leqslant \lambda \underline{u} - af(\cdot, \underline{u})\underline{u} \quad \text{and} \quad -\Delta \bar{u} \geqslant \lambda \bar{u} - af(\cdot, \bar{u})\bar{u} \quad \text{in } D,$$

$$\lim_{\operatorname{dist}(x,\partial D)\downarrow 0} \underline{u}(x) = \infty \quad \text{and} \quad \lim_{\operatorname{dist}(x,\partial D)\downarrow 0} \bar{u}(x) = \infty,$$

and

$$\underline{u} \leqslant \bar{u} \quad \text{in } D.$$

Then, the singular boundary value problem,

$$\begin{cases} -\Delta u = \lambda u - af(\cdot, u)u & \text{in } D, \\ u = \infty & \text{on } \partial D, \end{cases} \tag{3.3}$$

possesses a solution $u \in \mathcal{C}^{2+\mu}(D)$ such that $\underline{u} \leqslant u \leqslant \bar{u}$.

PROOF. For each sufficiently large $n \geqslant 1$, say $n \geqslant n_0$, we consider

$$D_n := \left\{ x \in D \colon \operatorname{dist}(x, \partial D) > \frac{1}{n} \right\}.$$

The integer $n_0 \geqslant 1$ must be chosen so that ∂D_n inherits the regularity of ∂D. Thanks to Theorem 3.2, for each $n \geqslant n_0$, the problem

$$\begin{cases} -\Delta u = \lambda u - af(\cdot, u)u & \text{in } D_n, \\ u = \frac{\underline{u}+\bar{u}}{2} & \text{on } \partial D_n \end{cases}$$

possesses a solution $u_n \in \mathcal{C}^{2+\mu}(\overline{D}_n)$ such that

$$\underline{u}|_{D_n} \leqslant u_n \leqslant \bar{u}|_{D_n} \quad \text{in } D_n.$$

Thanks to these estimates, there exists a subsequence $\{u_{n_m}\}_{m\geqslant 1}$ of $\{u_n\}_{n\geqslant n_0}$ such that

$$\lim_{m\to\infty} \|u_{n_m} - u_0\|_{\mathcal{C}^{2+\mu}(\overline{D}_{n_0})} = 0$$

for some solution $u_0 \in \mathcal{C}^{2+\mu}(\overline{D}_{n_0})$ of

$$\begin{cases} -\Delta u = \lambda u - af(\cdot, u)u & \text{in } D_{n_0}, \\ u = \frac{\underline{u}+\bar{u}}{2} & \text{on } \partial D_{n_0}. \end{cases}$$

Now, consider the new sequence $\{u_{n_m}|_{D_{n_0+1}}\}_{m\geqslant 1}$. The previous argument also shows the existence of a subsequence of $\{u_{n_m}|_{D_{n_0+1}}\}_{m\geqslant 1}$, labeled again by n_m, such that, for some $u_1 \in \mathcal{C}^{2+\mu}(\overline{D}_{n_0+1})$,

$$\lim_{m\to\infty} \|u_{n_m} - u_1\|_{\mathcal{C}^{2+\mu}(\overline{D}_{n_0+1})} = 0.$$

Necessarily, $u_1|_{D_{n_0}} = u_0$. Repeating this procedure infinitely many times, the point-wise limit of the diagonal sequence provides us with a solution of (3.3) satisfying all requirements. □

The following result collects some important properties that are going to be used throughout the remaining of this work.

LEMMA 3.4. *Suppose f satisfies* (Af), $g \in \mathcal{C}^{2+\mu}(\partial D)$, $g \geqslant 0$, *and* $\bar{u} \in \mathcal{C}^{2+\mu}(\overline{D})$, $\bar{u} > 0$, *is a supersolution of* (3.2). *Then,* $\bar{u} \gg 0$, *i.e.,* $\bar{u}(x) > 0$ *for each* $x \in D$ *and* $\frac{\partial \bar{u}}{\partial n_x}(x) < 0$ *for each* $x \in \bar{u}^{-1}(0) \cap \partial D$, *where* n_x *stands for the outward unit normal at* $x \in \partial D$. *In particular, any positive solution u of* (3.2) *satisfies* $u \gg 0$. *Moreover,*

$$\sigma\big[-\Delta - \lambda + af(\cdot, \bar{u}), D\big] \geqslant 0.$$

Actually, if $g = 0$ *and* $\bar{u}$ *is a solution of* (3.2), *then*

$$\lambda = \sigma\big[-\Delta + af(\cdot, \bar{u}), D\big]. \tag{3.4}$$

Furthermore, for each $\kappa > 1$, $\kappa\bar{u}$ *also provides us with a supersolution of* (3.2).

PROOF. Since $\bar{u}|_{\partial D} \geqslant 0$ and

$$\big(-\Delta - \lambda + af(\cdot, \bar{u})\big)\bar{u} \geqslant 0 \quad \text{in } D,$$

$\bar{u} > 0$ provides us with a positive supersolution of $-\Delta - \lambda + af(\cdot, \bar{u})$ in D under Dirichlet boundary conditions, and two different situations can occur.

If either $g > 0$ on ∂D, or $g = 0$ on ∂D but $\bar{u}$ is a positive strict supersolution of $-\Delta - \lambda + af(\cdot, \bar{u})$ in D under Dirichlet boundary conditions, then it follows from Theorem 3.1 that

$$\sigma\big[-\Delta - \lambda + af(\cdot, \bar{u}), D\big] > 0 \tag{3.5}$$

and $\bar{u} \gg 0$.

If $g = 0$ on ∂D and $\bar{u}$ is not a strict supersolution of $-\Delta - \lambda + af(\cdot, \bar{u})$ in D, then $\bar{u}$ provides us with a positive eigenfunction associated with the principal eigenvalue

$$\sigma\big[-\Delta - \lambda + af(\cdot, \bar{u}), D\big] = 0 \tag{3.6}$$

and hence, by Krein–Rutman theorem, $u \gg 0$. Note that in such case $\bar{u}$ is a solution of (3.2) and that (3.4) holds from (3.6).

Now, pick $\kappa > 1$. Then,

$$\kappa\bar{u}|_{\partial D} \geqslant \bar{u}|_{\partial D} \geqslant g,$$

and, in D, we have that

$$-\Delta(\kappa\bar{u}) \geqslant \lambda\kappa\bar{u} - af(\cdot,\bar{u})\kappa\bar{u} \geqslant \lambda\kappa\bar{u} - af(\cdot,\kappa\bar{u})\kappa\bar{u},$$

since, by (Af),

$$f(\cdot,\kappa\bar{u}) > f(\cdot,\bar{u}),$$

which concludes the proof. □

The following theorem will extraordinarily simplify the mathematical analysis of the next sections.

THEOREM 3.5. *Suppose f satisfies* (Af), $g \in \mathcal{C}^{2+\mu}(\partial D)$, $g \geqslant 0$, $\lambda > \sigma[-\Delta, D]$ *if* $g = 0$, *and* (3.2) *possesses a supersolution* $\bar{u} > 0$. *Then,* (3.2) *has a unique positive solution. Moreover, if we denote it by* $\theta_{[\lambda,D,g]}$, *then for any positive subsolution* (*resp. supersolution*) $\underline{u}$ (*resp.* $\bar{u}$) *of* (3.2), *one has that* $\underline{u} \leqslant \theta_{[\lambda,D,g]}$ (*resp.* $\theta_{[\lambda,D,g]} \leqslant \bar{u}$). *Furthermore, for each* $u_0 > 0$,

$$\lim_{t\uparrow\infty} \left\| u_{[\lambda,D,g]}(\cdot,t;u_0) - \theta_{[\lambda,D,g]} \right\|_{\mathcal{C}(\bar{D})} = 0, \tag{3.7}$$

where $u_{[\lambda,D,g]}(x,t;u_0)$ *stands for the unique solution of the parabolic problem*

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = \lambda u - af(\cdot,u)u & \text{in } D \times (0,\infty), \\ u = g & \text{on } \partial D \times (0,\infty), \\ u(\cdot,0) = u_0 & \text{in } D. \end{cases} \tag{3.8}$$

Suppose (Af), (3.2) *has a supersolution* $\bar{u} > 0$, $g = 0$ *and* $\lambda \leqslant \sigma[-\Delta, D]$. *Then,* (3.2) *cannot admit a positive subsolution and*

$$\lim_{t\uparrow\infty} \left\| u_{[\lambda,D,g]}(\cdot,t;u_0) \right\|_{\mathcal{C}(\bar{D})} = 0. \tag{3.9}$$

PROOF. In case $g > 0$, $\underline{u} := 0$ provides us with a strict subsolution of (3.2), and hence, $(0,\bar{u})$ provides us with an ordered sub–supersolution pair. Thus, thanks to Theorem 3.2, (3.2) possesses a solution $0 < u \leqslant \underline{u}$. Due to Lemma 3.4, $u \gg 0$.

Suppose $g = 0$ and $\lambda > \sigma[-\Delta, D]$. Let $\varphi \gg 0$ denote any principal eigenfunction associated with $\sigma[-\Delta, D]$. Then, for each sufficiently small $\varepsilon > 0$, the function $\underline{u} := \varepsilon\varphi$ provides us with a positive strict subsolution of (3.2). Indeed, $\varepsilon\varphi|_{\partial D} = 0$ and, in D,

$$-\Delta(\varepsilon\varphi) = \varepsilon\sigma[-\Delta, D]\varphi < \lambda\varepsilon\varphi - af(\cdot,\varepsilon\varphi)\varepsilon\varphi$$

for sufficiently small $\varepsilon > 0$, since $\sigma[-\Delta, D] < \lambda$ and

$$\lim_{\varepsilon \downarrow 0} \left\| af(\cdot, \varepsilon\varphi) \right\|_{\mathcal{C}(\bar{D})} = 0.$$

Fix one of these values of ε. Due to Lemma 3.4, $\underline{u} \gg 0$ and $\kappa \bar{u}$ is a supersolution of (3.2) for every $\kappa > 1$. Pick a sufficiently large $\kappa > 1$ so that

$$\varepsilon\varphi < \kappa\bar{u}.$$

Then, $(\varepsilon\varphi, \kappa\bar{u})$ provides us with an ordered sub–supersolution pair of (3.2) and hence, thanks to Theorem 3.2, (3.2) possesses a solution u such that

$$\varepsilon\varphi < u \leqslant \kappa\bar{u}.$$

Note that, due to Lemma 3.4, $u \gg 0$.

To show the uniqueness of the positive solution of (3.2) we proceed by contradiction. Suppose (3.2) has two different positive solutions $u_1 \neq u_2$. The previous analysis shows that there exist $\varepsilon > 0$, $\kappa > 1$, and a strict subsolution

$$\underline{u} \in \{0, \varepsilon\varphi\}$$

such that

$$\underline{u} < \min\{u_1, u_2\} < \max\{u_1, u_2\} \leqslant \kappa\bar{u}.$$

Let u_* and u^* denote the minimal and the maximal positive solutions of (3.2) in $[\underline{u}, \kappa\bar{u}]$. Necessarily,

$$\underline{u} < u_* \leqslant \min\{u_1, u_2\} < \max\{u_1, u_2\} \leqslant u^* \leqslant \kappa\bar{u}$$

and therefore, (3.2) possesses two ordered positive solutions, $u_* < u^*$. Setting

$$w := u^* - u_* > 0,$$

the following linear boundary value problem is satisfied

$$\begin{cases} (-\Delta - \lambda + V)w = 0 & \text{in } D, \\ w = 0 & \text{on } \partial D, \end{cases} \tag{3.10}$$

where V is the potential defined by

$$\begin{aligned} V := {} & a\int_0^1 \frac{\partial f}{\partial u}\big(\cdot, tu^* + (1-t)u_*\big)\big(tu^* + (1-t)u_*\big)\,\mathrm{d}t \\ & + a\int_0^1 f\big(\cdot, tu^* + (1-t)u_*\big)\,\mathrm{d}t. \end{aligned} \tag{3.11}$$

By Lemma 3.4, $u^* \gg 0$ and $u_* \gg 0$. Hence, it follows from (Af) and (3.1) that

$$V > a \int_0^1 f\big(\cdot, tu^* + (1-t)u_*\big)\,\mathrm{d}t \geqslant af(\cdot, u_*),$$

since $u^* > u_*$. In particular, by the monotonicity of the principal eigenvalue with respect to the potential, we find that

$$\sigma[-\Delta - \lambda + V, D] > \sigma\big[-\Delta - \lambda + af(\cdot, u_*), D\big]. \tag{3.12}$$

Thus, thanks to (3.4), (3.12) gives

$$\sigma[-\Delta - \lambda + V, D] > 0.$$

As the principal eigenvalue is dominant, we conclude from (3.10) that $w = 0$, which contradicts $w > 0$. This contradiction concludes the proof of the uniqueness of the positive solution. Subsequently, we denote by $\theta_{[\lambda,D,g]}$ the unique positive solution of (3.2).

Suppose $\underline{u} > 0$ is a subsolution of (3.2). Then, since $\bar{u} \gg 0$, for sufficiently large $\kappa > 1$, we have that $\underline{u} < \kappa\bar{u}$ and hence, by Theorem 3.2, (3.2) has a positive solution in $[\underline{u}, \kappa\bar{u}]$. By the uniqueness, $\underline{u} \leqslant \theta_{[\lambda,D,g]} \leqslant \kappa\bar{u}$ and, in particular,

$$\underline{u} \leqslant \theta_{[\lambda,D,g]}.$$

Suppose $\bar{u} > 0$ is a supersolution of (3.2). By Lemma 3.4, $\bar{u} \gg 0$. If $g > 0$, then, $\underline{u} = 0$ provides us with a strict subsolution of (3.2) and, due to Theorem 3.2, (3.2) has a positive solution u in $[0, \bar{u}]$. By the uniqueness,

$$\theta_{[\lambda,D,g]} \leqslant \bar{u}. \tag{3.13}$$

Similarly, if $g = 0$, then, for sufficiently small $\varepsilon > 0$, $\underline{u} := \varepsilon\varphi$ provides us with a strict subsolution of (3.2). Thus, if ε is chosen so that $\varepsilon\varphi < \bar{u}$, by Theorem 3.2, (3.2) has a positive solution u in $[\varepsilon\varphi, \bar{u}]$. Therefore, by the uniqueness, (3.13) as well holds.

Now, we will prove (3.7). First, we suppose that

$$g = 0 \quad \text{and} \quad \lambda > \sigma[-\Delta, D].$$

Since $u_0 > 0$, by the parabolic maximum principle, for each $t > 0$ we have that

$$u_{[\lambda,D,g]}(\cdot, t; u_0) \gg 0.$$

Now, pick a sufficiently small $\varepsilon > 0$ and a sufficiently large $\kappa > 1$ so that $(\varepsilon\varphi, \kappa\bar{u})$ be an ordered sub–supersolution pair of (3.2) for which

$$\varepsilon\varphi < u_{[\lambda,D,g]}(\cdot, 1; u_0) < \kappa\bar{u}.$$

Note that $\kappa > 1$ can be chosen so that

$$\kappa \bar{u} > \theta_{[\lambda, D, g]},$$

and hence, by the uniqueness of the positive solution, $\kappa \bar{u}$ must be a positive strict supersolution. Now, thanks again to the parabolic maximum principle, for each $t > 0$ we have that

$$\begin{aligned} u_{[\lambda,D,g]}(\cdot, t; \varepsilon\varphi) &\leqslant u_{[\lambda,D,g]}\big(\cdot, t; u_{[\lambda,D,g]}(\cdot, 1; u_0)\big) \\ &= u_{[\lambda,D,g]}(\cdot, t+1; u_0) \\ &\leqslant u_{[\lambda,D,g]}(\cdot, t; \kappa\bar{u}). \end{aligned} \tag{3.14}$$

Moreover, since $\varepsilon\varphi$ is a strict subsolution, as t grows, $u_{[\lambda,D,g]}(\cdot, t; \varepsilon\varphi)$ increases approximating the minimal positive solution of (3.2) in $[\varepsilon\varphi, \kappa\bar{u}]$, while, since $\kappa\bar{u}$ is a strict supersolution, $u_{[\lambda,D,g]}(\cdot, t; \kappa\bar{u})$ decreases approximating the maximal positive solution of (3.2) in $[\varepsilon\varphi, \kappa\bar{u}]$ (cf. [65]). As $\theta_{[\lambda,D,g]}$ is the unique positive solution of (3.2), passing to the limit as $t \uparrow \infty$ in (3.14) gives

$$\lim_{t\uparrow\infty} u_{[\lambda,D,g]}(\cdot, t; u_0) = \theta_{[\lambda,D,g]} \quad \text{in } \mathcal{C}\big(\overline{D}\big),$$

so concluding the proof of the theorem in this case.

Now, suppose $g > 0$ and pick a sufficiently large $\kappa > 1$ for which

$$\kappa\bar{u} > \theta_{[\lambda,D,g]} \quad \text{and} \quad 0 < u_{[\lambda,D,g]}(\cdot, 1; u_0) < \kappa\bar{u}.$$

Then, arguing as above, we find that

$$\begin{aligned} u_{[\lambda,D,g]}(\cdot, t; 0) &\leqslant u_{[\lambda,D,g]}\big(\cdot, t; u_{[\lambda,D,g]}(\cdot, 1; u_0)\big) \\ &= u_{[\lambda,D,g]}(\cdot, t+1; u_0) \\ &\leqslant u_{[\lambda,D,g]}(\cdot, t; \kappa\bar{u}), \end{aligned} \tag{3.15}$$

and similarly, (3.7) follows by passing to the limit as $t \uparrow \infty$ in (3.15).

Finally, suppose

$$g = 0 \quad \text{and} \quad \lambda \leqslant \sigma[-\Delta, D]. \tag{3.16}$$

We claim that (3.2) cannot admit a positive subsolution; in particular, it cannot admit a positive solution. To prove this feature we proceed by contradiction. So, suppose (3.2) possesses a positive subsolution $\underline{u} > 0$. Since $\bar{u} \gg 0$, there exists $\kappa > 1$ such that $\underline{u} < \kappa\bar{u}$ and therefore, due to Theorem 3.2, (3.2) possesses a positive solution u. Moreover, thanks to Lemma 3.4, we have $u \gg 0$ and

$$\lambda = \sigma\big[-\Delta + af(\cdot, u), D\big].$$

Since $u \gg 0$, it follows from (Af) and (3.1) that $af(\cdot, u) > 0$ in D and hence, thanks to the monotonicity of the principal eigenvalue with respect to the potential,

$$\lambda = \sigma\big[-\Delta + af(\cdot, u), D\big] > \sigma[-\Delta, D],$$

which contradicts (3.16). Therefore, (3.2) cannot admit a positive subsolution in such case. In order to show that, under (3.16), condition (3.9) is satisfied, we consider $\kappa > 1$ such that

$$0 < u_{[\lambda,D,g]}(\cdot, 1; u_0) < \kappa \bar{u}.$$

Then, for each $t > 0$, we have that

$$0 < u_{[\lambda,D,g]}(\cdot, t+1; u_0) < u_{[\lambda,D,g]}(\cdot, t; \kappa\bar{u}). \tag{3.17}$$

As $\kappa\bar{u}$ is a supersolution of (3.2), $u_{[\lambda,D,g]}(\cdot, t; \kappa\bar{u})$ decreases approximating as $t \uparrow \infty$ the maximal nonnegative solution of (3.2) in $[0, \kappa\bar{u}]$, which is the zero solution. Therefore, passing to the limit as $t \uparrow \infty$ in (3.17) shows (3.9) and completes the proof. □

Actually, the following strong comparison result is satisfied.

LEMMA 3.6. *Suppose f satisfies* (Af), $g \in \mathcal{C}^{2+\mu}(\partial D)$, $g \geqslant 0$, $\lambda > \sigma[-\Delta, D]$ *if* $g = 0$, *and* (3.2) *possesses a supersolution* $\bar{u} > 0$. *Then, for any positive strict subsolution* (*resp. supersolution*) $\underline{u}$ (*resp.* $\bar{u}$) *of* (3.2), *one has that* $\underline{u} \ll \theta_{[\lambda,D,g]}$ (*resp.* $\theta_{[\lambda,D,g]} \ll \bar{u}$), *where* $\theta_{[\lambda,D,g]}$ *is the unique positive solution of* (3.2).

PROOF. Suppose $\underline{u} > 0$ is a strict subsolution of (3.2). Then, due to Theorem 3.5, $\underline{u} \leqslant \theta_{[\lambda,D,g]}$, and therefore,

$$\underline{u} < \theta_{[\lambda,D,g]},$$

since $\underline{u}$ cannot be a solution. Consequently,

$$w := \theta_{[\lambda,D,g]} - \underline{u} > 0. \tag{3.18}$$

Now, adapting the uniqueness argument of the proof of Theorem 3.5, it readily follows that

$$\begin{cases} (-\Delta - \lambda + V)w = 0 & \text{in } D, \\ w \geqslant 0 & \text{on } \partial D, \end{cases} \tag{3.19}$$

where V is the potential defined by

$$V := a\int_0^1 \frac{\partial f}{\partial u}\big(\cdot, t\theta_{[\lambda,D,g]} + (1-t)\underline{u}\big)\big(t\theta_{[\lambda,D,g]} + (1-t)\underline{u}\big)\,\mathrm{d}t$$
$$+ a\int_0^1 f\big(\cdot, t\theta_{[\lambda,D,g]} + (1-t)\underline{u}\big)\,\mathrm{d}t.$$

Now, we have to distinguish between two different situations. If $w|_{\partial D} > 0$, then $w > 0$ provides us with a positive strict supersolution of $-\Delta - \lambda + V$ in D under homogeneous Dirichlet boundary conditions. Thus, due to Theorem 3.1, $w \gg 0$ in D, and therefore $\underline{u} \ll \theta_{[\lambda,D,g]}$. Now, suppose $w|_{\partial D} = 0$. Then, $w > 0$ is a principal eigenfunction associated with

$$\sigma[-\Delta - \lambda + V, D] = 0$$

and, due to Krein–Rutman theorem, $w \gg 0$, which concludes the proof. □

4. The classical logistic equation. A priori bounds in Ω_-

In this section we analyze the dynamics of

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = \lambda u - a(x) f(x,u) u & \text{in } D \times (0,\infty), \\ u = M & \text{on } \partial D \times (0,\infty), \\ u(\cdot,0) = u_0 > 0 & \text{in } D, \end{cases} \tag{4.1}$$

where $M \in [0,\infty)$ is a constant, D is a smooth subdomain of Ω such that

$$D \subset \Omega_-, \tag{4.2}$$

and f satisfies (Af) and (Ag). The main result establishes that the dynamics of (4.1) is governed by its maximal nonnegative steady-state, which is the maximal nonnegative solution of

$$\begin{cases} -\Delta u = \lambda u - a(x) f(x,u) u & \text{in } D, \\ u = M & \text{on } \partial D. \end{cases} \tag{4.3}$$

Actually, for each $M > 0$, (4.3) possesses a unique positive solution which is a global attractor for the solutions of (4.1). Throughout the rest of this paper it will be denoted by $\theta_{[\lambda,D,M]}$. It turns out that this solution satisfies

$$\lim_{M \downarrow 0} \theta_{[\lambda,D,M]} = 0 \quad \text{if } \lambda \leqslant \sigma[-\Delta, D],$$

while

$$\theta_{[\lambda,D,0]} := \lim_{M \downarrow 0} \theta_{[\lambda,D,M]} \quad \text{if } \lambda > \sigma[-\Delta, D],$$

provides us with the unique positive solution of (4.3) for $M = 0$. So, there is a continuous transition between both dynamics as M perturbs from zero.

The analysis of (4.1) in case $\overline{D} \subset \Omega_-$ is much easier than the analysis of the general case when (4.2) is satisfied, where a might partially, or totally, vanish on ∂D, as a result of the fact that sufficiently large positive constants provide us with supersolutions if $\bar{D} \subset \Omega_-$, while no positive constant can be a supersolution if a vanishes on ∂D and $\lambda > 0$. Thus, in the first two sections we will focus our attention into the simplest case when $\bar{D} \subset \Omega_-$. Then, in Section 4.3, we will assume that, in addition, f satisfies (Ah) in order to refine the classical a priori bounds of Keller [38] and Osserman [61]. These bounds will allow us to obtain the existence of a minimal and a maximal positive solution for the singular problem

$$\begin{cases} -\Delta u = \lambda u - a(x) f(x,u) u & \text{in } D, \\ u = \infty & \text{on } \partial D. \end{cases} \tag{4.4}$$

As a consequence from these bounds, it will be apparent that

$$\theta_{[\lambda,D,\infty]} := \lim_{M \uparrow \infty} \theta_{[\lambda,D,M]}$$

provides us with the minimal positive solution of (4.4). Finally, in Section 4.4, these results will be used to study the general case when condition (4.2) is satisfied. The following consequence from Lemma 3.4 will be very useful.

LEMMA 4.1. *Suppose $u \in \mathcal{C}^{2+\mu}(\bar{D})$, $u > 0$, is a solution of* (4.3) *for some $M \in [0,\infty)$. Then $u \gg 0$. Moreover,*

$$\lambda = \sigma\big[-\Delta + af(\cdot,u), D\big] \tag{4.5}$$

if $M = 0$. In case $M = \infty$, any solution $u \in \mathcal{C}^{2+\mu}(D)$, $u > 0$, must satisfy $u(x) > 0$ for all $x \in D$.

PROOF. The fact that $u \gg 0$ for any positive solution of (4.3) with $M \in [0,\infty)$ is a consequence from Lemma 3.4, and (4.5) is a consequence from (3.4).

Now, suppose $M = \infty$. Then, u provides us with a positive supersolution of $-\Delta - \lambda + af(\cdot,u)$ in any subdomain of the form $D_n := \{x \in D\colon \operatorname{dist}(x,\partial D) > 1/n\}$, for sufficiently large n, and hence, thanks again to Lemma 3.4, $u \gg 0$ in D_n, which concludes the proof. □

4.1. *The classical logistic equation: $M = 0$ and $\bar{D} \subset \Omega_-$*

The following result provides us with the existence and the uniqueness of the positive solution to (4.3) in this case. Note that

$$a_{L,D} := \min_{\bar{D}} a > 0. \tag{4.6}$$

THEOREM 4.2. *Suppose $\bar{D} \subset \Omega_-$, $M = 0$, and f satisfies* (Af) *and* (Ag). *Then,* (4.3) *possesses a positive solution if and only if $\lambda > \sigma[-\Delta, D]$. Moreover, it is unique, and strongly positive, if it exists. Furthermore, if we denote it by $\theta_{[\lambda,D,0]}$ and $u_{[\lambda,D,0]}(x,t;u_0)$ stands for the unique solution of* (4.1), *then*

(a) $\lim_{t\uparrow\infty} u_{[\lambda,D,0]}(\cdot,t;u_0) = 0$ *in* $\mathcal{C}(\overline{D})$ *if* $\lambda \leqslant \sigma[-\Delta, D]$;
(b) $\lim_{t\uparrow\infty} u_{[\lambda,D,0]}(\cdot,t;u_0) = \theta_{[\lambda,D,0]}$ *in* $\mathcal{C}(\overline{D})$ *if* $\lambda > \sigma[-\Delta, D]$.

PROOF. It is a direct consequence from Theorem 3.5, since sufficiently large positive constants provide us with positive supersolutions of (4.3), by (Ag). □

The following result establishes that the set of positive solutions of (4.3) consists of a differentiable curve emanating from $u = 0$ at the value of the parameter $\lambda = \sigma[-\Delta, D]$, where the attractive character of the steady state $u = 0$, as a solution of (4.1), is lost.

PROPOSITION 4.3. *Suppose* $\overline{D} \subset \Omega_-$, $M = 0$, *and* f *satisfies* (Af) *and* (Ag). *Then the solution map*

$$\begin{aligned} &\big(\sigma[-\Delta, D], \infty\big) \xrightarrow{\theta} \mathcal{C}\big(\overline{D}\big), \\ &\lambda \mapsto \theta(\lambda) := \theta_{[\lambda,D,0]} \end{aligned} \tag{4.7}$$

is of class $\mathcal{C}^1$ *and point-wise increasing. Actually,*

$$\theta(\lambda) \gg \theta(\mu) \quad \text{if } \lambda > \mu > \sigma[-\Delta, D].$$

Moreover, $\theta(\lambda)$ *bifurcates from* $(\lambda, u) = (\lambda, 0)$ *at* $\lambda = \sigma[-\Delta, D]$, *i.e.,*

$$\lim_{\lambda\downarrow\sigma[-\Delta,D]} \theta(\lambda) = 0. \tag{4.8}$$

PROOF. The solutions of (4.3) are the zeros of the nonlinear operator

$$\mathfrak{F} : \mathbb{R} \times \mathcal{C}_0\big(\overline{D}\big) \to \mathcal{C}_0\big(\overline{D}\big)$$

defined by

$$\mathfrak{F}(\lambda, u) := u - (-\Delta)^{-1}\big[\lambda u - af(\cdot, u)u\big],$$

where $(-\Delta)^{-1}$ stands for the resolvent operator of $-\Delta$ in D under homogeneous Dirichlet boundary conditions. The operator $\mathfrak{F}$ is of class $\mathcal{C}^1$ and, by elliptic regularity, $\mathfrak{F}(\lambda, \cdot)$ is a nonlinear compact perturbation of the identity for each $\lambda \in \mathbb{R}$. Moreover,

$$\mathfrak{F}(\lambda, 0) = 0, \quad \lambda \in \mathbb{R},$$

and, for each $(\lambda, u) \in \mathbb{R} \times \mathcal{C}_0(\overline{D})$,

$$D_u\mathfrak{F}(\lambda, 0)u = u - (-\Delta)^{-1}(\lambda u).$$

Thus, $D_u\mathfrak{F}(\lambda, 0)$ is a Fredholm analytic pencil of index zero whose spectrum consists of the eigenvalues of $-\Delta$. In particular,

$$N\big[D_u\mathfrak{F}\big(\sigma[-\Delta, D], 0\big)\big] = \text{span}[\varphi],$$

where $\varphi \gg 0$ is any principal eigenfunction of $\sigma[-\Delta, D]$. We claim that

$$D_\lambda D_u \mathfrak{F}\big(\sigma[-\Delta, D], 0\big)\varphi \notin R\big[D_u \mathfrak{F}\big(\sigma[-\Delta, D], 0\big)\big]. \tag{4.9}$$

Consequently, the transversality condition of Crandall and Rabinowitz [19] holds true. To prove (4.9) we proceed by contradiction assuming that

$$D_\lambda D_u \mathfrak{F}\big(\sigma[-\Delta, D], 0\big)\varphi = -(-\Delta)^{-1}\varphi \in R\big[D_u \mathfrak{F}\big(\sigma[-\Delta, D], 0\big)\big].$$

Then there exists $u \in \mathcal{C}_0(\overline{D})$ such that

$$u - (-\Delta)^{-1}\big(\sigma[-\Delta, D]u\big) = -(-\Delta)^{-1}\varphi.$$

By elliptic regularity, $u \in \mathcal{C}_0^{2+\mu}(\overline{D})$ and

$$\big(-\Delta - \sigma[-\Delta, D]\big)u = -\varphi.$$

Multiplying this equation by φ, integrating in D and applying the formula of integration by parts gives $\int_D \varphi^2 = 0$, which is impossible. This contradiction shows the validity of (4.9) and therefore, by the main theorem of [19], $(\lambda, u) = (\sigma[-\Delta, D], 0)$ is a bifurcation point from $(\lambda, u) = (\lambda, 0)$ to a smooth curve of positive solutions of (4.3). By the uniqueness of the positive solution, as a result from Theorem 4.2, (4.8) holds true.

Subsequently we suppose that $(\lambda, u) = (\lambda_0, u_0)$ is a positive solution of (4.3). Then $\mathfrak{F}(\lambda_0, u_0) = 0$ and, by Lemma 4.1, $u_0 \gg 0$ and

$$\lambda_0 = \sigma\big[-\Delta + af(\cdot, u_0), D\big]. \tag{4.10}$$

Differentiating with respect to u we have that, for each $u \in \mathcal{C}_0(\overline{D})$,

$$D_u \mathfrak{F}(\lambda_0, u_0)u = u - (-\Delta)^{-1}\left[\lambda_0 u - a\frac{\partial f}{\partial u}(\cdot, u_0)u_0 u - af(\cdot, u_0)u\right].$$

In particular, $D_u \mathfrak{F}(\lambda_0, u_0)$ is a Fredholm operator of index zero. Moreover, it is a linear topological isomorphism, since it is inyective. Indeed, if there exists $u \in \mathcal{C}_0(\overline{D})$ for which

$$u - (-\Delta)^{-1}\left[\lambda_0 u - a\frac{\partial f}{\partial u}(\cdot, u_0)u_0 u - af(\cdot, u_0)u\right] = 0,$$

then, by elliptic regularity, $u \in \mathcal{C}_0^{2+\mu}(\overline{D})$ and

$$\left(-\Delta - \lambda_0 + a\frac{\partial f}{\partial u}(\cdot, u_0)u_0 + af(\cdot, u_0)\right)u = 0 \quad \text{in } D. \tag{4.11}$$

On the other hand, thanks to (4.10) and (Af), we have that

$$\sigma\left[-\Delta-\lambda_0+a\frac{\partial f}{\partial u}(\cdot,u_0)u_0+af(\cdot,u_0),\,D\right] > \sigma\big[-\Delta-\lambda_0+af(\cdot,u_0),\,D\big]=0.$$

Consequently, it follows from (4.11) that $u=0$. Therefore, $D_u\mathfrak{F}(\lambda_0,u_0)$ is a linear topological isomorphism, and the uniqueness of the positive solution, as a consequence from Theorem 4.2, combined with the implicit function theorem shows the regularity of the map θ defined by (4.7). Finally, differentiating the identity

$$\mathfrak{F}\big(\lambda,\theta(\lambda)\big)=0,\quad \lambda>\sigma[-\Delta,D],$$

with respect to λ gives

$$D_\lambda\theta=(-\Delta)^{-1}\left(\theta+\lambda D_\lambda\theta-a\frac{\partial f}{\partial u}(\cdot,\theta)\theta D_\lambda\theta-af(\cdot,\theta)D_\lambda\theta\right)$$

or equivalently,

$$\left(-\Delta-\lambda+a\frac{\partial f}{\partial u}(\cdot,\theta)\theta+af(\cdot,\theta)\right)D_\lambda\theta=\theta.$$

As $\theta\gg 0$ and for each $\lambda>\sigma[-\Delta,D]$,

$$\sigma\left[-\Delta-\lambda+a\frac{\partial f}{\partial u}(\cdot,\theta)\theta+af(\cdot,\theta),\,D\right]>0,$$

it follows from Theorem 3.1 that

$$D_\lambda\theta(\lambda)=\left(-\Delta-\lambda+a\frac{\partial f}{\partial u}\big(\cdot,\theta(\lambda)\big)\theta(\lambda)+af\big(\cdot,\theta(\lambda)\big)\right)^{-1}\theta(\lambda)\gg 0,$$

which concludes the proof. □

Note that if $\lambda>\mu>\sigma[-\Delta,D]$, then $\theta_{[\mu,D,0]}$ is a strict subsolution of (4.3) and therefore, the relation $\theta_{[\mu,D,0]}\ll\theta_{[\lambda,D,0]}$ also follows from Lemma 3.6.

In Figure 4 we have illustrated the results from Theorem 4.2 and Proposition 4.3. For a given value $x\in D$, we have represented the curve

$$\lambda\mapsto\theta_{[\lambda,D,0]}(x).$$

It bifurcates from the horizontal axis at the value of the parameter $\lambda=\sigma[-\Delta,D]$ and it increases for all further values of λ. The direction of the arrows indicate the flow of (4.1). According to Theorem 4.2, as time grows to infinity $u_{[\lambda,D,0]}(x,t;u_0)$ decays to zero if

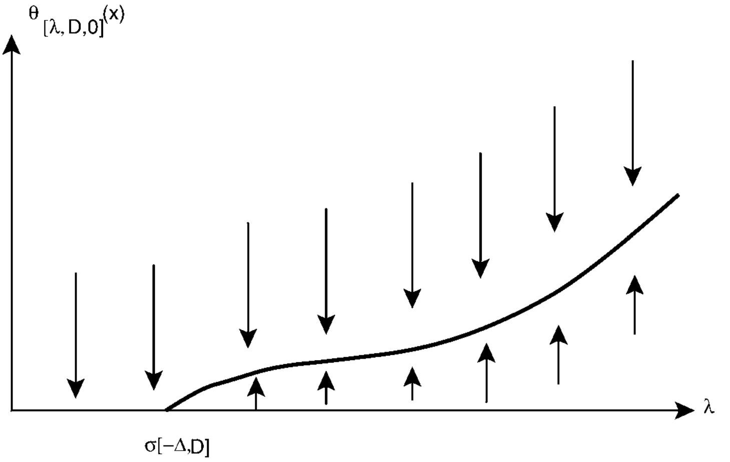

Fig. 4. The dynamics of (4.1) in case $\overline{D} \subset \Omega_-$ and $M = 0$.

$\lambda \leqslant \sigma[-\Delta, D]$, whereas it approximates $\theta_{[\lambda,D,0]}(x)$ if $\lambda > \sigma[-\Delta, D]$. It should be noted that the trivial steady-state $u = 0$ of (4.1) is linearly stable if $\lambda \leqslant \sigma[-\Delta, D]$, while it is linearly unstable if $\lambda > \sigma[-\Delta, D]$. Hence, the stability lost by $u = 0$ as λ crosses $\sigma[-\Delta, D]$ is gained by the positive solution $\theta_{[\lambda,D,0]}$ bifurcating from $u = 0$ at $\sigma[-\Delta, D]$, in complete agreement with the *exchange stability principle* of Crandall and Rabinowitz [20].

4.2. *The case $M > 0$ and $\overline{D} \subset \Omega_-$*

In this case, as an immediate consequence from Theorem 3.5, the next result holds true, because sufficiently large positive constants are supersolutions of (4.3).

THEOREM 4.4. *Suppose $\overline{D} \subset \Omega_-$, $M > 0$, and f satisfies* (Af) *and* (Ag). *Then, for each $\lambda \in \mathbb{R}$,* (4.3) *possesses a unique positive solution. Moreover, it is unique and strongly positive if it exists, and if we denote it by $\theta_{[\lambda,D,M]}$ and $u_{[\lambda,D,M]}(x,t;u_0)$ stands for the unique solution of* (4.1), *then*

$$\lim_{t\uparrow\infty} u_{[\lambda,D,M]}(\cdot,t;u_0) = \theta_{[\lambda,D,M]} \quad \text{in } \mathcal{C}\big(\overline{D}\big). \tag{4.12}$$

Further, thanks to Lemma 3.6, the following comparison result holds true.

LEMMA 4.5. *Suppose $\overline{D} \subset \Omega_-$, $M > 0$, and f satisfies* (Af) *and* (Ag). *Let $\underline{u} > 0$* (*resp.* $\bar{u} > 0$) *be a strict subsolution* (*resp. supersolution*) *of* (4.3). *Then*

$$\underline{u} \ll \theta_{[\lambda,D,M]} \quad (\textit{resp. } \theta_{[\lambda,D,M]} \ll \bar{u}).$$

Consequently, the estimates

$$0 < M_1 \leqslant M_2 < \infty, \qquad -\infty < \lambda_1 \leqslant \lambda_2 < \infty, \qquad M_2 - M_1 + \lambda_2 - \lambda_1 > 0$$

imply

$$\theta_{[\lambda_1, D, M_1]} \ll \theta_{[\lambda_2, D, M_2]}.$$

Moreover, for each $\lambda > \sigma[-\Delta, D]$ *and* $M > 0$,

$$\theta_{[\lambda, D, 0]} \ll \theta_{[\lambda, D, M]}.$$

As a consequence from Lemma 4.5, the point-wise limit

$$\theta_{[\lambda, D, \infty]} := \lim_{M \uparrow \infty} \theta_{[\lambda, D, M]} \quad \text{in } D \tag{4.13}$$

is well defined, though, without any further assumptions on f, it might be everywhere infinity. In the next section we shall see that (4.13) provides us with the minimal positive solution of (4.4) if, in addition, f condition (Ah). Actually, (Ah) is not only sufficient but also necessary for that, though this issue is outside the scope of this work.

The next result shows the *structural stability* of model (4.1) under perturbations of the parameter $M \in [0, \infty)$.

PROPOSITION 4.6. *Suppose* $\overline{D} \subset \Omega_-$ *and* f *satisfies* (Af), (Ag). *Then*

$$\lim_{M \downarrow 0} \theta_{[\lambda, D, M]} = \begin{cases} 0 & \text{if } \lambda \leqslant \sigma[-\Delta, D], \\ \theta_{[\lambda, D, 0]} & \text{if } \lambda > \sigma[-\Delta, D]. \end{cases}$$

PROOF. As a consequence from Lemma 4.5, the point-wise limit

$$\Theta_0 := \lim_{M \downarrow 0} \theta_{[\lambda, D, M]}$$

is well defined. Moreover, by the Schauder theory (e.g., [33]), it is easy to see that Θ_0 provides us with a classical solution of (4.3) for $M = 0$. Necessarily, $\Theta_0 \geqslant 0$. Thus, thanks to Theorem 4.2, $\Theta_0 = 0$ if $\lambda \leqslant \sigma[-\Delta, D]$, while, in case $\lambda > \sigma[-\Delta, D]$, it follows from Lemma 4.5 that $\Theta_0 \geqslant \theta_{[\lambda, D, 0]}$ and therefore, by the uniqueness of the positive solution, $\Theta_0 = \theta_{[\lambda, D, 0]}$, which concludes the proof. □

Thanks to Proposition 4.6, the curve of maximal nonnegative solutions of (4.3) for $M = 0$ perturbs into the curve of positive solutions of (4.3) when $M > 0$ separates away from $M = 0$. In Figure 5 we have fixed $M > 0$ and represented the curve

$$\lambda \mapsto \theta_{[\lambda, D, M]}(x)$$

for a generic $x \in D$, as well as the flow of (4.1). As $M \downarrow 0$ the curve approximates 0 if $\lambda \leqslant \sigma[-\Delta, D]$, and $\theta_{[\lambda, D, 0]}(x)$ if $\lambda > \sigma[-\Delta, D]$. Figure 5 should be compared with Figure 4; the dashed lines represent the nonnegative solutions of (4.3) with $M = 0$.

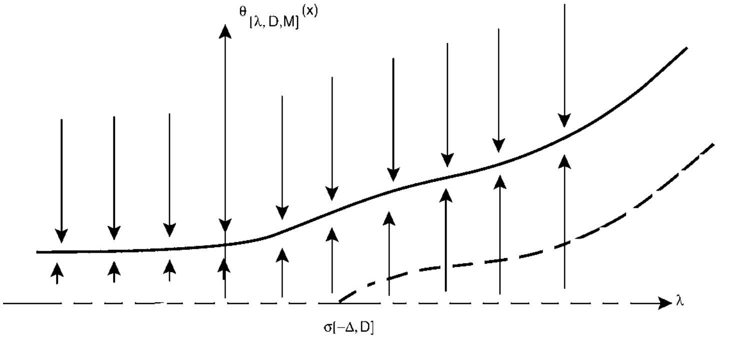

Fig. 5. The dynamics of (4.1) in case $\overline{D} \subset \Omega_-$ and $M > 0$.

4.3. *The radial case $M = \infty$ and $D = B_R(x_0)$ with $\overline{D} \subset \Omega_-$*

Subsequently, given $x_0 \in \mathbb{R}^N$ and $R > 0$, $B_R(x_0)$ stands for the open ball

$$B_R(x_0) := \{x \in \mathbb{R}^N : |x - x_0| < R\}.$$

The main result of this section is the following proposition.

PROPOSITION 4.7. *Pick $\lambda \in \mathbb{R}$, $x_0 \in \Omega_-$ and $R > 0$ such that $\overline{D} \subset \Omega_-$, where $D := B_R(x_0)$, and suppose f satisfies* (Af) *and* (Ag). *Consider the auxiliary function*

$$H(u) := a_{L,D}\, g(u)u - \lambda u, \quad u \in [0, \infty), \tag{4.14}$$

where $a_{L,D}$ is given by (4.6), *and let u_λ denote the unique positive zero of $H(u)$. Suppose, in addition, that*

$$I(u) := \int_u^\infty \left[\int_u^s H(z)\,\mathrm{d}z\right]^{-1/2} \mathrm{d}s < \infty \quad \textit{for each } u > u_\lambda, \tag{4.15}$$

and

$$\lim_{u \uparrow \infty} I(u) = 0. \tag{4.16}$$

Then, the point-wise limit (4.13) *is finite in D, and it provides us with the minimal positive solution of* (4.4).

PROOF. Thanks to Theorem 4.4, for each $M > 0$, (4.3) has a unique positive solution, which has been already denoted by $\theta_{[\lambda,D,M]}$. By (Ag), $\theta_{[\lambda,D,M]}$ provides us with a positive subsolution of the auxiliary problem

$$\begin{cases} -\Delta u = -H(u) & \text{in } D = B_R(x_0), \\ u = M & \text{on } \partial D. \end{cases} \tag{4.17}$$

Thanks again to Theorem 4.4, (4.17) possesses a unique positive solution. Let denote it Θ_M. By Lemma 4.5, we have

$$\theta_{[\lambda,D,M]} \leqslant \Theta_M \quad \text{for each } M > 0, \tag{4.18}$$

and

$$\Theta_{M_1} \ll \Theta_{M_2} \quad \text{if } 0 < M_1 < M_2.$$

Thus, the point-wise limit

$$\Theta_\infty := \lim_{M \uparrow \infty} \Theta_M \tag{4.19}$$

is well defined in D. Consequently, due to (4.18), in order to prove that the point-wise limit (4.13) is finite in D it suffices to show that Θ_∞ is finite in D.

Since (4.17) is invariant by rotations, Θ_M must be radially symmetric for each $M > 0$. Hence,

$$\Theta_M(x) = \Psi_M(r), \quad r := |x - x_0|, x \in D,$$

where Ψ_M is the unique positive solution of

$$\begin{cases} \psi''(r) + \frac{N-1}{r}\psi'(r) = H\big(\psi(r)\big), & 0 < r < R, \\ \psi'(0) = 0, \quad \psi(R) = M. \end{cases} \tag{4.20}$$

Indeed, the function $\underline{\psi} := 0$ is a subsolution of (4.20) and $\bar{\psi} := C$ is a supersolution of (4.20) for each sufficiently large $C > M$. Thus, (4.20) has a positive solution, since $\underline{\psi} < \bar{\psi}$; necessarily unique, because otherwise we would contradict the uniqueness of the positive solution of (4.17). Throughout the remaining of the proof, without lost of generality, we may assume that

$$M > u_\lambda.$$

For this choice, as $\underline{u} := u_\lambda$ is a positive strict subsolution of (4.17), we find from Lemma 4.5 that

$$u_\lambda < \Theta_M(x) = \Psi_M(r) \quad \text{for each } x \in \overline{D} \tag{4.21}$$

and, consequently,

$$H\big(\Psi_M(r)\big) > 0 \quad \text{for each } r \in [0, R], \tag{4.22}$$

since $H(z) > 0$ for each $z > u_\lambda$. Also, for each $z > u_\lambda$,

$$\begin{aligned} H'(z) &= a_{L,D}g(z) + a_{L,D}zg'(z) - \lambda \\ &> a_{L,D}g(u_\lambda) - \lambda + a_{L,D}zg'(z) \\ &= a_{L,D}zg'(z) > 0 \end{aligned}$$

since $H(u_\lambda) = 0$. Thus, H is increasing in (u_λ, ∞).

Now, multiplying the ψ-differential equation by r^{N-1} and rearranging terms gives

$$\big(r^{N-1}\Psi_M'(r)\big)' = r^{N-1}H\big(\Psi_M(r)\big), \quad 0 < r < R. \tag{4.23}$$

Hence, integrating (4.23) in $(0, r)$, yields to

$$\Psi_M'(r) = r^{1-N}\int_0^r s^{N-1}H\big(\Psi_M(s)\big)\,\mathrm{d}s > 0, \quad r \in (0, R), \tag{4.24}$$

where we have used (4.22), which, in particular, shows that $r \mapsto \Psi_M(r)$ is increasing, as well as $r \mapsto H(\Psi_M(r))$. Thus, it follows from (4.24) that

$$\Psi_M'(r) \leqslant r^{1-N}H\big(\Psi_M(r)\big)\int_0^r s^{N-1}\,\mathrm{d}s = \frac{r}{N}H\big(\Psi_M(r)\big), \quad 0 < r < R. \tag{4.25}$$

Now, note that (4.25) gives

$$H\big(\Psi_M(r)\big) = \Psi_M''(r) + \frac{N-1}{r}\Psi_M'(r) \leqslant \Psi_M''(r) + \frac{N-1}{N}H\big(\Psi_M(r)\big)$$

and hence,

$$\Psi_M''(r) \geqslant \frac{\Psi_M(r)}{N}, \quad 0 < r < R.$$

Similarly, since $\Psi_M' \geqslant 0$, we find that

$$\Psi_M''(r) \leqslant H\big(\Psi_M(r)\big)$$

and therefore,

$$H\big(\Psi_M(r)\big) \geqslant \Psi_M''(r) \geqslant \frac{\Psi_M(r)}{N}, \quad 0 < r < R. \tag{4.26}$$

We now multiply (4.26) by $\Psi_M'(r)$ an integrate in $(0, r)$ to obtain

$$2\int_{\Psi_M(0)}^{\Psi_M(r)} H(z)\,\mathrm{d}z \geqslant \big[\Psi_M'(r)\big]^2 \geqslant \frac{2}{N}\int_{\Psi_M(0)}^{\Psi_M(r)} H(z)\,\mathrm{d}z, \quad 0 < r < R. \tag{4.27}$$

Thus, taking the square root of the reciprocal of (4.27), multiplying by Ψ_M' and integrating the resulting expression in (r, R) shows that

$$\frac{1}{\sqrt{2}}\int_{\Psi_M(r)}^{M}\left[\int_{\Psi_M(0)}^{u} H\right]^{-1/2}\mathrm{d}u \leqslant R-r \leqslant \sqrt{\frac{N}{2}}\int_{\Psi_M(r)}^{M}\left[\int_{\Psi_M(0)}^{u} H\right]^{-1/2}\mathrm{d}u \tag{4.28}$$

for each $r \in [0, R)$.

Now pick $r \in [0, R)$. Then, for each $M > u_\lambda$, we have that $\Psi_M(r) \geqslant \Psi_M(0)$ and hence, for each $u > \Psi_M(r)$,

$$\int_{\Psi_M(0)}^{u} H \geqslant \int_{\Psi_M(r)}^{u} H.$$

Thus, it follows from the second inequality of (4.28) that

$$\begin{aligned} 0 < R - r &\leqslant \sqrt{\frac{N}{2}}\int_{\Psi_M(r)}^{M}\left[\int_{\Psi_M(r)}^{u} H\right]^{-1/2}\mathrm{d}u \\ &< \sqrt{\frac{N}{2}}\int_{\Psi_M(r)}^{\infty}\left[\int_{\Psi_M(r)}^{u} H\right]^{-1/2}\mathrm{d}u \\ &= \sqrt{\frac{N}{2}} I\big(\Psi_M(r)\big) < \infty \end{aligned}$$

because of (4.15). Therefore, due to (4.16), we find that

$$\lim_{M\uparrow\infty} \Psi_M(r) < \infty. \tag{4.29}$$

This shows that Θ_∞ in finite in D. Actually, setting

$$\Psi_\infty(r) := \lim_{M\uparrow\infty} \Psi_M(r), \quad r \in [0, R),$$

the monotone limit defined through (4.19) is given by

$$\Theta_\infty(x) = \Psi_\infty\big(|x - x_0|\big), \quad x \in D.$$

Moreover, by the continuous dependence with respect to the initial values, $\Psi_\infty(r)$ must be the unique solution of the Cauchy problem

$$\begin{cases} \psi''(r) + \frac{N-1}{r}\psi'(r) = H\big(\psi(r)\big), & 0 < r < R, \\ \psi(0) = \Psi_\infty(0), \qquad \psi'(0) = 0, \end{cases}$$

and

$$\lim_{r\uparrow R} \Psi_\infty(r) = \infty.$$

Consequently, Θ_∞ is a radially symmetric positive solution of

$$\begin{cases} -\Delta u = -H(u) & \text{in } D = B_R(x_0), \\ u = \infty & \text{on } \partial D. \end{cases} \tag{4.30}$$

Actually, Θ_∞ provides us with the minimal positive solution of (4.30), as it will become apparent later.

Since Θ_∞ is finite in D, it follows from (4.18) and (4.19) that the function $\theta_{[\lambda,D,\infty]}$ defined through (4.13) is finite in D. Now, pick $\varepsilon \in (0, R/2)$. Then, thanks to (4.18), for each $M > 0$,

$$\theta_{[\lambda,D,M]} \leqslant \Theta_\infty \quad \text{in } B_{R-\varepsilon}(x_0)$$

and hence, by Shauder's estimates, there is a constant $C(\varepsilon) > 0$ such that

$$\|\theta_{[\lambda,D,M]}\|_{\mathcal{C}^{2+\mu}(\overline{B}_{R-2\varepsilon}(x_0))} \leqslant C(\varepsilon) \quad \text{for each } M > 0.$$

Thus, combining the compactness of $(-\Delta)^{-1}$ with the uniqueness of the point-wise limit (4.13), we find that $\theta_{[\lambda,D,\infty]} \in \mathcal{C}^{2+\mu}(B_{R-2\varepsilon}(x_0))$ must be a solution of (1.9) in $B_{R-2\varepsilon}(x_0)$ and that

$$\lim_{M\uparrow\infty} \|\theta_{[\lambda,D,M]} - \theta_{[\lambda,D,\infty]}\|_{\mathcal{C}(\overline{B}_{R-2\varepsilon}(x_0))} = 0.$$

As this holds true for any sufficiently small $\varepsilon > 0$, $\theta_{[\lambda,D,\infty]}$ must solve (4.4). Actually, it is the minimal positive solution of (4.4). In particular, Θ_∞ is the minimal positive solution of (4.30). Indeed, let L be any positive solution of (4.4). Then, for each $M > 0$, there exists a constant $C > M$ and a sufficiently large $n \in \mathbb{N}$ such that

$$\theta_{[\lambda,D,M]} \leqslant C \leqslant L \quad \text{in } D \setminus B_{R-\frac{1}{n}}(x_0). \tag{4.31}$$

Thanks to Lemma 4.5, (4.31) implies

$$\theta_{[\lambda,D,M]} \leqslant \theta_{[\lambda,D_n,C]} \leqslant L \quad \text{in } D_n := B_{R-\frac{1}{n}}(x_0)$$

and, consequently, for each $M > 0$,

$$\theta_{[\lambda,D,M]} \leqslant L \quad \text{in } D. \tag{4.32}$$

Finally, passing to the limit as $M \uparrow \infty$ in (4.32) we find that

$$\theta_{[\lambda,D,\infty]} \leqslant L,$$

which shows the minimality of $\theta_{[\lambda,D,\infty]}$ and concludes the proof. □

4.4. *The case $M=\infty$ and $D\subset\Omega_-$ with $\overline{D}\subset\Omega_-$*

As a consequence from Proposition 4.7 the following result is satisfied for a general domain D such that $\overline{D}\subset\Omega_-$.

PROPOSITION 4.8. *Suppose $\overline{D}\subset\Omega_-$ and f satisfies* (Af), (Ag) *and* (Ah). *Then, for each $\lambda\in\mathbb{R}$, the point-wise limit* (4.13) *is finite in D and it provides us with the minimal positive solution of the singular problem* (4.4). *Moreover, for any positive solution $u\in\mathcal{C}^2(D)\cap\mathcal{C}(\overline{D})$ of* (1.9) *in D one has that*

$$u\leqslant\theta_{[\lambda,D,\infty]}\quad\text{in } D. \tag{4.33}$$

PROOF. Pick $x_0\in D$ and $R>0$ such that $\overline{B}_R(x_0)\subset D$, and set, for each $M>0$,

$$\alpha_M:=\max_{\partial B_R(x_0)}\theta_{[\lambda,D,M]}.$$

Then, thanks to Lemma 4.5, for each $M>0$ we have that

$$\theta_{[\lambda,D,M]}\ll\theta_{[\lambda,B_R(x_0),\alpha_M+1]}\quad\text{in } B_R(x_0),$$

and hence, by Proposition 4.7,

$$\theta_{[\lambda,D,M]}\ll\theta_{[\lambda,B_R(x_0),\infty]}<\infty\quad\text{in } B_R(x_0).$$

Therefore, passing to the limit as $M\uparrow\infty$, shows that

$$\theta_{[\lambda,D,\infty]}\leqslant\theta_{[\lambda,B_R(x_0),\infty]}<\infty\quad\text{in } B_R(x_0). \tag{4.34}$$

As (4.34) holds true in a small ball around each point $x_0\in D$, $\theta_{[\lambda,D,\infty]}$ must be finite in D. Moreover, for each compact subset $K\subset D$ there exists a constant $C(K)>0$ such that, for each $M>0$,

$$\theta_{[\lambda,D,M]}\leqslant C(K)\quad\text{in } K.$$

From these a priori estimates, adapting the last steps of the proof of Proposition 4.7, it is easy to see that $\theta_{[\lambda,D,\infty]}\in\mathcal{C}^{2+\mu}(D)$ must be the minimal positive solution of the singular problem (4.4).

To prove (4.33), let $u\in\mathcal{C}^2(D)\cap\mathcal{C}(\overline{D})$ be a positive solution of (1.9) and set

$$\alpha:=\max_{\partial D}u.$$

Then, for each $M > \alpha$,

$$u < M = \theta_{[\lambda,D,M]} \quad \text{in } \partial D,$$

and hence, thanks to Lemma 4.5, we find that

$$u \ll \theta_{[\lambda,D,M]} \leqslant \theta_{[\lambda,D,\infty]} \quad \text{in } D,$$

which concludes the proof. □

4.5. *The general case $M \in (0,\infty]$ and $D \subset \Omega_-$*

The next result extends Theorem 4.4 to cover the general case when $D \subset \Omega_-$. Note that, in this case, $a(x)$ might vanish on some piece of ∂D.

THEOREM 4.9. *Suppose $D \subset \Omega_-$, $M > 0$, and f satisfies* (Af), (Ag) *and* (Ah). *Then,* (4.3) *possesses a unique positive solution for each $\lambda \in \mathbb{R}$. Moreover, it is strongly positive and, if we denote it by $\theta_{[\lambda,D,M]}$, then the unique solution of* (4.1), $u_{[\lambda,D,M]}(x,t;u_0)$, *satisfies* (4.12).

PROOF. Subsequently, for each sufficiently large $n \in \mathbb{N}$, say $n \geqslant n_0$, we consider the subdomain of D defined by

$$D_n := \left\{ x \in D\colon \operatorname{dist}(x,\partial D) > \frac{1}{n} \right\}. \tag{4.35}$$

Then, for each $n \geqslant n_0$,

$$\bar{D}_n \subset D_{n+1} \subset D \subset \Omega_-, \qquad a_{L,D_n} := \min_{\bar{D}_n} a > 0 \tag{4.36}$$

and

$$D = \bigcup_{n=n_0}^{\infty} D_n. \tag{4.37}$$

Due to (4.36), Theorem 4.4 guarantees that $\theta_{[\lambda,D_n,M]}$ is well defined; recall that $\theta_{[\lambda,D_n,M]}$ stands for the unique positive solution of

$$\begin{cases} -\Delta u = \lambda u - af(\cdot,u)u & \text{in } D_n, \\ u = M & \text{on } \partial D_n. \end{cases}$$

Also, thanks to Proposition 4.8, for each $n \geqslant n_0$, the point-wise limit

$$\theta_{[\lambda,D_n,\infty]} := \lim_{M\uparrow\infty} \theta_{[\lambda,D_n,M]}$$

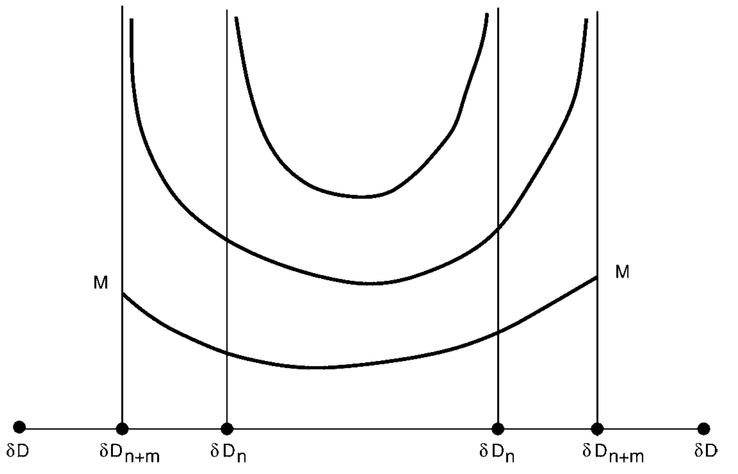

Fig. 6. The positive solutions of (4.38) for large M.

is finite and it provides us with the minimal positive solution of the singular problem

$$\begin{cases} -\Delta u = \lambda u - af(\cdot, u)u & \text{in } D_n, \\ u = \infty & \text{on } \partial D_n. \end{cases}$$

Moreover, for each $n \geqslant n_0$ and $m \geqslant 1$ (see Figure 6),

$$\theta_{[\lambda, D_{n+m}, M]} \ll \theta_{[\lambda, D_{n+m}, \infty]} \ll \theta_{[\lambda, D_n, \infty]} \quad \text{in } D_n. \tag{4.38}$$

In particular, due to (4.37), for each compact subset $K \subset D$ there exists a constant $C(K) > 0$ and an integer $n_K \in \mathbb{N}$ such that, for each $M > 0$ and $n \geqslant n_K$,

$$\theta_{[\lambda, D_n, M]} \leqslant C(K) \quad \text{in } K. \tag{4.39}$$

From (4.39), a diagonal argument combined with the compactness of $(-\Delta)^{-1}$, shows the existence of a positive solution of (4.3), denoted by $\theta_{[\lambda, D, M]}$. As the positive solution $\theta_{[\lambda, D, M]}$ itself provides us with a positive supersolution of (4.3), the remaining assertions of the theorem follow as direct consequences from Theorem 3.5. □

Also, thanks to Lemma 3.6, the following counterpart of Lemma 4.5 holds true.

LEMMA 4.10. *Suppose* $D \subset \Omega_-$, $M > 0$, *and* f *satisfies* (Af), (Ag) *and* (Ah). *Let* $\underline{u} > 0$ (*resp*. $\bar{u} > 0$) *be a strict subsolution* (*resp. supersolution*) *of* (4.3). *Then*

$$\underline{u} \ll \theta_{[\lambda, D, M]} \quad (\textit{resp. } \theta_{[\lambda, D, M]} \ll \bar{u}).$$

Consequently, the estimates

$$0 < M_1 \leqslant M_2 < \infty, \qquad -\infty < \lambda_1 \leqslant \lambda_2 < \infty, \qquad M_2 - M_1 + \lambda_2 - \lambda_1 > 0$$

imply

$$\theta_{[\lambda_1,D,M_1]} \ll \theta_{[\lambda_2,D,M_2]}.$$

Now, we can give the main existence result of positive solutions for the singular problem (4.4) in case $D \subset \Omega_-$. It should be noted that Theorem 2.1 is a direct consequence from it by making the choice $D = \Omega_-$.

THEOREM 4.11. *Suppose $D \subset \Omega_-$ and f satisfies* (Af), (Ag) *and* (Ah). *Then* (4.4) *possesses a minimal an a maximal positive solution, denoted by $L^{\min}_{[\lambda,D]}$ and $L^{\max}_{[\lambda,D]}$, respectively, i.e., any other positive solution L of* (4.4) *satisfies*

$$L^{\min}_{[\lambda,D]} \leqslant L \leqslant L^{\max}_{[\lambda,D]}.$$

Moreover,

$$L^{\min}_{[\lambda,D]} = \theta_{[\lambda,D,\infty]} := \lim_{M\uparrow\infty} \theta_{[\lambda,D,M]} \tag{4.40}$$

and

$$L^{\max}_{[\lambda,D]} = \lim_{n\uparrow\infty} \theta_{[\lambda,D_n,\infty]} = \lim_{n\uparrow\infty} L^{\min}_{[\lambda,D_n]}, \tag{4.41}$$

where D_n, $n \geqslant 1$, is the subdomain of D defined by (4.35).

PROOF. Thanks to Lemma 4.10, the point-wise limit (4.40) is well defined. Now, we shall see that it is finite. Pick $x_0 \in D$ and $R > 0$ such that

$$\overline{B}_R(x_0) \subset D \subset \Omega_-$$

and set

$$\alpha_M := \max_{\partial B_R(x_0)} \theta_{[\lambda,D,M]}, \qquad M > 0.$$

Thanks to Lemma 4.10,

$$\theta_{[\lambda,D,M]} \ll \theta_{[\lambda,B_R(x_0),\alpha_M+1]} \quad \text{in } B_R(x_0)$$

and therefore, for each $M > 0$,

$$\theta_{[\lambda,D,M]} \ll \theta_{[\lambda,B_R(x_0),\infty]} \quad \text{in } B_R(x_0).$$

As this argument is valid for any $x_0 \in D$ and, thanks to Proposition 4.7, the point-wise limit $\theta_{[\lambda,B_R(x_0),\infty]}$ is finite in $B_R(x_0)$, for each compact subset $K \subset D$ there exists a constant $C(K) > 0$ such that, for each $M > 0$,

$$\theta_{[\lambda,D,M]} \leqslant C(K) \quad \text{in } K.$$

This shows that the point-wise limit (4.40) is finite in D. Further, combining the monotonicity of the involved sequences with the compactness of $(-\Delta)^{-1}$, it is easy to see that $\theta_{[\lambda,D,\infty]} \in \mathcal{C}^{2+\mu}(D)$ provides us with a positive solution of (4.4). Actually, it is the minimal positive solution. Indeed, let L be any positive solution of (4.4). Then, for each $M > 0$, there exists a constant $C > 0$ and a sufficiently large $n \in \mathbb{N}$ such that

$$\theta_{[\lambda,D,M]} \leqslant C \leqslant L \quad \text{in } D \setminus D_n. \tag{4.42}$$

Also, thanks to Lemma 4.5, (4.42) implies

$$\theta_{[\lambda,D,M]} \leqslant \theta_{[\lambda,D_n,C]} \leqslant L \quad \text{in } D_n$$

and therefore,

$$\theta_{[\lambda,D,M]} \leqslant L \quad \text{in } D. \tag{4.43}$$

Consequently, passing to the limit as $M \uparrow \infty$ in (4.43) we find that

$$\theta_{[\lambda,D,\infty]} \leqslant L$$

which shows the minimality of $\theta_{[\lambda,D,\infty]}$ as a positive solution of the singular problem (4.4).

Similarly, thanks to (4.38) and (4.39), the point-wise limit (4.41) is well defined in D and it provides us with a positive solution of (4.4). To show its maximality, suppose L is any positive solution of (4.4). Then, for each $n \geqslant n_0$, there exists $M > 0$ such that $L|_{D_n}$ provides us with a positive subsolution of (4.3), and therefore, thanks to Lemma 4.5,

$$L \leqslant \theta_{[\lambda,D_n,M]} \leqslant \theta_{[\lambda,D_n,\infty]} \quad \text{in } D_n.$$

Consequently, passing to the limit as $n \uparrow \infty$ gives

$$L \leqslant L^{\max}_{[\lambda,D]},$$

which concludes the proof. □

In Figure 7 we have sketched the construction of $L^{\min}_{[\lambda,D]}$ and $L^{\max}_{[\lambda,D]}$ carried out in the proof of Theorem 4.11. Although we conjecture that, under the general assumptions of this section,

$$L^{\min}_{[\lambda,D]} = L^{\max}_{[\lambda,D]},$$

i.e., the positive solution of (4.4) is unique, we could not get a proof of this fact, except – essentially – in the case described by Theorem 2.7.

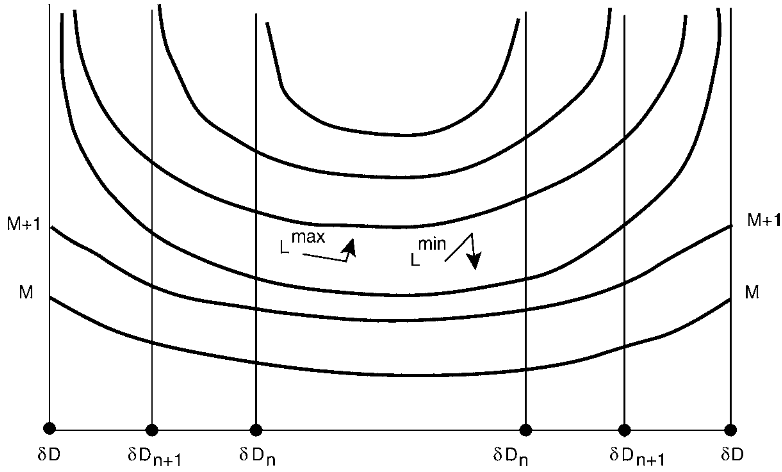

Fig. 7. The construction of $L^{\max}_{[\lambda,D]}$ and $L^{\min}_{[\lambda,D]}$.

5. Proofs of Theorems 2.1–2.3

Theorem 2.1 is a direct consequence from Theorem 4.11. Actually, thanks to Theorem 4.11, for each $\lambda \in \mathbb{R}$ the following relation is satisfied

$$L^{\min}_{[\lambda,\Omega_-]} = \lim_{M\uparrow\infty} \theta_{[\lambda,\Omega_-,M]}, \tag{5.1}$$

where $\theta_{[\lambda,\Omega_-,M]}$ stands for the unique positive solution of (2.1) with $D = \Omega_-$. Moreover,

$$L^{\max}_{[\lambda,\Omega_-]} = \lim_{n\uparrow\infty} L^{\min}_{[\lambda,\Omega^n_-]}, \tag{5.2}$$

where, for any sufficiently large $n \in \mathbb{N}$,

$$\Omega^n_- := \left\{ x \in \Omega_- \colon \operatorname{dist}(x, \partial\Omega_-) > \frac{1}{n} \right\}.$$

Now, we shall begin the proof of Theorem 2.3. First, we will characterize the existence of positive solutions of

$$\begin{cases} -\Delta u = \lambda u - af(\cdot,u)u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.3}$$

Suppose u is a positive solution of (5.3). Then, thanks to Lemma 3.4, $u \gg 0$ and

$$\lambda = \sigma\left[-\Delta + af(\cdot,u), \Omega\right]. \tag{5.4}$$

In particular, $af(\cdot,u)>0$ in Ω, and hence, by the monotonicity of the principal eigenvalue with respect to the potential, (5.4) implies

$$\lambda=\sigma\big[-\Delta+af(\cdot,u),\Omega\big]>\sigma[-\Delta,\Omega]=\sigma_0.$$

Also, by the monotonicity of the principal eigenvalue with respect to the domain, it is apparent that

$$\lambda=\sigma\big[-\Delta+af(\cdot,u),\Omega\big]<\sigma\big[-\Delta+af(\cdot,u),\Omega_{0,1}\big]=\sigma[-\Delta,\Omega_{0,1}]=\sigma_1$$

since $a=0$ in $\Omega_{0,1}$. Therefore, condition

$$\sigma_0<\lambda<\sigma_1 \tag{5.5}$$

is necessary for the existence of a positive solution of (5.3). Note that

$$\sigma_0=\sigma[-\Delta,\Omega]<\sigma[-\Delta,\Omega_{0,1}]=\sigma_1$$

by the monotonicity of the principal eigenvalue with respect to the domain.

Now, we will show that (5.5) is not only necessary but also sufficient for the existence of a positive solution of (5.3). Suppose (5.5). Then, due to Theorem 3.5, to show the existence of a positive solution it suffices to construct a positive supersolution of (5.3). Actually, thanks to Theorem 3.5, the existence of the positive supersolution itself entails its own uniqueness. It should be noted that, thanks to Theorem 3.5, (5.3) cannot admit a positive supersolution if $\lambda\geqslant\sigma_1$. To construct the supersolution we proceed as follows. For each $j\in\{1,2\}$ and sufficiently small $\delta>0$ consider the open δ-neighborhoods

$$\Omega_{\delta,j}:=\big\{x\in\Omega\colon\ \operatorname{dist}(x,\Omega_{0,j})<\delta\big\},$$

which have been represented in Figure 8. Note that $\Omega_{\delta,1}$ consists of $\Omega_{0,1}$, Γ_1 and the set of points $x\in\Omega_-$ such that $\operatorname{dist}(x,\Gamma_1)<\delta$. Similarly, $\Omega_{\delta,2}$ consists of $\bar{\Omega}_{0,2}$ and the set of points $x\in\Omega_-$ such that $\operatorname{dist}(x,\Gamma_2)<\delta$. By the continuous dependence of the principal eigenvalues with respect to the domain (e.g., [43], Theorem 4.2), we have that

$$\lim_{\delta\downarrow 0}\sigma[-\Delta,\Omega_{\delta,j}]=\sigma[-\Delta,\Omega_{0,j}]=\sigma_j,\qquad j\in\{1,2\}.$$

Thus, by the monotonicity of the principal eigenvalues with respect to the domains, it is apparent that, for each sufficiently small $\delta>0$,

$$\sigma_0<\lambda<\sigma[-\Delta,\Omega_{\delta,1}]<\sigma_1<\sigma[-\Delta,\Omega_{\delta,2}]<\sigma_2. \tag{5.6}$$

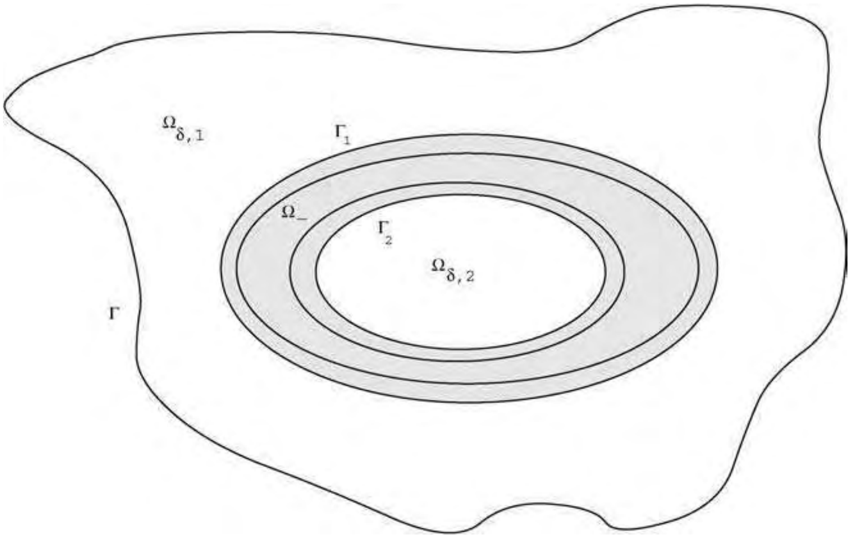

Fig. 8. The δ-neighborhoods $\Omega_{\delta,1}$ and $\Omega_{\delta,2}$.

Fix $\lambda \in (\sigma_0, \sigma_1)$ and pick one of these values of δ. Now, for each $j \in \{1,2\}$, let $\varphi_{\delta,j} \gg 0$ denote a principal eigenfunction associated with $\sigma[-\Delta, \Omega_{\delta,j}]$ – unique up to a multiplicative constant – and consider the function Φ defined through

$$\Phi := \begin{cases} \varphi_{\delta,1} & \text{in } \bar{\Omega}_{\delta/2,1}, \\ \varphi_{\delta,2} & \text{in } \bar{\Omega}_{\delta/2,2}, \\ \varphi_- & \text{in } \Omega \setminus \left(\bar{\Omega}_{\delta/2,1} \cup \bar{\Omega}_{\delta/2,2}\right), \end{cases} \tag{5.7}$$

where φ_- is any smooth extension of $\varphi_{\delta,1} \vee \varphi_{\delta,2}$ to

$$\Omega \setminus \left(\bar{\Omega}_{\delta/2,1} \cup \bar{\Omega}_{\delta/2,2}\right) = \left\{ x \in \Omega_- \colon \operatorname{dist}(x, \partial\Omega_-) > \frac{\delta}{2} \right\}$$

that it is positive and bounded away from zero. Note that φ_- exists, since $\varphi_{\delta,j}$ is positive and bounded away from zero on $\Omega_- \cap \partial\Omega_{\delta/2,j}$, $j \in \{1,2\}$. Figure 9 shows a genuine profile of Φ.

We claim that the function

$$\bar{u} := \kappa\Phi \tag{5.8}$$

is a supersolution of (5.3) for each sufficiently large $\kappa > 1$. Indeed,

$$\kappa\Phi = 0 \quad \text{on } \partial\Omega$$

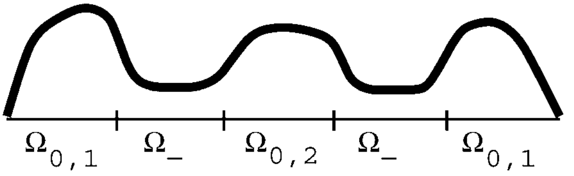

Fig. 9. The profile of the *supersolution element* Φ.

by construction. Moreover, for each $j \in \{1, 2\}$, we have that

$$-\Delta(\kappa\Phi) \geqslant \lambda\kappa\Phi - af(\cdot, \kappa\Phi)\kappa\Phi \quad \text{in } \Omega_{\delta/2,j}$$

if and only if

$$\kappa\sigma[-\Delta, \Omega_{\delta,j}]\varphi_{\delta,j} \geqslant \lambda\kappa\varphi_{\delta,j} - af(\cdot, \kappa\varphi_{\delta,j})\kappa\varphi_{\delta,j} \quad \text{in } \Omega_{\delta/2,j}$$

or equivalently,

$$af(\cdot, \kappa\varphi_{\delta,j}) \geqslant \lambda - \sigma[-\Delta, \Omega_{\delta,j}] \quad \text{in } \Omega_{\delta/2,j}$$

which holds true for every $\kappa > 0$ since, due to (5.6),

$$af(\cdot, \kappa\varphi_{\delta,j}) \geqslant 0 > \lambda - \sigma[-\Delta, \Omega_{\delta,j}] \quad \text{in } \Omega_{\delta,j}.$$

Further, we have that

$$-\Delta(\kappa\Phi) \geqslant \lambda\kappa\Phi - af(\cdot, \kappa\Phi)\kappa\Phi \quad \text{in } \Omega \setminus \left(\bar{\Omega}_{\delta/2,1} \cup \bar{\Omega}_{\delta/2,2}\right)$$

if and only if

$$\frac{-\Delta\varphi_-}{\varphi_-} \geqslant \lambda - af(\cdot, \kappa\varphi_-) \quad \text{in } \Omega \setminus \left(\bar{\Omega}_{\delta/2,1} \cup \bar{\Omega}_{\delta/2,2}\right)$$

which holds true for all sufficiently large $\kappa > 1$, by (Ag), since a and φ_- are positive and bounded away from zero in

$$\Omega \setminus \left(\bar{\Omega}_{\delta/2,1} \cup \bar{\Omega}_{\delta/2,2}\right) \subset\subset \Omega_-.$$

Therefore, (5.8) provides us with a supersolution of (5.3) for sufficiently large $\kappa > 1$. Consequently, by Theorem 3.5, (5.3) possesses a positive solution if and only if condition (5.5) is satisfied. Moreover, it is strongly positive and unique. Subsequently, the unique positive solution of (5.3) will be denoted by

$$\theta(\lambda) = \theta_{[\lambda,\Omega]} = \theta_{[\lambda,\Omega,0]}.$$

Now, the proof of Proposition 4.3 can be easily adapted to prove (2.6) and to show that the map (2.8) in point-wise increasing. Actually,

$$\theta_{[\lambda,\Omega]} \ll \theta_{[\mu,\Omega]} \quad \text{if } \sigma_0 < \lambda < \mu < \sigma_1,$$

which can be derived from Lemma 3.6. So, the technical details of the proofs of these features, by repetitive, will be omitted here in.

Now, we will prove the following property

$$\lim_{\lambda\uparrow\sigma_1} \theta_{[\lambda,\Omega]} = \infty \quad \text{uniformly in compact subsets of } \Omega_{0,1}. \tag{5.9}$$

Pick $\lambda_1 \in (\sigma_0, \sigma_1)$ and consider $\eta > 0$ such that

$$\theta_{[\lambda_1,\Omega]} > \eta\varphi_{0,1} \quad \text{in } \Omega_{0,1}, \tag{5.10}$$

where $\varphi_{0,1} \gg 0$ is a principal eigenfunction associated with $\sigma_1 = \sigma[-\Delta, \Omega_{0,1}]$. By differentiating the realization of (5.3) at $\theta(\lambda)$ with respect to λ, particularizing the result at $\theta(\lambda) = \theta_{[\lambda,\Omega]}$, and rearranging terms gives

$$\begin{cases} \big(-\Delta + a\frac{\partial f}{\partial u}\big(\cdot,\theta(\lambda)\big)\theta(\lambda) + af\big(\cdot,\theta(\lambda)\big) - \lambda\big)\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}(\lambda) = \theta(\lambda) & \text{in } \Omega, \\ \frac{\mathrm{d}\theta}{\mathrm{d}\lambda}(\lambda) = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.11}$$

It should be noted that (5.11) follows straight ahead from the proof of the differentiability of the mapping $\lambda \mapsto \theta(\lambda) := \theta_{[\lambda,\Omega]}$ through the adaptation of the proof of Proposition 4.3. Since

$$a\frac{\partial f}{\partial u}\big(\cdot,\theta(\lambda)\big)\theta(\lambda) > 0 \quad \text{in } \Omega$$

and

$$\lambda = \sigma\big[-\Delta + af\big(\cdot,\theta(\lambda)\big), \Omega\big],$$

we find, from the monotonicity of the principal eigenvalue with respect to the potential, that

$$\begin{aligned} &\sigma\left[-\Delta + a\frac{\partial f}{\partial u}\big(\cdot,\theta(\lambda)\big)\theta(\lambda) + af\big(\cdot,\theta(\lambda)\big) - \lambda, \Omega\right] \\ &\quad > \sigma\big[-\Delta + af\big(\cdot,\theta(\lambda)\big) - \lambda, \Omega\big] = 0 \end{aligned}$$

and therefore, thanks to Theorem 3.1, the differential operator

$$-\Delta + a\frac{\partial f}{\partial u}\big(\cdot,\theta(\lambda)\big)\theta(\lambda) + af\big(\cdot,\theta(\lambda)\big) - \lambda$$

satisfies the strong maximum principle in Ω under Dirichlet boundary conditions. In particular, it follows from (5.11) that $\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}(\lambda) \gg 0$ and, consequently, $\lambda \mapsto \theta(\lambda)$ is *strongly increasing*. Now, due to (5.10), for each $\lambda \in [\lambda_1, \sigma_1)$, we have that

$$\theta_{[\lambda,\Omega]} > \eta\varphi_{0,1} \quad \text{in } \Omega_{0,1},$$

and hence, (5.11) gives

$$\begin{cases} (-\Delta - \lambda)\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}(\lambda) > \eta\varphi_{0,1} & \text{in } \Omega_{0,1}, \\ \frac{\mathrm{d}\theta}{\mathrm{d}\lambda}(\lambda) > 0 & \text{on } \partial\Omega_{0,1} \end{cases} \tag{5.12}$$

since $\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}(\lambda)|_{\Gamma_1} > 0$. Moreover, for each $\lambda \in [\lambda_1, \sigma_1)$,

$$\sigma[-\Delta - \lambda, \Omega_{0,1}] = \sigma_1 - \lambda > 0$$

and therefore, thanks again to Theorem 3.1, we find from (5.12) that

$$\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}(\lambda) \gg \Psi_\lambda \quad \text{in } \Omega_{0,1}, \tag{5.13}$$

where Ψ_λ is the unique solution of

$$\begin{cases} (-\Delta - \lambda)\Psi_\lambda = \eta\varphi_{0,1} & \text{in } \Omega_{0,1}, \\ \Psi_\lambda = 0 & \text{on } \partial\Omega_{0,1}. \end{cases}$$

A direct calculation shows that

$$\Psi_\lambda = \frac{\eta}{\sigma_1 - \lambda}\varphi_{0,1},$$

and hence, (5.13) gives

$$\lim_{\lambda\uparrow\sigma_1} \frac{\mathrm{d}\theta}{\mathrm{d}\lambda}(\lambda) = \infty \quad \text{uniformly in compact subsets of } \Omega_{0,1}.$$

Consequently, (5.9) holds true.

Before ending the proof of Theorem 2.3, we shall begin the proof of Theorem 2.2. Subsequently, for each $M > 0$, we consider the following nonlinear boundary value problem

$$\begin{cases} -\Delta u = \lambda u - af(\cdot, u)u & \text{in } D, \\ u = M & \text{on } \partial D, \end{cases} \tag{5.14}$$

where

$$D := \Omega \setminus \bar{\Omega}_{0,1} = \Omega_- \cup \bar{\Omega}_{0,2}. \tag{5.15}$$

The notation (5.15) will be maintain in the remaining of this section. The following result is satisfied.

LEMMA 5.1. *Problem* (5.14) *possesses a positive solution if and only if* $\lambda < \sigma_2$. *Moreover, it is unique if it exists and if we denote it by* $\theta_{[\lambda,D,M]}$, *then*

$$0 \ll \theta_{[\lambda_1,D,M_1]} \ll \theta_{[\lambda_2,D,M_2]} \quad \text{in } D$$

if

$$-\infty < \lambda_1 \leqslant \lambda_2 < \sigma_2, \qquad 0 < M_1 \leqslant M_2, \qquad \lambda_2 - \lambda_1 + M_2 - M_1 > 0.$$

Also, for any $\lambda < \sigma_2$ *and any positive strong subsolution* (*resp. supersolution*) $\underline{u}$ (*resp.* $\bar{u}$) *of* (5.14), *one has that* $\underline{u} \ll \theta_{[\lambda,D,M]}$ (*resp.* $\bar{u} \gg \theta_{[\lambda,D,M]}$).

Furthermore, for each $M > 0$, $\lambda < \sigma_2$ *and* $u_0 > 0$,

$$\lim_{t\uparrow\infty} u_{[\lambda,D,M]}(\cdot,t;u_0) = \theta_{[\lambda,D,M]} \quad \text{in } \mathcal{C}(\overline{D}),$$

where $u_{[\lambda,D,M]}(x,t;u_0)$ *stands for the unique solution of the parabolic problem*

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = \lambda u - af(\cdot,u)u & \text{in } D \times (0,\infty), \\ u = M & \text{on } \partial D \times (0,\infty), \\ u(\cdot,0) = u_0 & \text{in } D. \end{cases} \tag{5.16}$$

PROOF. For any sufficiently small $\delta > 0$, consider the δ-neighborhood of Γ_1

$$\Gamma_{1,\delta} := \Gamma_1 + B_\delta(0) = \{x \in \Omega \colon \operatorname{dist}(x,\Gamma_1) < \delta\} \tag{5.17}$$

and the open subdomain of Ω defined by

$$\Omega_{1,\delta} := D \cup \Gamma_{1,\delta}; \tag{5.18}$$

$\delta > 0$ must be chosen sufficiently small so that

$$\partial\Omega_{1,\delta} \subset \Omega_{0,1} \quad \text{and} \quad \sigma[-\Delta,\Gamma_{1,\delta}] > \sigma_2 = \sigma[-\Delta,\Omega_{0,2}]. \tag{5.19}$$

The second relation of (5.19) can be got since the Lebesgue measure of $\Gamma_{1,\delta}$ decays to zero as $\delta \downarrow 0$, by Faber–Krahn inequality (cf. [43], if necessary).

Suppose $u > 0$ is a solution of (5.14). Then, thanks to Lemma 3.4, $u \gg 0$. Moreover, since u is a positive strict supersolution of $-\Delta - \lambda + af(\cdot,u)$ in D under homogeneous Dirichlet boundary conditions, it follows from Theorem 3.1 that

$$0 < \sigma[-\Delta - \lambda + af(\cdot,u), D] \leqslant \sigma[-\Delta - \lambda + af(\cdot,u), \Omega_{0,2}] = \sigma_2 - \lambda$$

by the monotonicity of the principal eigenvalue with respect to the domain. Note that $a = 0$ in $\Omega_{0,2}$. Thus, $\lambda < \sigma_2$ is necessary for the existence of a positive solution of (5.14). Consequently, in the remaining of the proof we suppose

$$\lambda < \sigma_2.$$

Thanks to Theorem 3.5 and Lemma 3.6, in order to complete the proof of Lemma 5.1 it suffices to construct a positive supersolution for (5.14).

Let $\varphi_{1,\delta} \gg 0$ be a principal eigenfunction associated with $\sigma[-\Delta, \Gamma_{1,\delta}]$. Note that, due to (5.19),

$$\lambda < \sigma_2 < \sigma[-\Delta, \Gamma_{1,\delta}]. \tag{5.20}$$

Now, reduce $\delta > 0$, if necessary, so that the δ-neighborhood of $\Omega_{0,2}$ defined by

$$\Omega_{\delta,2} := \Omega_{0,2} + B_\delta(0) = \{x \in \Omega\colon \operatorname{dist}(x, \Omega_{0,2}) < \delta\}$$

satisfy

$$\lambda < \sigma[-\Delta, \Omega_{\delta,2}] < \sigma_2, \tag{5.21}$$

and pick a principal eigenfunction associated with $\sigma[-\Delta, \Omega_{\delta,2}]$, say $\varphi_{\delta,2} \gg 0$. Fix a sufficiently small $\delta > 0$ satisfying the previous requirements and consider the function Ψ defined by

$$\Psi := \begin{cases} \varphi_{1,\delta} & \text{in } \bar{\Omega}_{1,\delta} \setminus \{x \in D\colon \operatorname{dist}(x, \Gamma_1) > \frac{\delta}{2}\}, \\ \varphi_{\delta,2} & \text{in } \bar{\Omega}_{\frac{\delta}{2},2}, \\ \varphi_- & \text{in } \{x \in \Omega_-\colon \operatorname{dist}(x, \partial\Omega_-) > \frac{\delta}{2}\}, \end{cases} \tag{5.22}$$

where φ_- is any positive smooth extension, bounded away from zero, of $\varphi_{1,\delta} \vee \varphi_{\delta,2}$ to

$$\Omega_{-,\delta} := \left\{x \in \Omega_-\colon \operatorname{dist}(x, \partial\Omega_-) > \frac{\delta}{2}\right\}.$$

We claim that the function

$$\bar{u} := \kappa\Psi$$

is a positive supersolution of (5.14) in D if $\kappa > 1$ is sufficiently large. Indeed, by construction, $\bar{u} \gg 0$, and $\kappa\Psi > M$ on $\Gamma_1 = \partial D$ for sufficiently large $\kappa > 1$ since

$$\min_{\Gamma_1} \varphi_{1,\delta} > 0.$$

Moreover, in the set

$$\left\{x \in \Omega_-\colon \operatorname{dist}(x, \Gamma_1) \leqslant \frac{\delta}{2}\right\}$$

we have that

$$-\Delta(\kappa\Psi) \geqslant \lambda\kappa\Psi - af(\cdot, \kappa\Psi)\kappa\Psi \tag{5.23}$$

if and only if

$$\kappa\sigma[-\Delta, \Gamma_{1,\delta}]\varphi_{1,\delta} \geqslant \lambda\kappa\varphi_{1,\delta} - af(\cdot, \kappa\varphi_{1,\delta})\kappa\varphi_{1,\delta},$$

or equivalently,

$$af(\cdot, \kappa\varphi_{1,\delta}) \geqslant \lambda - \sigma[-\Delta, \Gamma_{1,\delta}]$$

which is true because, due to (5.20),

$$af(\cdot, \kappa\varphi_{1,\delta}) \geqslant 0 > \lambda - \sigma[-\Delta, \Gamma_{1,\delta}].$$

Similarly, in $\Omega_{\delta/2,2}$, (5.23) is satisfied if and only if

$$af(\cdot, \kappa\varphi_{\delta,2}) \geqslant \lambda - \sigma[-\Delta, \Omega_{\delta,2}],$$

which holds true by (5.21). Finally, in $\Omega_{-,\delta}$, (5.23) is satisfied if and only if

$$af(\cdot, \kappa\varphi_{-}) \geqslant \lambda + \frac{\Delta\varphi_{-}}{\varphi_{-}},$$

which holds true for sufficiently large $\kappa > 1$, since φ_{-} and a are positive and bounded away from zero. Consequently, $\bar{u} = \kappa\Psi$ provides us with a positive supersolution of (5.14) for any sufficiently large $\kappa > 1$, which ends the proof. □

Thanks to Lemma 5.1, the point-wise limit

$$\theta_{[\lambda,D,\infty]} := \lim_{M\uparrow\infty} \theta_{[\lambda,D,M]} \tag{5.24}$$

is well defined in D for each $\lambda < \sigma_2$. We claim that it is finite everywhere in D. Indeed, setting

$$\alpha_M := \max\Big\{M, \max_{\Gamma_2} \theta_{[\lambda,D,M]}\Big\} + 1, \quad M > 0,$$

we have that

$$\theta_{[\lambda,D,M]}|_{\partial\Omega_{-}} < \alpha_M, \quad M > 0,$$

and hence, $\theta_{[\lambda,D,M]}|_{\Omega_{-}}$ provides us with a positive strong subsolution of

$$\begin{cases} -\Delta u = \lambda u - af(\cdot, u)u & \text{in } \Omega_{-}, \\ u = \alpha_M & \text{on } \partial\Omega_{-}. \end{cases} \tag{5.25}$$

Thus, thanks to Lemma 4.10 and Theorem 4.11, we have that, for each $M > 0$,

$$\theta_{[\lambda,D,M]} \ll \theta_{[\lambda,\Omega_{-},\alpha_M]} \ll L^{\min}_{[\lambda,\Omega_{-}]} \quad \text{in } \Omega_{-},$$

where $L^{\min}_{[\lambda,\Omega_-]}$ stands for the minimal large solution of (1.9) in Ω_-. Therefore, passing to the limit as $M \uparrow \infty$ gives

$$\theta_{[\lambda,D,\infty]} \leqslant L^{\min}_{[\lambda,\Omega_-]} \quad \text{in } \Omega_-. \tag{5.26}$$

This shows that (5.24) is finite in Ω_-. To prove that it is finite in $\bar{\Omega}_{0,2}$ we can use the following result.

LEMMA 5.2. *For each sufficiently large $n \in \mathbb{N}$, say $n \geqslant n_0$, let D_n denote the open set defined in* (4.35), *where D is given by* (5.15), *and, for each $M > 0$, consider the boundary value problem*

$$\begin{cases} -\Delta u = \lambda u - af(\cdot,u)u & \text{in } D_n, \\ u = M & \text{on } \partial D_n. \end{cases} \tag{5.27}$$

Then, (5.27) *possesses a positive solution if and only if $\lambda < \sigma_2$. Moreover, it is unique if it exists and if we denote it by $\theta_{[\lambda,D_n,M]}$, then, for each $n \geqslant n_0$,*

$$0 \ll \theta_{[\lambda_1,D_n,M_1]} \ll \theta_{[\lambda_2,D_n,M_2]} \quad \text{in } D$$

provided

$$-\infty < \lambda_1 \leqslant \lambda_2 < \sigma_2, \qquad 0 < M_1 \leqslant M_2, \qquad \lambda_2 - \lambda_1 + M_2 - M_1 > 0.$$

Moreover, for any $\lambda < \sigma_2$ and any positive strict subsolution (*resp. supersolution*) $\underline{u}$ (*resp.* $\bar{u}$) *of* (5.27) *one has that $\underline{u} \ll \theta_{[\lambda,D_n,M]}$ (resp. $\bar{u} \gg \theta_{[\lambda,D_n,M]}$).*

Furthermore, for each $M > 0$, $\lambda < \sigma_2$, and $u_0 > 0$,

$$\lim_{t\uparrow\infty} u_{[\lambda,D_n,M]}(\cdot,t;u_0) = \theta_{[\lambda,D_n,M]},$$

where $u_{[\lambda,D_n,M]}(x,t;u_0)$ stands for the unique solution of the parabolic problem

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = \lambda u - af(\cdot,u)u & \text{in } D_n \times (0,\infty), \\ u = M & \text{on } \partial D_n \times (0,\infty), \\ u(\cdot,0) = u_0 & \text{in } D_n. \end{cases}$$

The proof of Lemma 5.2 is easier than the proof of Lemma 5.1 since a is positive and bounded away from zero on ∂D_n, which simplifies the construction of the basic supersolution pattern Ψ. Therefore, its technical details are omitted here in by repetitive.

Now, fix $n \geqslant n_0$. As ∂D_n is a compact subset of Ω_-, due to (5.26), there exists a constant $C(n) > 0$ such that, for each $M > 0$,

$$\theta_{[\lambda,D,M]} \leqslant C(n) \quad \text{on } \partial D_n.$$

Thus, thanks to Lemma 5.2, for each $M > 0$ we find that

$$\theta_{[\lambda,D,M]} \leqslant \theta_{[\lambda,D_n,C(n)]} \quad \text{in } D_n$$

and, consequently, $\theta_{[\lambda,D,\infty]}$ is finite in D_n. As this argument is valid for each $n \geqslant n_0$ and

$$D = \bigcup_{n=n_0}^{\infty} D_n, \tag{5.28}$$

necessarily $\theta_{[\lambda,D,\infty]}$ is finite in D.

The previous argument can be easily adapted to show that, for any $n \geqslant n_0$, each of the point-wise limits

$$\theta_{[\lambda,D_n,\infty]} := \lim_{M\uparrow\infty} \theta_{[\lambda,D_n,M]} \tag{5.29}$$

is well defined and finite in D_n.

Now, adapting the argument of the final part of the proof of Proposition 4.7, it is easy to see that $\theta_{[\lambda,D,\infty]} \in \mathcal{C}^{2+\mu}(D)$ provides us with a positive solution of

$$\begin{cases} -\Delta u = \lambda u - af(\cdot,u)u & \text{in } D, \\ u = \infty & \text{on } \partial D. \end{cases} \tag{5.30}$$

Actually, it is the minimal positive solution. Indeed, let L be any positive solution of (5.30). Then, for each $M > 0$, there exist a constant $C > 0$ and a sufficiently large $n \in \mathbb{N}$ such that

$$\theta_{[\lambda,D,M]} \leqslant C \leqslant L \quad \text{in } D \setminus \bar{D}_n. \tag{5.31}$$

Moreover, thanks to Lemma 5.2, (5.31) implies

$$\theta_{[\lambda,D,M]} \leqslant \theta_{[\lambda,D_n,C]} \leqslant L \quad \text{in } D_n,$$

and consequently,

$$\theta_{[\lambda,D,M]} \leqslant L \quad \text{in } D. \tag{5.32}$$

Therefore, passing to the limit as $M \uparrow \infty$ in (5.32) we find that

$$\theta_{[\lambda,D,\infty]} \leqslant L,$$

which shows its minimality. Consequently, the solution $\theta_{[\lambda,D,\infty]}$ provides us with the large solution $L^{\min}_{[\lambda,D]}$ referred to in the statement of Theorem 2.2.

Similarly, for each $n \geqslant n_0$, the point-wise limit (5.29) provides us with the minimal positive solution of

$$\begin{cases} -\Delta u = \lambda u - af(\cdot,u)u & \text{in } D_n, \\ u = \infty & \text{on } \partial D_n, \end{cases} \tag{5.33}$$

which will be subsequently denoted by $L^{\min}_{[\lambda,D_n]}$.

We now show that the point-wise limit

$$L^{\max}_{[\lambda,D]} := \lim_{n\uparrow\infty} L^{\min}_{[\lambda,D_n]} \tag{5.34}$$

is well defined and that, actually, it provides us with the maximal solution of (5.30). Indeed, for each $M>0$, $n \geqslant n_0$ and $m \geqslant 1$, we have that

$$\theta_{[\lambda,D_{n+m},M]} \leqslant \theta_{[\lambda,D_{n+m},\infty]} \leqslant \theta_{[\lambda,D_n,\infty]} \quad \text{in } D_n. \tag{5.35}$$

Thus, due to (5.28), for each compact subset $K \subset D$, there exist $C(K)>0$ and $n_K \in \mathbb{N}$ such that, for each $M>0$ and $n \geqslant n_K$,

$$\theta_{[\lambda,D_n,M]} \leqslant C(K) \quad \text{in } K,$$

and hence,

$$L^{\min}_{[\lambda,D_n]} \leqslant C(K) \quad \text{in } K. \tag{5.36}$$

From these a priori estimates, a diagonal argument combined with the compactness of $(-\Delta)^{-1}$, shows that (5.34) is finite in D and that it provides us with a solution of (5.30). To show the maximality of $L^{\max}_{[\lambda,D]}$, let L be any positive solution of (5.30). Then, for each $n \geqslant n_0$, there exists $M>0$ such that $L|_{D_n}$ provides us with a positive subsolution of (5.27), and therefore, thanks to Lemma 5.2,

$$L \leqslant \theta_{[\lambda,D_n,M]} \leqslant \theta_{[\lambda,D_n,\infty]} \quad \text{in } D_n.$$

Consequently, passing to the limit as $n \uparrow \infty$ gives

$$L \leqslant L^{\max}_{[\lambda,D]}.$$

To conclude the proof of the existence of Theorem 2.2 it remains to shows that $\lambda < \sigma_2$ is necessary for the existence of a positive solution of (5.30). Suppose L is a solution of (5.30) for some $\lambda \in \mathbb{R}$. Then, for each $n \geqslant n_0$, $L|_{D_n}$ provides us with a positive strict supersolution of $-\Delta + af(\cdot,L) - \lambda$ in D_n under homogeneous Dirichlet boundary conditions, and hence, thanks to Theorem 3.1,

$$\sigma\big[-\Delta + af(\cdot,L) - \lambda,\, D_n\big] > 0.$$

Therefore, by the monotonicity of the principal eigenvalue with respect to the domain

$$\lambda < \sigma\big[-\Delta + af(\cdot,L),\, D_n\big] < \sigma\big[-\Delta + af(\cdot,L),\, \Omega_{0,2}\big] = \sigma_2$$

since $a=0$ in $\Omega_{0,2}$. Note that, in order to complete the proof of Theorem 2.2, it remains to show (2.5).

Now, we shall show that

$$\lim_{\lambda\uparrow\sigma_2} L^{\min}_{[\lambda,D]} = \infty \quad \text{uniformly in compact subsets of } \Omega_{0,2}. \tag{5.37}$$

To prove (5.37) we need the following result.

LEMMA 5.3. *Consider the problem*

$$\begin{cases} -\Delta u = \lambda u - af(\cdot,u)u & \text{in } D, \\ u = 0 & \text{on } \partial D, \end{cases} \tag{5.38}$$

where $D = \Omega \setminus \overline{\Omega}_{0,1}$. *Then,* (5.38) *possesses a positive solution if and only if*

$$\sigma[-\Delta, D] < \lambda < \sigma_2. \tag{5.39}$$

Moreover, it is unique if it exists and if we denote it by $\theta_{[\lambda,D]}$, *then the mapping*

$$\begin{aligned} &\big(\sigma[-\Delta, D], \sigma_2\big) \xrightarrow{\theta} \mathcal{C}\big(\overline{\Omega}\big), \\ &\lambda \mapsto \theta(\lambda) := \theta_{[\lambda,D]} \end{aligned} \tag{5.40}$$

is point-wise increasing and of class $\mathcal{C}^1$. *Furthermore,*

$$\lim_{\lambda\uparrow\sigma_2} \theta_{[\lambda,D]} = \infty \quad \text{uniformly in compact subsets of } \Omega_{0,2}. \tag{5.41}$$

PROOF. Suppose $u > 0$ is a solution of (5.38). Then, by Lemma 3.4, $u \gg 0$ and

$$\lambda = \sigma\big[-\Delta + af(\cdot,u), D\big].$$

Thus,

$$\sigma[-\Delta, D] < \lambda = \sigma\big[-\Delta + af(\cdot,u), D\big] < \sigma[-\Delta, \Omega_{0,2}] = \sigma_2$$

since $a = 0$ in $\Omega_{0,2}$. Therefore, (5.39) is necessary for the existence of a positive solution. Suppose (5.39). Then, thanks to Lemma 5.1, the problem

$$\begin{cases} -\Delta u = \lambda u - af(\cdot,u)u & \text{in } D, \\ u = 1 & \text{on } \partial D \end{cases}$$

possesses a unique positive solution, $\theta_{[\lambda,D,1]}$. Clearly, $\bar{u} := \theta_{[\lambda,D,1]}$ is a positive supersolution of (5.38) and consequently, the lemma is a direct consequence from Theorem 3.5 and Lemma 3.6, except for the regularity of the map defined in (5.40) and (5.41), which can be easily obtained by adapting the argument used to prove the regularity of the map (2.8) and the proof of (5.9), respectively. By repetitive, we shall omit the technical details here in. □

Now, we are ready to prove (5.37). Thanks to Lemma 5.1, for each $M > 0$ and $\lambda, \mu \in \mathbb{R}$, $\lambda < \mu < \sigma_2$, we have that

$$\theta_{[\lambda,D,M]} \ll \theta_{[\mu,D,M]}$$

and hence, passing to the limit as $M \uparrow \infty$ gives

$$L^{\min}_{[\lambda,D]} \leqslant L^{\min}_{[\mu,D]}.$$

Thus, the point-wise limit

$$L_{\sigma_2} := \lim_{\lambda\uparrow\sigma_2} L^{\min}_{[\lambda,D]} \tag{5.42}$$

is well defined in $\bar{D}$. Moreover, for each $M > 0$, we have that

$$L^{\min}_{[\lambda,D]} \gg \theta_{[\lambda,D,M]} \gg \theta_{[\lambda,D]},$$

and therefore, (5.37) follows from (5.41).

Subsequently, we shall complete the proof of (2.7), which concludes the proof of Theorem 2.3. For a fixed $n \geqslant n_0$ we will consider the open set D_n defined in (4.35) with $D = \Omega \setminus \bar{\Omega}_{0,1}$. For each $M > 0$, let $\theta_{[\sigma_1,D_n,M]}$ be the unique positive solution of (5.27) at $\lambda = \sigma_1$. Such solution exists by Lemma 5.2. For each $\lambda \in (\sigma_0, \sigma_1)$, set

$$M_\lambda := \max_{\partial D_n} \theta_{[\lambda,\Omega]}.$$

Then $\theta_{[\lambda,\Omega]}|_{\partial D_n} \leqslant M_\lambda$ and, thanks to Lemma 5.2, we find that

$$\theta_{[\lambda,\Omega]} \leqslant \theta_{[\sigma_1,D_n,M_\lambda]} \quad \text{in } D_n.$$

Thus,

$$\theta_{[\lambda,\Omega]} \leqslant L^{\min}_{[\sigma_1,D_n]} \quad \text{in } D_n \tag{5.43}$$

since

$$L^{\min}_{[\sigma_1,D_n]} = \lim_{M\uparrow\infty} \theta_{[\sigma_1,D_n,M]}.$$

As the mapping $\lambda \mapsto \theta_{[\lambda,\Omega]}$ is increasing and (5.43) holds for each $\lambda \in (\sigma_0, \sigma_1)$ and $n \geqslant n_0$, it is apparent that the point-wise limit

$$L_{\sigma_1} := \lim_{\lambda\uparrow\sigma_1} \theta_{[\lambda,\Omega]} \quad \text{in } D \tag{5.44}$$

is well defined and finite. Actually, $L_{\sigma_1} \in \mathcal{C}^{2+\mu}(D)$ provides us with a solution of (1.9) in D. In order to prove the identity

$$L_{\sigma_1} = L^{\min}_{[\sigma_1, D]},$$

it suffices to show that

$$\lim_{\lambda \uparrow \sigma_1} \min_{\partial D} \theta_{[\lambda, \Omega]} = \infty \tag{5.45}$$

(note that $\partial D = \Gamma_1$). Indeed, suppose (5.45) has been proven. Then, setting

$$m_\lambda := \min_{\partial D} \theta_{[\lambda, \Omega]}, \quad \lambda \in (\sigma_0, \sigma_1),$$

it follows from Lemma 5.1 that

$$\theta_{[\lambda, \Omega]} \geqslant \theta_{[\lambda, D, m_\lambda]} \quad \text{in } D.$$

In particular, for each $\varepsilon > 0$ and $\lambda \in (\sigma_1 - \varepsilon, \sigma_1)$,

$$\theta_{[\lambda, \Omega]} \geqslant \theta_{[\sigma_1 - \varepsilon, D, m_\lambda]} \quad \text{in } D.$$

Thus, passing to the limit as $\lambda \uparrow \sigma_1$, (5.44) implies

$$L_{\sigma_1} \geqslant L^{\min}_{[\sigma_1 - \varepsilon, D]} \quad \text{in } D$$

for each $\varepsilon > 0$. This shows that L_{σ_1} provides us with a large solution of (1.9) in D. Similarly, we have that

$$\theta_{[\lambda, \Omega]} \leqslant \theta_{[\lambda, D, \max_{\partial D} \theta_{[\lambda, \Omega]}]} \leqslant L^{\min}_{[\sigma_1, D]} \quad \text{in } D$$

and therefore,

$$L_{\sigma_1} \leqslant L^{\min}_{[\sigma_1, D]} \quad \text{in } D.$$

Consequently,

$$L_{\sigma_1} = L^{\min}_{[\sigma_1, D]} \quad \text{in } D,$$

and the proof of Theorem 2.3 will be completed if we show (5.45). Note that (5.9) and (5.45) imply

$$\lim_{\lambda \uparrow \sigma_1} \theta_{[\lambda, \Omega]} = \infty \quad \text{in } \bar{\Omega}_{0,1} \setminus \partial \Omega.$$

Thus, $1/\theta_{[\lambda, \Omega]}$, $\lambda \in (\sigma_0, \sigma_1)$, provides us with a decreasing family of continuous functions point-wise converging to zero in $\bar{\Omega}_{0,1} \setminus \partial \Omega$, and therefore, it approximates zero uniformly in compact subsets of $\bar{\Omega}_{0,1} \setminus \partial\Omega$. Consequently, $\theta_{[\lambda,\Omega]}$ approximates ∞ as $\lambda \uparrow \sigma_1$ uniformly in compact subsets of $\bar{\Omega}_{0,1} \setminus \partial\Omega$.

To prove (5.45) we argue by contradiction. Suppose (5.45) is not satisfied. Then

$$b := \min_{\partial D} L_{\sigma_1} \in (0, \infty). \tag{5.46}$$

As

$$\partial D = \Gamma_1 = \partial\Omega_{0,1} \setminus \partial\Omega$$

is of class $\mathcal{C}^2$, there exist $R > 0$ and a map $Y : \partial D \to \Omega_{0,1}$ such that, for each $x \in \partial D = \Gamma_1$,

$$B_R\big(Y(x)\big) \subset \Omega_{0,1}, \qquad \bar{B}_R\big(Y(x)\big) \cap \partial\Omega = \emptyset, \qquad \bar{B}_R\big(Y(x)\big) \cap \partial D = \{x\}.$$

Moreover, thanks to (5.46), for each $\lambda \in (\sigma_0, \sigma_1)$, there exists $x_\lambda \in \partial D$ such that

$$\theta_{[\lambda,\Omega]}(x_\lambda) = \min_{\partial D} \theta_{[\lambda,\Omega]} \leqslant \min_{\partial D} L_{\sigma_1} = b.$$

By construction,

$$\operatorname{dist}\big(x_\lambda, Y(x_\lambda)\big) = R \quad \text{for each } \lambda \in (\sigma_0, \sigma_1).$$

Moreover, the manifold Γ_1^R defined by

$$\Gamma_1^R \doteq \big\{y \in \Omega_{0,1} \colon \operatorname{dist}(y, \partial D) = 2R\big\}$$

is a compact subset of $\Omega_{0,1}$, and hence, thanks to (5.9),

$$\lim_{\lambda\uparrow\sigma_1} \theta_{[\lambda,\Omega]} = \infty \quad \text{uniformly in } \Gamma_1^R. \tag{5.47}$$

Setting

$$\Omega_{0,1}^R := \big\{y \in \Omega_{0,1} \colon \operatorname{dist}(y, \partial D) < 2R\big\},$$

it turns out that Γ_1^R is one of the components of $\partial\Omega_{0,1}^R$, while the other one consists of ∂D. Let $\tilde{x}_\lambda \in \bar{\Omega}_{0,1}^R$ be such that

$$\theta_{[\lambda,\Omega]}(\tilde{x}_\lambda) = \min_{\bar{\Omega}_{0,1}^R} \theta_{[\lambda,\Omega]}$$

and suppose $\tilde{x}_\lambda \in \Omega_{0,1}^R$. Then

$$\nabla\theta_{[\lambda,\Omega]}(\tilde{x}_\lambda) = 0, \qquad \Delta\theta_{[\lambda,\Omega]}(\tilde{x}_\lambda) \geqslant 0,$$

and, since

$$(-\Delta-\lambda)\theta_{[\lambda,\Omega]}=0 \quad \text{in } \Omega^R_{0,1},$$

we find that

$$-\Delta\theta_{[\lambda,\Omega]}(\tilde{x}_\lambda)=\lambda\theta_{[\lambda,\Omega]}(\tilde{x}_\lambda)>0$$

for each $\lambda\in(\sigma_0,\sigma_1)$ because $\lambda>0$, which is contradictory. Consequently, for each $\lambda\in(\sigma_0,\sigma_1)$,

$$\tilde{x}_\lambda\in\partial\Omega^R_{0,1}=\Gamma_1\cup\Gamma_1^R.$$

Therefore, thanks to (5.47), there exists $\hat{\sigma}_1\in(\sigma_0,\sigma_1)$ such that

$$\min_{\overline{\Omega}^R_{0,1}}\theta_{[\lambda,\Omega]}=\min_{\partial D}\theta_{[\lambda,\Omega]}=\theta_{[\lambda,\Omega]}(x_\lambda)\leqslant b$$

for each $\lambda\in(\hat{\sigma}_1,\sigma_1)$. In particular,

$$\theta_{[\lambda,\Omega]}(x)\geqslant\theta_{[\lambda,\Omega]}(x_\lambda) \quad \text{for each } x\in\overline{B}_R\big(Y(x_\lambda)\big). \tag{5.48}$$

Now, for each $\alpha>0$ and $\lambda\in(\hat{\sigma}_1,\sigma_1)$, we consider the auxiliary function ψ_λ defined by

$$\psi_\lambda(x):=\mathrm{e}^{-\alpha|x-Y(x_\lambda)|^2}-\mathrm{e}^{-\alpha R^2} \quad \text{for each } x\in\overline{B}_R\big(Y(x_\lambda)\big).$$

A direct calculation shows that, for each $x\in B_R(Y(x_\lambda))$,

$$(-\Delta-\lambda)\psi_\lambda(x)=\big(2\alpha N-4\alpha^2\big|x-Y(x_\lambda)\big|^2-\lambda\big)\mathrm{e}^{-\alpha|x-Y(x_\lambda)|^2}+\lambda\mathrm{e}^{-\alpha R^2}$$

and hence, there exist $\alpha>0$ and $\omega>0$ such that, for each $\lambda\in(\hat{\sigma}_1,\sigma_1)$,

$$(-\Delta-\lambda)\psi_\lambda\leqslant-\omega \quad \text{in } A_R:=B_R\big(Y(x_\lambda)\big)\setminus\overline{B}_{R/2}\big(Y(x_\lambda)\big). \tag{5.49}$$

Subsequently, we will assume that α has been chosen to satisfy (5.49). Since $\overline{B}_{R/2}(Y(x_\lambda))$ is a compact subset of $\Omega_{0,1}$, it follows from (5.9) that

$$\lim_{\lambda\uparrow\sigma_1}\min_{\overline{B}_{R/2}(Y(x_\lambda))}\theta_{[\lambda,\Omega]}=\infty$$

and hence, setting

$$c_\lambda:=\frac{\min_{\overline{B}_{R/2}(Y(x_\lambda))}\theta_{[\lambda,\Omega]}-\theta_{[\lambda,\Omega]}(x_\lambda)}{\mathrm{e}^{-\alpha R^2/4}-\mathrm{e}^{-\alpha R^2}}$$

we have that

$$\lim_{\lambda\uparrow\sigma_1} c_\lambda = \infty \tag{5.50}$$

since

$$\lim_{\lambda\uparrow\sigma_1} \theta_{[\lambda,\Omega]}(x_\lambda) = b.$$

By the definition of c_λ, for each $\lambda \in (\hat\sigma_1, \sigma_1)$, we have that

$$\theta_{[\lambda,\Omega]}(x) \geqslant \theta_{[\lambda,\Omega]}(x_\lambda) + c_\lambda\bigl(\mathrm{e}^{-\alpha R^2/4} - e^{-\alpha R^2}\bigr) \quad \forall x \in \overline{B}_{R/2}\bigl(Y(x_\lambda)\bigr). \tag{5.51}$$

Now, for each $\lambda \in (\hat\sigma_1, \sigma_1)$, we consider the auxiliary function

$$v_\lambda := \theta_{[\lambda,\Omega]} - \theta_{[\lambda,\Omega]}(x_\lambda) - c_\lambda \psi_\lambda \quad \text{in } A_R = B_R\bigl(Y(x_\lambda)\bigr) \setminus \overline{B}_{R/2}\bigl(Y(x_\lambda)\bigr).$$

Thanks to (5.51),

$$v_\lambda \geqslant 0 \quad \text{on } \partial B_{R/2}\bigl(Y(x_\lambda)\bigr).$$

Moreover, since $\psi_\lambda = 0$ on $\partial B_R(Y(x_\lambda))$, it follows from (5.48) that

$$v_\lambda = \theta_{[\lambda,\Omega]} - \theta_{[\lambda,\Omega]}(x_\lambda) \geqslant 0 \quad \text{on } \partial B_R\bigl(Y(x_\lambda)\bigr).$$

Furthermore, due to (5.49), in A_R we have that

$$(-\Delta - \lambda)v_\lambda = \lambda\theta_{[\lambda,\Omega]}(x_\lambda) - c_\lambda(-\Delta - \lambda)\psi_\lambda \geqslant \lambda\theta_{[\lambda,\Omega]}(x_\lambda) + c_\lambda\omega$$

and hence, by (5.50),

$$(-\Delta - \lambda)v_\lambda > 0 \quad \text{in } A_R = B_R\bigl(Y(x_\lambda)\bigr) \setminus \overline{B}_{R/2}\bigl(Y(x_\lambda)\bigr)$$

provided $\lambda < \sigma_1$ is sufficiently close to σ_1. Therefore, since

$$\lambda < \sigma_1 = \sigma[-\Delta, \Omega_{0,1}] < \sigma[-\Delta, A_R],$$

it follows from Theorem 3.1 that

$$v_\lambda(x) > 0 \quad \text{if } \frac{R}{2} < \bigl|x - Y(x_\lambda)\bigr| < R,$$

and consequently,

$$\theta_{[\lambda,\Omega]}(x) \geqslant \theta_{[\lambda,\Omega]}(x_\lambda) + c_\lambda\psi_\lambda(x) \quad \text{for each } x \in A_R. \tag{5.52}$$

Now, setting

$$n_\lambda := \frac{Y(x_\lambda) - x_\lambda}{R},$$

we have that

$$\frac{\partial \theta_{[\lambda,\Omega]}}{\partial n_\lambda}(x_\lambda) = \lim_{t\downarrow 0} \frac{\theta_{[\lambda,\Omega]}(x_\lambda + tn_\lambda) - \theta_{[\lambda,\Omega]}(x_\lambda)}{t}.$$

Moreover, thanks to (5.52), for each $t \in (0, \frac{R}{2})$, we find that

$$\begin{aligned} &\frac{\theta_{[\lambda,\Omega]}(x_\lambda + tn_\lambda) - \theta_{[\lambda,\Omega]}(x_\lambda)}{t} \\ &\qquad \geqslant \frac{c_\lambda \psi_\lambda(x_\lambda + tn_\lambda)}{t} \\ &\qquad = \frac{c_\lambda(\mathrm{e}^{-\alpha|x_\lambda + tn_\lambda - Y(x_\lambda)|^2} - \mathrm{e}^{-\alpha R^2})}{t} \\ &\qquad = \frac{c_\lambda(\mathrm{e}^{-\alpha|tn_\lambda - Rn_\lambda|^2} - \mathrm{e}^{-\alpha R^2})}{t} \\ &\qquad = \frac{c_\lambda(\mathrm{e}^{-\alpha(R-t)^2} - \mathrm{e}^{-\alpha R^2})}{t}, \end{aligned}$$

and hence, since

$$\lim_{t\downarrow 0} \frac{\mathrm{e}^{-\alpha(R-t)^2} - \mathrm{e}^{-\alpha R^2}}{t} = 2\alpha R \mathrm{e}^{-\alpha R^2},$$

we obtain that

$$\frac{\partial \theta_{[\lambda,\Omega]}}{\partial n_\lambda}(x_\lambda) \geqslant 2\alpha R \mathrm{e}^{-\alpha R^2} c_\lambda.$$

Therefore, thanks to (5.50),

$$\lim_{\lambda\uparrow\sigma_1} \frac{\partial \theta_{[\lambda,\Omega]}}{\partial n_\lambda}(x_\lambda) = \infty. \tag{5.53}$$

Subsequently, for each $\lambda < \sigma_1$, $\lambda \sim \sigma_1$, we consider the auxiliary boundary value problem

$$\begin{cases} -\Delta u = \lambda u - af(\cdot, u)u & \text{in } D, \\ u = \theta_{[\lambda,\Omega]}(x_\lambda) & \text{on } \partial D. \end{cases} \tag{5.54}$$

Since $\lambda < \sigma_2$, thanks to Lemma 5.1, (5.54) possesses a unique positive solution, denoted by

$$\vartheta_\lambda := \theta_{[\lambda, D, \theta_{[\lambda,\Omega]}(x_\lambda)]}.$$

Moreover, since $\theta_{[\lambda,\Omega]}|_D$ provides us with a positive supersolution of (5.54), we have that

$$\vartheta_\lambda \leqslant \theta_{[\lambda,\Omega]} \quad \text{in } \overline{D}.$$

Therefore, since $\vartheta_\lambda(x_\lambda) = \theta_{[\lambda,\Omega]}(x_\lambda)$, we find that

$$\frac{\partial \vartheta_\lambda}{\partial n_\lambda}(x_\lambda) \geqslant \frac{\partial \theta_{[\lambda,\Omega]}}{\partial n_\lambda}(x_\lambda),$$

and consequently, by (5.53),

$$\lim_{\lambda \uparrow \sigma_1} \frac{\partial \vartheta_\lambda}{\partial n_\lambda}(x_\lambda) = \infty. \tag{5.55}$$

This is impossible, since the functions ϑ_λ approximate in $\mathcal{C}^1(\overline{D})$, as $\lambda \uparrow \sigma_1$, the unique positive solution, $\vartheta_{\sigma_1} := \theta_{[\sigma_1, D, b]}$, of

$$\begin{cases} -\Delta u = \sigma_1 u - af(\cdot, u)u & \text{in } D, \\ u = b & \text{on } \partial D. \end{cases}$$

Therefore, (5.45) get shown, which completes the proof of Theorem 2.3.

Finally, we will complete the proof of Theorem 2.2. It remains to prove (2.5). Thanks to (5.37), it suffices to prove that

$$\lim_{\lambda \uparrow \sigma_2} \min_{\partial \Omega_{0,2}} L^{\min}_{[\lambda, D]} = \infty, \tag{5.56}$$

where $D = \Omega \setminus \overline{\Omega}_{0,1}$, and that, for some large solution L of (1.9) in Ω_-,

$$\lim_{\lambda \uparrow \sigma_2} L^{\min}_{[\lambda, D]} = L \quad \text{in } \Omega_-. \tag{5.57}$$

We already know that, for each $\lambda < \sigma_2$,

$$L^{\min}_{[\lambda, D]} \geqslant \theta_{[\lambda, D]}.$$

Moreover, the proof of (5.45) can be easily adapted to show that

$$\lim_{\lambda \uparrow \sigma_2} \min_{\partial \Omega_{0,2}} \theta_{[\lambda, D]} = \infty;$$

now, one must work with $\partial \Omega_{0,1}$, instead of ∂D, though the technical details can be adapted *mutatis mutandis*. Therefore (5.56) holds true.

In order to prove (5.57), we consider the point-wise limit (5.42), which is well defined and equals ∞ in $\bar{\Omega}_{0,2}$, and, for sufficiently large $n \in \mathbb{N}$, the open sets

$$\Omega_-^n := \left\{x \in \Omega_- \colon \operatorname{dist}(x, \partial\Omega_-) > n^{-1}\right\}.$$

Subsequently, for each $\lambda < \sigma_2$, we set

$$M_\lambda := \max_{\partial\Omega_-^n} L^{\min}_{[\lambda,D]}.$$

By definition, $L^{\min}_{[\lambda,D]}|_{\partial\Omega_-^n} \leqslant M_\lambda$. Hence, thanks to Lemma 4.10,

$$L^{\min}_{[\lambda,D]} \leqslant \theta_{[\lambda,\Omega_-^n,M_\lambda]} \leqslant \theta_{[\sigma_2,\Omega_-^n,M_\lambda]} \leqslant L^{\min}_{[\sigma_2,\Omega_-^n]} \quad \text{in } \Omega_-^n.$$

Therefore, due to Theorem 4.11, we obtain that, for each $\lambda < \sigma_2$,

$$L^{\min}_{[\lambda,D]} \leqslant L^{\max}_{[\sigma_2,\Omega_-]} \quad \text{in } \Omega_-. \tag{5.58}$$

Consequently, the point-wise limit (5.42) is finite in Ω_-. Moreover, by the monotonicity in λ and the compactness of $(-\Delta)^{-1}$, it provides us with a large solution of (1.9) in Ω_-, which concludes the proof of Theorem 2.2.

6. Proof of Theorem 2.4

Part (a) is a direct consequence from the last assertion of Theorem 3.5, as well as part (b) since, due to Theorem 2.3, (1.1) possesses a positive steady state for each $\lambda \in (\sigma_0, \sigma_1)$. So, it remains to prove parts (c) and (d). Suppose

$$\sigma_1 \leqslant \lambda < \sigma_2.$$

Thanks to the parabolic maximum principle, for each $\varepsilon > 0$ and $t \geqslant 0$,

$$u_{[\lambda,\Omega]}(\cdot, t; u_0) \geqslant u_{[\sigma_1-\varepsilon,\Omega]}(\cdot, t; u_0)$$

since $\lambda > \sigma_1 - \varepsilon$, and hence, thanks to part (b),

$$\liminf_{t\uparrow\infty} u_{[\lambda,\Omega]}(\cdot, t; u_0) \geqslant \lim_{t\uparrow\infty} u_{[\sigma_1-\varepsilon,\Omega]}(\cdot, t; u_0) = \theta_{[\sigma_1-\varepsilon,\Omega]}. \tag{6.1}$$

As (6.1) holds true for each sufficiently small $\varepsilon > 0$, it follows from Theorem 2.3 that

$$\liminf_{t\uparrow\infty} u_{[\lambda,\Omega]}(\cdot, t; u_0) \geqslant \lim_{\varepsilon\downarrow 0} \theta_{[\sigma_1-\varepsilon,\Omega]} = \mathfrak{M}^{\min}_{[\sigma_1,\Omega\setminus\bar{\Omega}_{0,1}]}. \tag{6.2}$$

In particular,

$$\liminf_{t\uparrow\infty} u_{[\lambda,\Omega]}(\cdot,t;u_0)=\infty \quad \text{uniformly in compact subsets of } \bar{\Omega}_{0,1}\setminus\partial\Omega, \tag{6.3}$$

and hence, for each $M>0$ there exists a constant $T_M>0$ such that

$$u_{[\lambda,\Omega]}(x,t;u_0)\geqslant M \quad \text{for each } (x,t)\in\Gamma_1\times[T_M,\infty).$$

Thus, $u_{[\lambda,\Omega]}(\cdot,t;u_0)$ provides us with a supersolution of the parabolic problem

$$\begin{cases} \frac{\partial u}{\partial t}-\Delta u=\lambda u-af(\cdot,u)u & \text{in } \big(\Omega\setminus\bar{\Omega}_{0,1}\big)\times(T_M,\infty),\\ u=M & \text{on } \Gamma_1\times(T_M,\infty),\\ u(\cdot,T_M)=u_{[\lambda,\Omega]}(\cdot,T_M;u_0) & \text{in } \Omega\setminus\bar{\Omega}_{0,1}. \end{cases}$$

By the parabolic maximum principle, we have that

$$u_{[\lambda,\Omega]}(x,t;u_0)\geqslant u_{[\lambda,\Omega\setminus\bar{\Omega}_{0,1},M]}\big(x,t-T_M;u_{[\lambda,\Omega]}(\cdot,T_M;u_0)\big)$$

for each $(x,t)\in\Omega\setminus\bar{\Omega}_{0,1}\times(T_M,\infty)$. So, thanks to Lemma 5.1,

$$\begin{aligned}\liminf_{t\uparrow\infty} u_{[\lambda,\Omega]}(x,t;u_0)&\geqslant \lim_{t\uparrow\infty} u_{[\lambda,\Omega\setminus\bar{\Omega}_{0,1},M]}\big(x,t-T_M;u_{[\lambda,\Omega]}(\cdot,T_M;u_0)\big)\\ &=\theta_{[\lambda,\Omega\setminus\bar{\Omega}_{0,1},M]}.\end{aligned}$$

Therefore, passing to the limit as $M\uparrow\infty$,

$$\liminf_{t\uparrow\infty} u_{[\lambda,\Omega]}(x,t;u_0)\geqslant L^{\min}_{[\lambda,\Omega\setminus\bar{\Omega}_{0,1}]} \quad \text{in } \Omega\setminus\bar{\Omega}_{0,1}$$

since

$$L^{\min}_{[\lambda,\Omega\setminus\bar{\Omega}_{0,1}]}=\lim_{M\uparrow\infty}\theta_{[\lambda,\Omega\setminus\bar{\Omega}_{0,1},M]}$$

(see the proof of Theorem 2.2). Consequently, bringing together this estimate with (6.3) we find that

$$\liminf_{t\uparrow\infty} u_{[\lambda,\Omega]}(x,t;u_0)\geqslant \mathfrak{M}^{\min}_{[\lambda,\Omega\setminus\bar{\Omega}_{0,1}]} \quad \text{in } \Omega. \tag{6.4}$$

This estimate provides us with Theorem 2.4(c)(i) and the lower estimate of Theorem 2.4(c)(ii).

Now, we assume, in addition, that u_0 is a subsolution of (1.9) in Ω. Then, for each $t>0$, the function

$$x\mapsto u_{[\lambda,\Omega]}(x,t;u_0),\quad x\in\bar{\Omega},$$

is a subsolution of (1.9) in Ω, since $t \mapsto u_{[\lambda,\Omega]}(\cdot,t;u_0)$ is increasing. Fix $t>0$ and set

$$M_t := \max_{\overline{D}} u_{[\lambda,\Omega]}(\cdot,t;u_0), \qquad D := \Omega \setminus \overline{\Omega}_{0,1}.$$

Thanks to Lemma 5.1, for each $M \geqslant M_t$, we have that

$$u_{[\lambda,\Omega]}(\cdot,t;u_0) \leqslant \theta_{[\lambda,\Omega\setminus\overline{\Omega}_{0,1},M]} \quad \text{in } \Omega \setminus \overline{\Omega}_{0,1}.$$

Hence, by the construction of the minimal large solution,

$$u_{[\lambda,\Omega]}(\cdot,t;u_0) \leqslant \lim_{M\uparrow\infty} \theta_{[\lambda,\Omega\setminus\overline{\Omega}_{0,1},M]} = L^{\min}_{[\lambda,\Omega\setminus\overline{\Omega}_{0,1}]} \quad \text{in } \Omega \setminus \overline{\Omega}_{0,1},$$

and therefore, passing to the limit as $t \uparrow \infty$,

$$\limsup_{t\uparrow\infty} u_{[\lambda,\Omega]}(\cdot,t;u_0) \leqslant L^{\min}_{[\lambda,\Omega\setminus\overline{\Omega}_{0,1}]} \quad \text{in } \Omega \setminus \overline{\Omega}_{0,1}.$$

Consequently, due to (6.4),

$$\lim_{t\uparrow\infty} u_{[\lambda,\Omega]}(\cdot,t;u_0) = \mathfrak{M}^{\min}_{[\lambda,\Omega\setminus\overline{\Omega}_{0,1}]} \quad \text{in } \Omega, \tag{6.5}$$

which concludes the proof of Theorem 2.4(c)(iii).

Now, suppose u_0 is arbitrary – not necessarily a subsolution of (1.9) in Ω – and

$$\lambda \geqslant \sigma_2.$$

Then, thanks to the parabolic maximum principle, we have that

$$u_{[\lambda,\Omega]}(\cdot,t;u_0) \geqslant u_{[\sigma_2-\varepsilon,\Omega]}(\cdot,t;u_0) \quad \text{in } \Omega,$$

and hence, it follows from (6.4) that

$$\liminf_{t\uparrow\infty} u_{[\lambda,\Omega]}(\cdot,t;u_0) \geqslant \liminf_{t\uparrow\infty} u_{[\sigma_2-\varepsilon,\Omega]}(\cdot,t;u_0) \geqslant \mathfrak{M}^{\min}_{[\sigma_2-\varepsilon,\Omega\setminus\overline{\Omega}_{0,1}]} \tag{6.6}$$

in Ω for each sufficiently small $\varepsilon>0$ since

$$\sigma_1 < \sigma_2 - \varepsilon < \sigma_2.$$

Note that we cannot apply (6.5) since u_0 might not be a subsolution of (1.9) in Ω. As (6.6) is satisfied for every sufficiently small $\varepsilon>0$, and, thanks to Theorem 2.2, there exists a metasolution of

$$-\Delta u = \sigma_2 u - af(\cdot,u)u$$

supported in Ω_-, say $\mathfrak{M}_{[\sigma_2,\Omega_-]}$, such that

$$\lim_{\varepsilon\downarrow 0}\mathfrak{M}^{\min}_{[\sigma_2-\varepsilon,\Omega\setminus\bar{\Omega}_{0,1}]}=\mathfrak{M}_{[\sigma_2,\Omega_-]},$$

it follows from (6.6) that

$$\liminf_{t\uparrow\infty} u_{[\lambda,\Omega]}(\cdot,t;u_0)\geqslant \mathfrak{M}_{[\sigma_2,\Omega_-]}\quad \text{in } \Omega.$$

In particular,

$$\liminf_{t\uparrow\infty} u_{[\lambda,\Omega]}(\cdot,t;u_0)=\infty\quad \text{in } \Omega\setminus\Omega_- \tag{6.7}$$

which concludes the proof of Theorem 2.4(d)(i). Moreover, since (6.7) occurs uniformly on $\partial\Omega_-$, for each $M>0$, there exists a time $T_M>0$ for which

$$u_{[\lambda,\Omega]}(x,t;u_0)\geqslant M\quad \text{if } (x,t)\in\partial\Omega_-\times(T_M,\infty).$$

Thus, $u_{[\lambda,\Omega]}(\cdot,t;u_0)$ provides us with a supersolution of the parabolic problem

$$\begin{cases}\frac{\partial u}{\partial t}-\Delta u=\lambda u-auf(\cdot,u) & \text{in } \Omega_-\times(T_M,\infty),\\ u=M & \text{on } \partial\Omega_-\times(T_M,\infty),\\ u(\cdot,T_M)=u_{[\lambda,\Omega]}(\cdot,T_M;u_0) & \text{in } \Omega_-.\end{cases}$$

By the parabolic maximum principle, we have that, for each $(x,t)\in\Omega_-\times(T_M,\infty)$,

$$u_{[\lambda,\Omega]}(x,t;u_0)\geqslant u_{[\lambda,\Omega_-,M]}\big(x,t-T_M;u_{[\lambda,\Omega]}(\cdot,T_M;u_0)\big),$$

and hence, thanks to Theorem 4.9,

$$\begin{aligned}\liminf_{t\uparrow\infty} u_{[\lambda,\Omega]}(x,t;u_0)&\geqslant \lim_{t\uparrow\infty} u_{[\lambda,\Omega_-,M]}\big(x,t-T_M;u_{[\lambda,\Omega]}(\cdot,T_M;u_0)\big)\\ &=\theta_{[\lambda,\Omega_-,M]}.\end{aligned}$$

Therefore, passing to the limit as $M\uparrow\infty$ gives

$$\liminf_{t\uparrow\infty} u_{[\lambda,\Omega]}(\cdot,t;u_0)\geqslant L^{\min}_{[\lambda,\Omega_-]}\quad \text{in } \Omega_-$$

since

$$L^{\min}_{[\lambda,\Omega_-]}=\lim_{M\uparrow\infty}\theta_{[\lambda,\Omega_-,M]}.$$

Consequently, bringing together this estimate with (6.7) gives

$$\liminf_{t\uparrow\infty} u_{[\lambda,\Omega]}(\cdot,t;u_0)\geqslant \mathfrak{M}^{\min}_{[\lambda,\Omega_-]}\quad \text{in } \Omega \tag{6.8}$$

which provides us with the lower estimate of Theorem 2.4(d)(ii).

Now, besides $\lambda \geqslant \sigma_2$, suppose u_0 is a subsolution of (1.9) in Ω. Then, for each $t > 0$, the function $u_{[\lambda,\Omega]}(\cdot,t;u_0)$ is a subsolution of (1.9) in Ω, since $t \mapsto u_{[\lambda,\Omega]}(\cdot,t;u_0)$ is increasing. Fix $t > 0$ and set

$$M_t := \max_{\bar{\Omega}_-} u_{[\lambda,\Omega]}(\cdot,t;u_0).$$

Thanks to Lemma 4.10, for each $M \geqslant M_t$, we have that

$$u_{[\lambda,\Omega]}(\cdot,t;u_0) \leqslant \theta_{[\lambda,\Omega_-,M]} \quad \text{in } \Omega_-,$$

and hence,

$$u_{[\lambda,\Omega]}(\cdot,t;u_0) \leqslant \lim_{M\uparrow\infty} \theta_{[\lambda,\Omega_-,M]} = L^{\min}_{[\lambda,\Omega_-]} \quad \text{in } \Omega_-.$$

Therefore, passing to the limit as $t \uparrow \infty$,

$$\limsup_{t\uparrow\infty} u_{[\lambda,\Omega]}(\cdot,t;u_0) \leqslant L^{\min}_{[\lambda,\Omega_-]} \quad \text{in } \Omega_-$$

and, consequently, due to (6.7) and (6.8),

$$\lim_{t\uparrow\infty} u_{[\lambda,\Omega]}(\cdot,t;u_0) = \mathfrak{M}^{\min}_{[\lambda,\Omega_-]} \quad \text{in } \Omega,$$

which concludes the proof of Theorem 2.4(d)(iii).

To conclude the proof of Theorem 2.4, it remains to obtain the upper estimates for an arbitrary $u_0 > 0$, in order to get the upper estimates of parts (c)(ii) and (d)(ii). To do this, the strategy adopted here consists in obtaining a priori bounds in Ω_- for the solutions of (1.1). These a priori bounds can be obtained as follows.

Fix $\lambda \geqslant \sigma_1$ and consider the function $u_{[\lambda,\Omega]}(\cdot,1;u_0) \gg 0$. Then, there exists $\kappa > 1$ such that

$$u_{[\lambda,\Omega]}(\cdot,1;u_0) < \kappa\varphi, \tag{6.9}$$

where $\varphi \gg 0$ is a principal eigenfunction associated with σ_0. We claim that there exists

$$\Lambda > \max\{\lambda,\sigma_2\} \tag{6.10}$$

for which the function $\kappa\varphi$ is a subsolution of

$$\begin{cases} -\Delta u = \Lambda u - af(\cdot,u)u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.11}$$

Indeed, since $\kappa\varphi = 0$ on $\partial\Omega$, $\kappa\varphi$ is a subsolution of (6.11) if and only if

$$-\Delta(\kappa\varphi) \leqslant \Lambda\kappa\varphi - af(\cdot,\kappa\varphi)\kappa\varphi \quad \text{in } \Omega,$$

or equivalently,

$$af(\cdot,\kappa\varphi) \leqslant \Lambda - \sigma_0 \quad \text{in } \Omega$$

which is satisfied for any sufficiently large Λ satisfying (6.10).

Now, thanks to the parabolic maximum principle, it follows from (6.9) that, for any $(x,t) \in \Omega \times (0,\infty)$,

$$u_{[\lambda,\Omega]}(x,t+1;u_0) = u_{[\lambda,\Omega]}\bigl(x,t;u_{[\lambda,\Omega]}(\cdot,1;u_0)\bigr) \leqslant u_{[\lambda,\Omega]}(\cdot,t;\kappa\varphi).$$

Similarly, (6.10) implies

$$u_{[\lambda,\Omega]}(\cdot,t;\kappa\varphi) < u_{[\Lambda,\Omega]}(\cdot,t;\kappa\varphi).$$

Thus, for each $t > 0$,

$$u_{[\lambda,\Omega]}(\cdot,t+1;u_0) \leqslant u_{[\Lambda;\Omega]}(\cdot,t;\kappa\varphi) \quad \text{in } \Omega. \tag{6.12}$$

As $\kappa\varphi$ is a subsolution of (6.11), it follows from part (d)(iii) that

$$\lim_{t\uparrow\infty} u_{[\Lambda,\Omega]}(\cdot,t;\kappa\varphi) = L^{\min}_{[\Lambda,\Omega_-]} \quad \text{in } \Omega_-,$$

and hence, (6.12) implies

$$\limsup_{t\uparrow\infty} u_{[\lambda,\Omega]}(\cdot,t;u_0) \leqslant L^{\min}_{[\Lambda,\Omega_-]} \quad \text{in } \Omega_-. \tag{6.13}$$

Consequently, $u_{[\lambda,\Omega]}(\cdot,t;u_0)$ is uniformly bounded above in any compact subset of Ω_- for each $t > 0$, which provides us with the necessary a priori bounds to complete the proof of the theorem.

Subsequently, we suppose

$$\sigma_1 \leqslant \lambda < \sigma_2,$$

and for each sufficiently large $n \in \mathbb{N}$, we consider the open set

$$D_n := \left\{x \in D := \Omega \setminus \overline{\Omega}_{0,1}\colon \operatorname{dist}(x,\partial D) > \frac{1}{n}\right\}.$$

Fix one of these values of n. Since $\partial D_n \subset \Omega_-$, it follows from (6.13) that there exists a constant $M_0 > 0$ such that, for each $M \geqslant M_0$ and $t > 0$,

$$u_{[\lambda,\Omega]}(\cdot,t;u_0) \leqslant M \quad \text{on } \partial D_n,$$

and hence, the parabolic maximum principle shows that, for each $t > 0$,

$$u_{[\lambda,\Omega]}(\cdot,t;u_0) \leqslant u_{[\lambda,D_n,M]}(\cdot,t;u_0) \quad \text{in } D_n,$$

where $u_{[\lambda,D_n,M]}(\cdot,t;u_0)$ is the solution defined in Lemma 5.2. Due to Lemma 5.2, we have that

$$\lim_{t\uparrow\infty} u_{[\lambda,D_n,M]}(\cdot,t;u_0) = \theta_{[\lambda,D_n,M]} \quad \text{in } D_n$$

and therefore,

$$\limsup_{t\uparrow\infty} u_{[\lambda,\Omega]}(\cdot,t;u_0) \leqslant \theta_{[\lambda,D_n,M]} \quad \text{in } D_n. \tag{6.14}$$

Consequently, passing to the limit as $M \uparrow \infty$ in (6.14) gives

$$\limsup_{t\uparrow\infty} u_{[\lambda,\Omega]}(\cdot,t;u_0) \leqslant L^{\min}_{[\lambda,D_n]} \quad \text{in } D_n. \tag{6.15}$$

As (6.15) is valid for all sufficiently large $n \in \mathbb{N}$, and, thanks to the analysis carried out in the proof of Theorem 2.2, we already know that

$$L^{\max}_{[\lambda,D]} = \lim_{n\uparrow\infty} L^{\min}_{[\lambda,D_n]},$$

we find from (6.15) that

$$\limsup_{t\uparrow\infty} u_{[\lambda,\Omega]}(\cdot,t;u_0) \leqslant L^{\max}_{[\lambda,D]} \quad \text{in } D,$$

which concludes the proof of part (c)(ii).

Finally, suppose

$$\lambda \geqslant \sigma_2,$$

and, for each sufficiently large $n \in \mathbb{N}$, consider the open subset of Ω_- defined by

$$\Omega^n_- := \left\{ x \in \Omega_- :\ \mathrm{dist}(x, \partial\Omega_-) > \frac{1}{n} \right\}$$

and fix one of these values of n. Since $\partial\Omega^n_- \subset \Omega_-$, it follows from (6.13) that there exists a constant $M_0 > 0$ such that, for each $M \geqslant M_0$ and $t > 0$,

$$u_{[\lambda,\Omega]}(\cdot,t;u_0) \leqslant M \quad \text{on } \partial\Omega^n_-.$$

Thus, by the parabolic maximum principle, we have that, for each $t > 0$,

$$u_{[\lambda,\Omega]}(\cdot,t;u_0) \leqslant u_{[\lambda,\Omega^n_-,M]}(\cdot,t;u_0) \quad \text{in } \Omega^n_-,$$

where $u_{[\lambda,\Omega_-^n,M]}(\cdot,t;u_0)$ is the unique solution of the parabolic problem

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = \lambda u - af(\cdot,u)u & \text{in } \Omega_-^n \times (0,\infty), \\ u = M & \text{on } \partial\Omega_-^n \times (0,\infty), \\ u(\cdot,0) = u_0 & \text{in } \Omega_-^n. \end{cases}$$

Thanks to Theorem 4.9,

$$\lim_{t\uparrow\infty} u_{[\lambda,\Omega_-^n,M]}(\cdot,t;u_0) = \theta_{[\lambda,\Omega_-^n,M]} \quad \text{in } \Omega_-^n$$

and therefore,

$$\limsup_{t\uparrow\infty} u_{[\lambda,\Omega]}(\cdot,t;u_0) \leqslant \theta_{[\lambda,\Omega_-^n,M]} \quad \text{in } \Omega_-^n. \tag{6.16}$$

Consequently, passing to the limit as $M \uparrow \infty$ in (6.16) gives

$$\limsup_{t\uparrow\infty} u_{[\lambda,\Omega]}(\cdot,t;u_0) \leqslant L^{\min}_{[\lambda,\Omega_-^n]} \quad \text{in } \Omega_-^n. \tag{6.17}$$

As (6.17) is valid for all sufficiently large $n \in \mathbb{N}$ and we already know that

$$L^{\max}_{[\lambda,\Omega_-]} = \lim_{n\uparrow\infty} L^{\min}_{[\lambda,\Omega_-^n]},$$

(6.17) implies

$$\limsup_{t\uparrow\infty} u_{[\lambda,\Omega]}(\cdot,t;u_0) \leqslant L^{\max}_{[\lambda,\Omega_-]} \quad \text{in } \Omega_-$$

which concludes the proof of part (d)(ii). The proof of the theorem is completed.

7. Proofs of Theorems 2.6–2.8

The proofs of Theorems 2.6 and 2.7 are based upon the following estimates for the boundary growth rate of the large positive solutions of

$$-\Delta u = \lambda u - au^p \tag{7.1}$$

in $D \in \{\Omega \setminus \bar{\Omega}_{0,1}, \Omega_-\}$.

PROPOSITION 7.1. *Under the assumptions of Theorem* 2.6, *for each* $\lambda < \sigma_2$ *and any large positive solution* $L(x)$ *of* (7.1) *in* $\Omega \setminus \bar{\Omega}_{0,1}$, *one has that*

$$\lim_{t\downarrow 0} \frac{L(x_1 - t\mathbf{n}_{x_1})}{B(x_1)[\operatorname{dist}(x_1 - t\mathbf{n}_{x_1}, \Gamma_1)]^{-\alpha(x_1)}} = 1 \quad \textit{for each } x_1 \in \Gamma_1, \tag{7.2}$$

where $\mathbf{n}_{x_1}$ *stands for the outward unit normal to* $\Omega \setminus \bar{\Omega}_{0,1}$ *at* x_1 *and*

$$\alpha(x_1) := \frac{\gamma(x_1)+2}{p-1}, \qquad B(x_1) := \left[\frac{\alpha(x_1)(\alpha(x_1)+1)}{\beta(x_1)}\right]^{1/(p-1)}. \tag{7.3}$$

Therefore, for any pair (L_1, L_2) *of large positive solutions of* (7.1) *in* $\Omega \setminus \bar{\Omega}_{0,1}$, *one has that*

$$\lim_{t\downarrow 0} \frac{L_1(x_1 - t\mathbf{n}_{x_1})}{L_2(x_1 - t\mathbf{n}_{x_1})} = 1 \quad \textit{for each } x_1 \in \Gamma_1. \tag{7.4}$$

Notice that, for each $x_1 \in \Gamma_1$ and sufficiently small $t > 0$,

$$\operatorname{dist}(x_1 - t\mathbf{n}_{x_1}, \Gamma_1) = t.$$

PROPOSITION 7.2. *Under the assumptions of Theorem* 2.7, *for each* $\lambda \in \mathbb{R}$ *and any large positive solution* $L(x)$ *of* (7.1) *in* Ω_-, *one has that*

$$\lim_{t\downarrow 0} \frac{L(x_1 - t\mathbf{n}_{x_1})}{B(x_1)[\operatorname{dist}(x_1 - t\mathbf{n}_{x_1}, \partial\Omega_-)]^{-\alpha(x_1)}} = 1 \quad \textit{for each } x_1 \in \partial\Omega_-, \tag{7.5}$$

where $\mathbf{n}_{x_1}$ *stands for the outward unit normal to* Ω_- *at* x_1 *and* $\alpha(x_1)$ *and* $B(x_1)$ *are given through* (7.3). *Therefore, for any pair* (L_1, L_2) *of large positive solutions of* (7.1) *in* Ω_-, *one has that*

$$\lim_{t\downarrow 0} \frac{L_1(x_1 - t\mathbf{n}_{x_1})}{L_2(x_1 - t\mathbf{n}_{x_1})} = 1 \quad \textit{for each } x_1 \in \partial\Omega_-. \tag{7.6}$$

The distribution of this section is as follows. In Section 7.1 we show that in one spatial dimension the estimates (7.2) and (7.5) follow in a rather natural manner. In Section 7.2 we characterize the boundary blow-up rates of the large solutions of (7.1) for some related radially symmetric problems. In Section 7.3, we prove Propositions 7.1 and 7.2. In Section 7.4 we complete the proofs of Theorems 2.6 and 2.7. Finally, in Section 7.5 we prove Theorem 2.8.

7.1. *Finding out the boundary blow-up rate of a large solution*

In one space dimension, $N = 1$, as well as in the radially symmetric case, the fact that (2.9) and (2.10) imply (7.2) follows in a very natural manner. Suppose that we want ascertaining the blow up rate at $R > 0$ of any solution $u(x)$ of the one-dimensional singular boundary value problem

$$\begin{cases} -u'' = \lambda u - au^p & \text{in } (0, R), \\ u(0) = 0, & \lim_{x\uparrow R} u(x) = \infty, \end{cases} \tag{7.7}$$

where, for some $\gamma \geqslant 0$,

$$a(x) = \beta(x)(R-x)^{\gamma}, \quad x \in (0, R), \beta(R) \neq 0.$$

Then, the change of variable

$$u(x) = (R-x)^{-\alpha}\psi(x), \quad x \in [0, R],$$

where $\alpha > 0$ has to be determined, transforms (7.7) into the differential equation

$$\begin{aligned}(R-x)^{-\alpha}\psi''(x) + 2\alpha(R-x)^{-\alpha-1}\psi'(x) + \alpha(\alpha+1)(R-x)^{-\alpha-2}\psi(x)\\ = \beta(x)(R-x)^{\gamma-\alpha p}\psi^{p}(x) - \lambda(R-x)^{-\alpha}\psi(x)\end{aligned} \tag{7.8}$$

subject to the boundary conditions

$$\psi(0) = 0, \qquad \psi(R) \in (0, \infty),$$

so that α provides us with the exact blow-up rate of u at R. Multiplication by $(R-x)^{\alpha+2}$ transforms (7.8) into

$$\begin{aligned}(R-x)^{2}\psi''(x) + 2\alpha(R-x)\psi'(x) + \alpha(\alpha+1)\psi(x)\\ = \beta(x)(R-x)^{\gamma-\alpha p+\alpha+2}\psi^{p}(x) - \lambda(R-x)^{2}\psi(x).\end{aligned}$$

Thus, assuming

$$\lim_{x\uparrow R}\big[(R-x)^{2}\psi''(x)\big] = \lim_{x\uparrow R}\big[(R-x)\psi'(x)\big] = 0,$$

one is driven to impose

$$\gamma - \alpha p + \alpha + 2 = 0, \qquad \alpha(\alpha+1)\psi(R) = \beta(R)\psi^{p}(R),$$

which provides us with the values of α and $\psi(R)$. Namely,

$$\alpha := \frac{\gamma+2}{p-1} \quad \text{and} \quad \psi(R) = \left[\frac{\alpha(\alpha+1)}{\beta(R)}\right]^{1/(p-1)}$$

in complete agreement with the statement of Propositions 7.1 and 7.2.

7.2. *Two auxiliary radially symmetric problems*

The main result of this section is the following lemma.

LEMMA 7.3. *Suppose* $x_0 \in \mathbb{R}^N$, $R > 0$, $\lambda \in \mathbb{R}$, $p > 1$, $\gamma \geqslant 0$ *and*

$$\beta(x) = \beta(r), \quad r := |x - x_0|, \beta \in \mathcal{C}\big([0, R]; (0, \infty)\big).$$

Then, for each $\varepsilon > 0$, *the singular boundary value problem*

$$\begin{cases} -\Delta u = \lambda u - \beta(x)\big[\operatorname{dist}\big(x, \partial B_R(x_0)\big)\big]^\gamma u^p & \text{in } B_R(x_0), \\ u = \infty & \text{on } \partial B_R(x_0) \end{cases} \tag{7.9}$$

possesses a positive solution L_ε *such that*

$$1 - \varepsilon \leqslant \liminf_{d(x)\downarrow 0} \frac{L_\varepsilon(x)}{B[d(x)]^{-\alpha}} \leqslant \limsup_{d(x)\downarrow 0} \frac{L_\varepsilon(x)}{B[d(x)]^{-\alpha}} \leqslant 1 + \varepsilon, \tag{7.10}$$

where

$$d(x) := \operatorname{dist}\big(x, \partial B_R(x_0)\big) = R - |x - x_0| = R - r$$

and

$$\alpha := \frac{\gamma + 2}{p - 1}, \qquad B := \left[\frac{\alpha(\alpha + 1)}{\beta(R)}\right]^{1/(p-1)}. \tag{7.11}$$

PROOF. Note that the radially symmetric solutions of (7.9) are given by

$$u(x) := \psi(r), \quad r = |x - x_0|,$$

where ψ satisfies

$$\begin{cases} -\psi'' - \frac{N-1}{r}\psi' = \lambda\psi - \beta(r)(R - r)^\gamma \psi^p & \text{in } (0, R), \\ \psi'(0) = 0, \qquad \lim_{r\uparrow R} \psi(r) = \infty. \end{cases} \tag{7.12}$$

First, we show that, for each sufficiently small $\varepsilon > 0$, there exists a constant $A_\varepsilon > 0$ such that, for every $A > A_\varepsilon$, the function

$$\bar{\psi}_\varepsilon(r) := A + B_+ \left(\frac{r}{R}\right)^2 (R - r)^{-\alpha}, \quad B_+ := (1 + \varepsilon)B, \tag{7.13}$$

provides us with a positive supersolution of (7.12). Indeed,

$$\bar{\psi}_\varepsilon'(0) = 0 \quad \text{and} \quad \lim_{r\uparrow R} \bar{\psi}_\varepsilon(r) = \infty$$

since $\alpha > 0$. Thus, $\bar{\psi}_\varepsilon$ is a supersolution of (7.12) if and only if

$$-2N\frac{B_+}{R^2}(R-r)^{-\alpha} - \alpha(N+3)\frac{B_+}{R^2}r(R-r)^{-\alpha-1} - \alpha(\alpha+1)B_+\left(\frac{r}{R}\right)^2(R-r)^{-\alpha-2} \geqslant \lambda(R-r)^{-\alpha}\left[A(R-r)^\alpha + B_+\left(\frac{r}{R}\right)^2\right] - \beta(r)(R-r)^{\gamma-\alpha p}\left[A(R-r)^\alpha + B_+\left(\frac{r}{R}\right)^2\right]^p.$$

Thus, multiplying this inequality by $(R-r)^{\alpha+2}$ and taking into account that

$$\alpha + 2 + \gamma - \alpha p = 0,$$

we find that $\bar{\psi}_\varepsilon$ is a supersolution of (7.12) if and only if

$$-2N\frac{B_+}{R^2}(R-r)^2 - \alpha(N+3)\frac{B_+}{R^2}r(R-r) - \alpha(\alpha+1)B_+\left(\frac{r}{R}\right)^2 \geqslant \lambda(R-r)^2\left[A(R-r)^\alpha + B_+\left(\frac{r}{R}\right)^2\right] - \beta(r)\left[A(R-r)^\alpha + B_+\left(\frac{r}{R}\right)^2\right]^p. \tag{7.14}$$

At $r = R$, (7.14) becomes into

$$-\alpha(\alpha+1)B_+ \geqslant -\beta(R)B_+^p,$$

which is satisfied if and only if

$$B_+^{p-1} \geqslant \frac{\alpha(\alpha+1)}{\beta(R)}.$$

Therefore, by making the choice (7.13), the inequality (7.14) is satisfied in a left neighborhood of $r = R$, say $(R-\delta, R]$, for some $\delta = \delta(\varepsilon) > 0$. Finally, by choosing a sufficiently large A, it is clear that the inequality is satisfied in the whole interval $[0, R]$ since $p > 1$ and β is bounded away from zero. This concludes the proof of the claim above.

Now, we will construct an appropriate subsolution for problem (7.12). For doing this we shall distinguish two different cases according to the sign of the parameter λ. First, we assume

$$\lambda \geqslant 0.$$

Then, for each sufficiently small $\varepsilon > 0$, there exists $C < 0$ for which the function

$$\underline{\psi}_\varepsilon(r) := \max\left\{0, C + B_-\left(\frac{r}{R}\right)^2 (R-r)^{-\alpha}\right\}$$

provides us with a nonnegative subsolution of (7.12) if

$$B_- = (1-\varepsilon)B. \tag{7.15}$$

Indeed, it is easy to see that $\underline{\psi}_\varepsilon$ is a subsolution of (7.12) if in the region where

$$C + B_-\left(\frac{r}{R}\right)^2 (R-r)^{-\alpha} \geqslant 0,$$

the following inequality is satisfied

$$\begin{aligned}
-2N\frac{B_-}{R^2}(R-r)^{-\alpha} &- \alpha(N+3)\frac{B_-}{R^2} r(R-r)^{-\alpha-1} \\
&- \alpha(\alpha+1)B_-\left(\frac{r}{R}\right)^2 (R-r)^{-\alpha-2} \\
&\leqslant \lambda(R-r)^{-\alpha}\left[C(R-r)^\alpha + B_-\left(\frac{r}{R}\right)^2\right] \\
&- \beta(r)(R-r)^{\gamma-\alpha p}\left[C(R-r)^\alpha + B_-\left(\frac{r}{R}\right)^2\right]^p.
\end{aligned}$$

Equivalently,

$$\begin{aligned}
-2N\frac{B_-}{R^2}(R-r)^2 &- \alpha(N+3)\frac{B_-}{R^2} r(R-r) - \alpha(\alpha+1)B_-\left(\frac{r}{R}\right)^2 \\
&\leqslant \lambda(R-r)^2\left[C(R-r)^\alpha + B_-\left(\frac{r}{R}\right)^2\right] - \beta(r)\left[C(R-r)^\alpha + B_-\left(\frac{r}{R}\right)^2\right]^p.
\end{aligned} \tag{7.16}$$

Now, note that for each $C < 0$ there exists a constant

$$z = z(C) \in (0, R)$$

such that

$$C + B_-\left(\frac{r}{R}\right)^2 (R-r)^{-\alpha} < 0 \quad \text{if } r \in \big[0, z(C)\big)$$

and

$$C+B_-\left(\frac{r}{R}\right)^2(R-r)^{-\alpha}\geqslant 0\quad \text{if } r\in\big[z(C),R\big)$$

because the mapping

$$r\mapsto\left(\frac{r}{R}\right)^2(R-r)^{-\alpha}$$

is increasing. Moreover, $z(C)$ is decreasing with respect to C and

$$\lim_{C\downarrow-\infty}z(C)=R,\qquad \lim_{C\uparrow 0}z(C)=0. \tag{7.17}$$

Thus, since $\lambda\geqslant 0$, for each $r\in[z(C),R)$, the following condition implies (7.16)

$$\beta(r)\left[C(R-r)^\alpha+B_-\left(\frac{r}{R}\right)^2\right]^p\leqslant\alpha(\alpha+1)B_-\left(\frac{r}{R}\right)^2. \tag{7.18}$$

Note that, since $C<0$, for (7.18) to be satisfied it suffices that

$$\beta(r)B_-^{p-1}\left(\frac{r}{R}\right)^{2(p-1)}\leqslant\alpha(\alpha+1) \tag{7.19}$$

for each $r\in[z(C),R)$. Making the choice (7.15) and using the continuity of $\beta(r)$, it is easy to see that there exists a constant $\delta(\varepsilon)>0$ for which (7.19) is satisfied in $[R-\delta(\varepsilon),R)$. Moreover, thanks to (7.17), there exists $C<0$ such that $z(C)=R-\delta(\varepsilon)$. For this choice of C, it readily follows that $\underline{\psi}_\varepsilon$ provides us with a subsolution of (7.12).

Similarly, it is easy to see that there exists $C<0$ for which the function

$$\underline{\psi}_\varepsilon(r):=\mathrm{e}^{\sqrt{-\lambda}(r-R)}\max\left\{0,C+B_-\left(\frac{r}{R}\right)^2(R-r)^{-\alpha}\right\}$$

provides us with a subsolution of (7.12) if $\lambda<0$.

Note that in each of these cases

$$\lim_{r\uparrow R}\frac{\bar{\psi}_\varepsilon(r)}{B_+(R-r)^{-\alpha}}=1,\qquad \lim_{r\uparrow R}\frac{\underline{\psi}_\varepsilon(r)}{B_-(R-r)^{-\alpha}}=1,$$

where B_+ and B_- are the constants defined in (7.13) and (7.15). Moreover, for any sufficiently large $A>A_\varepsilon$,

$$\underline{\psi}_\varepsilon\leqslant\bar{\psi}_\varepsilon,$$

and hence,

$$1-\varepsilon=\lim_{r\uparrow R}\frac{\underline{\psi}_\varepsilon(r)}{B(R-r)^{-\alpha}}\leqslant\lim_{r\uparrow R}\frac{\bar{\psi}_\varepsilon(r)}{B(R-r)^{-\alpha}}=1+\varepsilon.$$

Therefore, setting

$$\underline{L}_\varepsilon(x):=\underline{\psi}_\varepsilon(r),\qquad \overline{L}_\varepsilon(x):=\bar{\psi}_\varepsilon(r),\quad r:=|x-x_0|,$$

$(\underline{L}_\varepsilon,\overline{L}_\varepsilon)$ provides us with an ordered sub–supersolution pair of (7.9) such that

$$1-\varepsilon=\lim_{d(x)\downarrow 0}\frac{\underline{L}_\varepsilon(x)}{B[d(x)]^{-\alpha}}\leqslant\lim_{d(x)\downarrow 0}\frac{\overline{L}_\varepsilon(x)}{B[d(x)]^{-\alpha}}=1+\varepsilon. \tag{7.20}$$

Consequently, by Theorem 3.3, there exists a positive solution L_ε of (7.9) such that

$$\underline{L}_\varepsilon\leqslant L_\varepsilon\leqslant\overline{L}_\varepsilon.$$

Moreover, due to (7.20), L_ε must satisfy (7.10), which concludes the proof. □

Now, adapting the previous argument together with a reflection around $r_0:=\frac{R_1+R_2}{2}$, Lemma 7.3 provides us with the corresponding result for each of the annuli

$$A_{R_1,R_2}(x_0):=\{x\in\mathbb{R}^N:\ 0<R_1<|x-x_0|<R_2\}.$$

LEMMA 7.4. *Suppose* $x_0\in\mathbb{R}^N$, $R_2>R_1>0$, $\lambda\in\mathbb{R}$, $p>1$, $\gamma\geqslant 0$ *and*

$$\beta(x)=\beta(r),\quad r:=|x-x_0|,\ \beta\in\mathcal{C}\big([R_1,R_2];(0,\infty)\big),$$

is the reflection around $r_0:=\frac{R_1+R_2}{2}$ *of some function* $\tilde{\beta}\in\mathcal{C}([r_0,R_2];(0,\infty))$, *so that, in particular,* $\beta(R_1)=\beta(R_2)$. *Then, for each* $\varepsilon>0$, *the singular problem,*

$$\begin{cases}-\Delta u=\lambda u-\beta(x)\big[\operatorname{dist}\big(x,\partial A_{R_1,R_2}(x_0)\big)\big]^\gamma u^p & \text{in } A_{R_1,R_2}(x_0),\\ u=\infty & \text{on } \partial A_{R_1,R_2}(x_0),\end{cases} \tag{7.21}$$

possesses a positive solution $L_\varepsilon(x)$ *satisfying*

$$1-\varepsilon\leqslant\liminf_{\delta(x)\downarrow 0}\frac{L_\varepsilon(x)}{B[\delta(x)]^{-\alpha}}\leqslant\limsup_{\delta(x)\downarrow 0}\frac{L_\varepsilon(x)}{B[\delta(x)]^{-\alpha}}\leqslant 1+\varepsilon, \tag{7.22}$$

where α *and* B *are given by* (7.11) *and*

$$\delta(x):=\operatorname{dist}\big(x,\partial A_{R_1,R_2}(x_0)\big)=\begin{cases}R_2-|x-x_0| & \text{if } r_0\leqslant|x-x_0|<R_2,\\ |x-x_0|-R_1 & \text{if } R_1<|x-x_0|<r_0.\end{cases}$$

7.3. *Proof of Propositions 7.1 and 7.2*

As it will be clear later, the proof of Proposition 7.1 can be easily adapted to prove Proposition 7.2, as it is of a local nature around each of the points of the boundary of the underlying domain. So, suppose (2.9), (2.10), $\lambda < \sigma_2$ and L is a large positive solution of (7.1) in $\Omega \setminus \overline{\Omega}_{0,1}$. Pick

$$x_1 \in \Gamma_1 = \partial\big(\Omega \setminus \overline{\Omega}_{0,1}\big).$$

Since Γ_1 is of class $\mathcal{C}^2$, there exist $R > 0$ and $\delta_0 > 0$ such that

$$B_R(x_1 - R\mathbf{n}_{x_1}) \subset \Omega_-, \qquad \overline{B}_R(x_1 - R\mathbf{n}_{x_1}) \cap \Gamma_1 = \{x_1\}, \tag{7.23}$$

and, for each $\delta \in (0, \delta_0]$,

$$\overline{B}_R\big(x_1 - (R+\delta)\mathbf{n}_{x_1}\big) \subset \Omega_-. \tag{7.24}$$

On the other hand, due to (2.10), given any $\eta \in (0, \beta(x_1))$, $R > 0$ can be shortened, if necessary, so that, for each $\delta \in [0, \delta_0]$,

$$a \geqslant \big(\beta(x_1) - \eta\big)\big[\operatorname{dist}(\cdot, \Gamma_1)\big]^{\gamma(x_1)} \quad \text{in } B_R\big(x_1 - (R+\delta)\mathbf{n}_{x_1}\big). \tag{7.25}$$

As for each $\delta \in [0, \delta_0]$ and $x \in B_R(x_1 - (R+\delta)\mathbf{n}_{x_1})$, we have that

$$\begin{aligned}\operatorname{dist}(x, \Gamma_1) &\geqslant \operatorname{dist}\big(x, B_R\big(x_1 - (R+\delta)\mathbf{n}_{x_1}\big)\big)\\ &= R - \big|x - \big[x_1 - (R+\delta)\mathbf{n}_{x_1}\big]\big|,\end{aligned}$$

(7.25) implies

$$a(x) \geqslant \big(\beta(x_1) - \eta\big)\big(R - r_\delta(x)\big)^{\gamma(x_1)}, \quad x \in B_R\big(x_1 - (R+\delta)\mathbf{n}_{x_1}\big), \tag{7.26}$$

where

$$r_\delta(x) := \big|x - \big[x_1 - (R+\delta)\mathbf{n}_{x_1}\big]\big|.$$

Thanks to (7.24) and (7.26), for each $\delta \in (0, \delta_0]$, the restriction

$$\underline{L}^\delta := L|_{\overline{B}_R(x_1-(R+\delta)\mathbf{n}_{x_1})} \tag{7.27}$$

provides us with a classical subsolution (bounded) of the singular problem

$$\begin{cases} -\Delta u = \lambda u - \big(\beta(x_1) - \eta\big)(R - r_\delta)^{\gamma(x_1)} u^p & \text{in } B_R\big(x_1 - (R+\delta)\mathbf{n}_{x_1}\big),\\ u = \infty & \text{on } \partial B_R\big(x_1 - (R+\delta)\mathbf{n}_{x_1}\big).\end{cases} \tag{7.28}$$

Now, consider the limiting problem at $\delta = 0$

$$\begin{cases} -\Delta u = \lambda u - \big(\beta(x_1) - \eta\big)(R - r_0)^{\gamma(x_1)} u^p & \text{in } B_R(x_1 - R\mathbf{n}_{x_1}), \\ u = \infty & \text{on } \partial B_R(x_1 - R\mathbf{n}_{x_1}). \end{cases} \tag{7.29}$$

Subsequently, for each $\delta \in [0, \delta_0]$, we shall denote

$$\begin{aligned} d_\delta(x) &:= \operatorname{dist}\big(x, \partial B_R\big(x_1 - (R+\delta)\mathbf{n}_{x_1}\big)\big) \\ &= R - \big|x - \big(x_1 - (R+\delta)\mathbf{n}_{x_1}\big)\big| \\ &= R - r_\delta(x). \end{aligned}$$

Thanks to Lemma 7.3, for each $\varepsilon > 0$, the problem (7.29) possesses a solution, say L_ε, such that

$$\limsup_{d_0(x)\downarrow 0} \frac{L_\varepsilon(x)}{B_\eta(x_1)[d_0(x)]^{-\alpha(x_1)}} \leqslant 1 + \varepsilon, \tag{7.30}$$

where $\alpha(x_1)$ is given by (7.3) and

$$B_\eta(x_1) := \left[\frac{\alpha(x_1)(\alpha(x_1)+1)}{\beta(x_1) - \eta}\right]^{1/(p-1)}.$$

Subsequently, we fix $\varepsilon > 0$ and, for each $\delta \in (0, \delta_0]$, consider the function L_ε^δ defined by

$$L_\varepsilon^\delta(x) := L_\varepsilon(x + \delta\mathbf{n}_{x_1}), \quad x \in B_R\big(x_1 - (R+\delta)\mathbf{n}_{x_1}\big).$$

By construction,

$$\lim_{\delta\downarrow 0} L_\varepsilon^\delta = L_\varepsilon \tag{7.31}$$

and, for each $\delta \in (0, \delta_0]$, L_ε^δ provides us with a positive solution of (7.28). Thus, since the function $\underline{L}^\delta$ defined in (7.27) is a positive classical subsolution of (7.28), it follows from Lemma 3.6 that, for each $\delta \in (0, \delta_0]$,

$$\underline{L}^\delta \leqslant L_\varepsilon^\delta \quad \text{in } B_R\big(x_1 - (R+\delta)\mathbf{n}_{x_1}\big).$$

Consequently, passing to the limit as $\delta \downarrow 0$, (7.27) and (7.31) imply

$$L \leqslant L_\varepsilon \quad \text{in } B_R(x_1 - R\mathbf{n}_{x_1}).$$

In particular, for each $t \in (0, R)$,

$$L(x_1 - t\mathbf{n}_{x_1}) \leqslant L_\varepsilon(x_1 - t\mathbf{n}_{x_1})$$

and hence,

$$\frac{L(x_1 - t\mathbf{n}_{x_1})}{B_\eta(x_1)[\operatorname{dist}(x_1 - t\mathbf{n}_{x_1}, \Gamma_1)]^{-\alpha(x_1)}} \leqslant \frac{L_\varepsilon(x_1 - t\mathbf{n}_{x_1})}{B_\eta(x_1)[\operatorname{dist}(x_1 - t\mathbf{n}_{x_1}, \Gamma_1)]^{-\alpha(x_1)}}.$$

On the other hand, by construction, we have that

$$\operatorname{dist}(x_1 - t\mathbf{n}_{x_1}, \Gamma_1) = t = d_0(x_1 - t\mathbf{n}_{x_1})$$

and so, passing to the limit as $t \downarrow 0$ in the previous inequality, (7.30) gives

$$\limsup_{t\downarrow 0} \frac{L(x_1 - t\mathbf{n}_{x_1})}{B_\eta(x_1)[\operatorname{dist}(x_1 - t\mathbf{n}_{x_1}, \Gamma_1)]^{-\alpha(x_1)}} \leqslant 1 + \varepsilon.$$

As this inequality holds for each sufficiently small $\eta > 0$ and $\varepsilon > 0$, it is apparent that

$$\limsup_{t\downarrow 0} \frac{L(x_1 - t\mathbf{n}_{x_1})}{B(x_1)[\operatorname{dist}(x_1 - t\mathbf{n}_{x_1}, \Gamma_1)]^{-\alpha(x_1)}} \leqslant 1, \tag{7.32}$$

where α and B are given by (7.3). Consequently, to complete the proof of (7.2) it remains to show that

$$1 \leqslant \liminf_{t\downarrow 0} \frac{L(x_1 - t\mathbf{n}_{x_1})}{B(x_1)[\operatorname{dist}(x_1 - t\mathbf{n}_{x_1}, \Gamma_1)]^{-\alpha(x_1)}}. \tag{7.33}$$

To prove (7.33) we will construct a large subsolution having the appropriate growth at $x_1 \in \Gamma_1$. Since Γ_1 is smooth, there exist $R_2 > R_1 > 0$ and $\delta_0 > 0$ such that

$$\bar{A}_{R_1,R_2}(x_1 + R_1\mathbf{n}_{x_1}) \cap \Gamma_1 = \{x_1\},$$
$$\Omega \setminus \bar{\Omega}_{0,1} \subset \bigcap_{\delta\in[0,\delta_0]} A_{R_1,R_2}\bigl(x_1 + (R_1+\delta)\mathbf{n}_{x_1}\bigr)$$

and

$$\Omega \setminus \bar{\Omega}_{0,1} \subset A_{R_1,(R_1+R_2)/2}(x_1 + R\mathbf{n}_{x_1}),$$

which can be accomplished by taking a sufficiently small $R_1 > 0$ and a sufficiently large $R_2 > R_1$. Now, fix a sufficiently small $\eta > 0$. Thanks to (2.10), there exists a radially symmetric function

$$\tilde{a} : \bar{A}_{R_1,R_2}(x_1 + R_1\mathbf{n}_{x_1}) \to [0, \infty)$$

such that

$$a \leqslant \tilde{a} \leqslant \sup_{\Omega\setminus\bar{\Omega}_{0,1}} a + 1 \quad \text{in } \Omega \setminus \bar{\Omega}_{0,1},$$

and, for each $x \in A_{R_1,R_2}(x_1 + R_1\mathbf{n}_{x_1})$,

$$\tilde{a}(x) = b\big(|x - x_1 - R_1\mathbf{n}_{x_1}|\big)\big[\mathrm{dist}\big(x, \partial A_{R_1,R_2}(x_1 + R_1\mathbf{n}_{x_1})\big)\big]^{\gamma(x_1)}$$

for some $b \in \mathcal{C}([R_1, R_2]; (0,\infty))$ satisfying

$$b(R_1) = \beta(x_1) + \eta.$$

Moreover, by enlarging R_2, if necessary, one can assume that b is the reflection around $\frac{R_1+R_2}{2}$ of a continuous positive function. Actually, by shortening δ_0, if necessary, we can assume that it satisfies all the requirements of the proof of (7.32).

Subsequently, for each $\delta \in [0, \delta_0]$, we consider the singular auxiliary problem

$$\begin{cases} -\Delta u = \lambda u - \tilde{a}(\cdot + \delta\mathbf{n}_{x_1})u^p & \text{in } A_{R_1,R_2}\big(x_1 + (R_1+\delta)\mathbf{n}_{x_1}\big), \\ u = \infty & \text{on } \partial A_{R_1,R_2}\big(x_1 + (R+\delta)\mathbf{n}_{x_1}\big). \end{cases} \tag{7.34}$$

Fix a sufficiently small $\varepsilon > 0$. Thanks to Lemma 7.4, for $\delta = 0$, (7.34) possesses a positive solution $\mathcal{L}_\varepsilon$ such that

$$1 - \varepsilon \leqslant \liminf_{\mathcal{D}_0(x)\downarrow 0} \frac{\mathcal{L}_\varepsilon(x)}{\mathcal{B}_\eta(x_1)[\mathcal{D}_0(x)]^{-\alpha(x_1)}}, \tag{7.35}$$

where $\alpha(x_1)$ is given by (7.3),

$$\mathcal{B}_\eta(x_1) := \left[\frac{\alpha(x_1)(\alpha(x_1)+1)}{\beta(x_1)+\eta}\right]^{1/(p-1)},$$

and, for each $x \in A_{R_1,R_2}(x_1 + R_1\mathbf{n}_{x_1})$ sufficiently close to Γ_1,

$$\mathcal{D}_0(x) := \mathrm{dist}\big(x, \partial A_{R_1,R_2}(x_1 + R_1\mathbf{n}_{x_1})\big) = \big|x - (x_1 + R_1\mathbf{n}_{x_1})\big| - R_1.$$

Now, for each $\delta \in (0, \delta_0]$, consider the function $\mathcal{L}_\varepsilon^\delta$ defined by

$$\mathcal{L}_\varepsilon^\delta(x) := \mathcal{L}_\varepsilon(x - \delta\mathbf{n}_{x_1}), \quad x \in A_{R_1,R_2}\big(x_1 + (R+\delta)\mathbf{n}_{x_1}\big).$$

By construction,

$$\lim_{\delta\downarrow 0} \mathcal{L}_\varepsilon^\delta = \mathcal{L}_\varepsilon. \tag{7.36}$$

Moreover, for each $\delta \in (0, \delta_0]$, the restriction

$$\mathcal{L}_\varepsilon^\delta|_{\Omega\setminus\bar{\Omega}_{0,1}}$$

is a classical (bounded) subsolution of the singular problem

$$\begin{cases} -\Delta u = \lambda u - a u^p & \text{in } \Omega \setminus \bar{\Omega}_{0,1}, \\ u = \infty & \text{on } \Gamma_1 \end{cases}$$

since $\tilde{a} \geqslant a$. Thus, it follows from Lemma 3.6 that, for each $\delta \in (0, \delta_0]$,

$$\mathcal{L}_\varepsilon^\delta \leqslant L \quad \text{in } \Omega \setminus \bar{\Omega}_{0,1}.$$

Consequently, passing to the limit as $\delta \downarrow 0$, (7.36) implies

$$\mathcal{L}_\varepsilon \leqslant L \quad \text{in } \Omega \setminus \bar{\Omega}_{0,1}.$$

In particular, for each $t \in (0, R)$,

$$\mathcal{L}_\varepsilon(x_1 - t\mathbf{n}_{x_1}) \leqslant L(x_1 - t\mathbf{n}_{x_1})$$

and hence,

$$\frac{\mathcal{L}_\varepsilon(x_1 - t\mathbf{n}_{x_1})}{\mathcal{B}_\eta(x_1)[\operatorname{dist}(x_1 - t\mathbf{n}_{x_1}, \Gamma_1)]^{-\alpha(x_1)}} \leqslant \frac{L(x_1 - t\mathbf{n}_{x_1})}{\mathcal{B}_\eta(x_1)[\operatorname{dist}(x_1 - t\mathbf{n}_{x_1}, \Gamma_1)]^{-\alpha(x_1)}}.$$

On the other hand, by construction, we have that

$$\operatorname{dist}(x_1 - t\mathbf{n}_{x_1}, \Gamma_1) = t = \mathcal{D}_0(x_1 - t\mathbf{n}_{x_1})$$

and so, passing to the limit as $t \downarrow 0$ in the previous inequality, (7.35) implies

$$1 - \varepsilon \leqslant \liminf_{t\downarrow 0} \frac{L(x_1 - t\mathbf{n}_{x_1})}{\mathcal{B}_\eta(x_1)[\operatorname{dist}(x_1 - t\mathbf{n}_{x_1}, \Gamma_1)]^{-\alpha(x_1)}}.$$

As this inequality holds true for each sufficiently small $\eta > 0$ and $\varepsilon > 0$, (7.33) holds. Therefore, by (7.32), the proof of (7.2) is concluded. Condition (7.4) is a direct consequence from (7.2). This concludes the proof of Proposition 7.1. At this stage, the validity of Proposition 7.2 should be apparent.

7.4. *Proof of Theorems 2.6 and 2.7*

Suppose (2.9), (2.10), and pick $\lambda < \sigma_2$. Subsequently, we set

$$L_1 := L^{\min}_{[\lambda, \Omega \setminus \bar{\Omega}_{0,1}]}, \qquad L_2 := L^{\max}_{[\lambda, \Omega \setminus \bar{\Omega}_{0,1}]}, \qquad D := \Omega \setminus \bar{\Omega}_{0,1}.$$

Theorem 2.6 establishes that $L_1 = L_2$. By construction, we already know that

$$L_1 \leqslant L_2.$$

Moreover, thanks to Lemma 5.1, (5.24) and (5.26) imply

$$L_1(x) > 0, \quad x \in D.$$

LEMMA 7.5. *Suppose $L_1 < L_2$. Then, for each $x \in D$,*

$$L_1(x) < L_2(x). \tag{7.37}$$

PROOF. We proceed by contradiction. In the contrary case, there exist $x_0 \in D$ and $R > 0$ such that $\overline{B}_R(x_0) \subset D$, $L_1(x) < L_2(x)$ for each $x \in B_R(x_0)$ and $L_1(y) = L_2(y)$ for some $y \in \partial B_R(x_0)$. Now, consider the problem

$$\begin{cases} -\Delta u = \lambda u - a u^p & \text{in } B_R(x_0), \\ u = L_2 & \text{on } \partial B_R(x_0). \end{cases} \tag{7.38}$$

If $L_1 = L_2$ on $\partial B_R(x_0)$ then, thanks to Theorem 3.5, $L_1 = L_2$ in $\overline{B}_R(x_0)$, which contradicts $L_1 < L_2$ in $B_R(x_0)$. Thus, $L_1 < L_2$ on $\partial B_R(x_0)$, and hence, L_1 is a positive strict subsolution of (7.38), whose unique solution is L_2. So, thanks to Lemma 3.6, $L_2 - L_1 \gg 0$. Consequently, since $L_2(y) - L_1(y) = 0$, we have that

$$\frac{\partial (L_2 - L_1)}{\partial n_y}(y) < 0, \quad n_y := \frac{y - x_0}{R},$$

and therefore, for each sufficiently small $t > 0$,

$$L_2(y + t n_y) < L_1(y + t n_y),$$

which contradicts $L_1 \leqslant L_2$, so concluding the proof of (7.37). □

Subsequently, we consider the function q defined through

$$q(x) := \frac{L_1(x)}{L_2(x)}, \quad x \in D. \tag{7.39}$$

As an immediate consequence from the previous properties, the function q is well defined in D and, actually, it is a function of class $\mathcal{C}^{2+\mu}(D)$ such that

$$0 < q(x) \leqslant 1 \quad \text{for each } x \in D. \tag{7.40}$$

Moreover, thanks to Proposition 7.1,

$$\lim_{t \downarrow 0} q(x_1 - t\mathbf{n}_{x_1}) = 1 \quad \text{for each } x_1 \in \varGamma_1 = \partial D, \tag{7.41}$$

and, due to Lemma 7.5,

$$0 < q(x) < 1 \quad \text{for each } x \in D \tag{7.42}$$

if $L_1 < L_2$.

The following lemma is the basic technical tool to prove Theorems 2.6 and 2.7.

LEMMA 7.6. *Suppose q admits a continuous extension $Q \in \mathcal{C}(\overline{D})$. Then, $L_1 = L_2$.*

PROOF. Let $Q \in \mathcal{C}(\overline{D})$ be a continuous extension of q. Then, thanks to (7.40) and (7.41), $0 < Q = q \leqslant 1$ in D and $Q = 1$ on ∂D. Moreover, differentiating the identity $L_1 = QL_2$ gives

$$-\Delta L_1 = -L_2 \Delta Q - 2\langle \nabla L_2, \nabla Q\rangle - Q\Delta L_2,$$

and hence,

$$\lambda L_1 - aL_1^p = -L_2 \Delta Q - 2\langle \nabla L_2, \nabla Q\rangle + Q\big(\lambda L_2 - aL_2^p\big).$$

Consequently, dividing by L_2 and rearranging terms, we find that

$$-\Delta Q - 2\left\langle \frac{\nabla L_2}{L_2}, \nabla Q\right\rangle = aQ\big(L_2^{p-1} - L_1^{p-1}\big) \geqslant 0.$$

Now, for each sufficiently large $n \in \mathbb{N}$, say $n \geqslant n_0$, we consider the approximating open sets

$$D_n := \left\{ x \in D\colon \operatorname{dist}(x, \partial D) > \frac{1}{n} \right\}.$$

Thanks to the maximum principle, for each $n \geqslant n_0$, $\min_{\overline{D}_n} Q$ must be taken at some point $x_1^n \in \partial D_n$. Actually, this is a consequence from Theorem 3.1 applied to $Q - \min_{\overline{D}_n} Q$. By construction, the sequence x_1^n approximates $\Gamma_1 = \partial D$ as $n \uparrow \infty$. Thus, there exists $x_1 \in \Gamma_1$ and a subsequence of x_1^n, again labeled by n, such that

$$\lim_{n\uparrow\infty} x_1^n = x_1.$$

Therefore,

$$1 = Q(x_1) = \lim_{n\uparrow\infty} Q\big(x_1^n\big) = \lim_{n\uparrow\infty} \min_{\overline{D}_n} Q = \min_{\overline{D}} Q,$$

and consequently, $Q = 1$ in D, which concludes the proof. □

As an immediate consequence from the proof of Lemma 7.6, it is apparent that

$$\liminf_{x\to x_1} q = \inf_{\overline{D}} q < \limsup_{x\to x_1} q = \sup_{\overline{D}} q = 1$$

if $L_1 < L_2$. Moreover, the following uniqueness result holds.

LEMMA 7.7. *Under the assumptions of Lemma* 7.3 *and Lemma* 7.4, *each of the problems* (7.9) *and* (7.21) *possesses a unique solution. Moreover, the solution is radially symmetric.*

PROOF. Throughout this proof, we consider

$$\Omega \in \{B_R(x_0), A_{R_1,R_2}(x_0)\}$$

and, for sufficiently large n,

$$\Omega_n := \left\{x \in \Omega\colon\ \mathrm{dist}(x, \partial\Omega) > \frac{1}{n}\right\}.$$

Going back to the proof of Proposition 4.7, it is apparent, by construction, that $L_1 := L^{\min}_{[\lambda,\Omega]}$ and $L^{\min}_{[\lambda,\Omega_n]}$ are radially symmetric. Thus,

$$L_2 := L^{\max}_{[\lambda,\Omega]} = \lim_{n\uparrow\infty} L^{\min}_{[\lambda,\Omega_n]}$$

also is radially symmetric. Therefore, the quotient of these solutions, $q = L_1/L_2$, is radially symmetric as well.

On the other hand, adapting the proof of Propositions 7.1 and 7.2 to the domain Ω, it is rather clear that, for each $x_1 \in \partial\Omega$,

$$\lim_{t\downarrow 0} \frac{L_j(x_1 - t\mathbf{n}_{x_1})}{Bt^{-\alpha}} = 1, \qquad j \in \{1, 2\}, \tag{7.43}$$

where α and B are given by (7.11). Hence,

$$\lim_{t\downarrow 0} q(x_1 - t\mathbf{n}_{x_1}) = 1 \quad \text{for each } x_1 \in \partial\Omega. \tag{7.44}$$

As q is radially symmetric, due to (7.44), q admits a continuous extension $Q \in \mathcal{C}(\bar{\Omega})$, and therefore, adapting Lemma 7.6 to cover the present situation, we obtain that $L_1 = L_2$, which concludes the proof. □

Subsequently, in order to refine it, we go back to the proof of Proposition 7.1. By Lemma 7.7, we already know that, for every $x_1 \in \Gamma_1$, each of the auxiliary problems, (7.29) and

$$\begin{cases} -\Delta u = \lambda u - \tilde{a}u^p & \text{in } A_{R_1,R_2}(x_1 + R_1\mathbf{n}_{x_1}), \\ u = \infty & \text{on } \partial A_{R_1,R_2}(x_1 + R\mathbf{n}_{x_1}), \end{cases} \tag{7.45}$$

possesses a unique positive solution. Let denote them by $L_0^{x_1}$ and $\mathcal{L}_0^{x_1}$, respectively. Note that in the proof of Proposition 7.1 the constants δ_0, R, R_1 and R_2 can be chosen to be independent of $x_1 \in \Gamma_1$ since Γ_1 is of class $\mathcal{C}^2$. Subsequently, instead of arguing with the large solutions L_ε and $\mathcal{L}_\varepsilon$ whose existence was guaranteed by Lemmas 7.3 and 7.4, we repeat the argument of the proof of Proposition 7.1 using $L_0^{x_1}$ and $\mathcal{L}_0^{x_1}$. Then, for every $\eta > 0$, $x_1 \in \Gamma_1$ and $t \in (0, R)$, we find that

$$\frac{\mathcal{L}_0^{x_1}(x_1 - t\mathbf{n}_{x_1})}{B(x_1)t^{-\alpha(x_1)}} \leqslant \frac{L(x_1 - t\mathbf{n}_{x_1})}{B(x_1)t^{-\alpha(x_1)}} \leqslant \frac{L_0^{x_1}(x_1 - t\mathbf{n}_{x_1})}{B(x_1)t^{-\alpha(x_1)}}$$

and, consequently,

$$\frac{\mathcal{B}_\eta(x_1)}{B(x_1)} \frac{\mathcal{L}_0^{x_1}(x_1 - t\mathbf{n}_{x_1})}{\mathcal{B}_\eta(x_1)t^{-\alpha(x_1)}} \leqslant \frac{L(x_1 - t\mathbf{n}_{x_1})}{B(x_1)t^{-\alpha(x_1)}} \leqslant \frac{B_\eta(x_1)}{B(x_1)} \frac{L_0^{x_1}(x_1 - t\mathbf{n}_{x_1})}{B_\eta(x_1)t^{-\alpha(x_1)}}. \tag{7.46}$$

Note that in the special case when $\alpha(x_1)$ and $B(x_1)$ are constant the large solutions $L_0^{x_1}$ and $\mathcal{L}_0^{x_1}$ are translations of a fixed profile along each of the points at a distance R from Γ_1, and therefore, by (7.43),

$$\lim_{t\downarrow 0} \frac{\mathcal{L}_0^{x_1}(x_1 - t\mathbf{n}_{x_1})}{\mathcal{B}_\eta(x_1)t^{-\alpha(x_1)}} = 1 = \lim_{t\downarrow 0} \frac{L_0^{x_1}(x_1 - t\mathbf{n}_{x_1})}{B_\eta(x_1)t^{-\alpha(x_1)}} \tag{7.47}$$

uniformly in $x_1 \in \Gamma_1$. Thus, passing to the limit as $t \downarrow 0$ in (7.46) we obtain that

$$\frac{\mathcal{B}_\eta(x_1)}{B(x_1)} \leqslant \liminf_{t\downarrow 0} \frac{L(x_1 - t\mathbf{n}_{x_1})}{B(x_1)t^{-\alpha(x_1)}} \leqslant \limsup_{t\downarrow 0} \frac{L(x_1 - t\mathbf{n}_{x_1})}{B(x_1)t^{-\alpha(x_1)}} \leqslant \frac{B_\eta(x_1)}{B(x_1)} \tag{7.48}$$

uniformly in $x_1 \in \Gamma_1$. Consequently, since

$$\lim_{\eta\downarrow 0} \frac{\mathcal{B}_\eta(x_1)}{B(x_1)} = 1 = \lim_{\eta\downarrow 0} \frac{B_\eta(x_1)}{B(x_1)} \quad \text{uniformly in } x_1 \in \Gamma_1,$$

it follows from (7.48) that

$$\lim_{t\downarrow 0} \frac{L(x_1 - t\mathbf{n}_{x_1})}{B(x_1)t^{-\alpha(x_1)}} = 1 \quad \text{uniformly in } x_1 \in \Gamma_1, \tag{7.49}$$

which, in particular, entails the continuity of the quotient function q defined in (7.39) and, thanks to Lemma 7.6, concludes the proof of the uniqueness in this special case.

In our general setting, the proof is completed if we are able to show that (7.47) is as well satisfied uniformly in $x_1 \in \Gamma_1$. This is easily realized by having a careful look at the proof of Lemma 7.3, which is based upon the construction of the supersolution

$$\bar{\psi}_\varepsilon(r) := A + B_+ \left(\frac{r}{R}\right)^2 (R - r)^{-\alpha}, \quad B_+ := (1 + \varepsilon)B,$$

and the subsolution

$$\underline{\psi}_\varepsilon(r) := \max\left\{0, C + B_- \left(\frac{r}{R}\right)^2 (R - r)^{-\alpha}\right\}, \quad B_- := (1 - \varepsilon)B.$$

Rereading the technical details of their constructions, it is rather obvious that the constants A and C can be chosen to be the same for α and $\beta(R)$, and, hence, B, varying in any compact subset of $(0,\infty)$. Therefore,

$$\lim_{r\uparrow R}\frac{\bar{\psi}_\varepsilon(r)}{B(R-r)^{-\alpha}}=\frac{B_+}{B}\quad\text{and}\quad\lim_{r\uparrow R}\frac{\underline{\psi}_\varepsilon(r)}{B(R-r)^{-\alpha}}=\frac{B_-}{B}$$

for α and $\beta(R)$ varying in any compact subset of $(0,\infty)$. As $\alpha(x_1)$ and $\beta(x_1)$ are continuous functions of $x_1\in\Gamma_1$ and Γ_1 is compact, (7.47) occurs uniformly in $x_1\in\Gamma_1$, which concludes the proof of Theorem 2.6. The proof of Theorem 2.7 follows by repeating the argument along the remaining component of $\partial\Omega_-$.

7.5. *Proof of Theorem 2.8*

Throughout this subsection we suppose the assumptions of Theorem 2.7 are satisfied. Under these assumptions, Theorem 2.6 can be applied and hence, for each $\lambda\in\mathbb{R}$, (1.9) has a unique large solution in Ω_-, denoted by $L_{[\lambda,\Omega_-]}$, and, for each $\lambda<\sigma_2$, it has a unique large solution in $\Omega\setminus\bar{\Omega}_{0,1}$, denoted by $L_{[\lambda,\Omega\setminus\bar{\Omega}_{0,1}]}$. Theorem 2.8 will follow after a series of lemmas. The next result shows the strong monotonicity of these large solutions as functions of the parameter λ.

LEMMA 7.8. *For each $\lambda,\mu\in\mathbb{R}$, $\lambda<\mu$, one has that*

(a) $L_{[\lambda,\Omega_-]}(x)<L_{[\mu,\Omega_-]}(x)$ *for every* $x\in\Omega_-$;

(b) $L_{[\lambda,\Omega\setminus\bar{\Omega}_{0,1}]}(x)<L_{[\mu,\Omega\setminus\bar{\Omega}_{0,1}]}(x)$ *for every* $x\in\Omega\setminus\bar{\Omega}_{0,1}$ *provided* $\mu<\sigma_2$.

PROOF. By construction, we already know that

$$L_{[\lambda,\Omega_-]}=\lim_{M\uparrow\infty}\theta_{[\lambda,\Omega_-,M]}\quad\text{and}\quad L_{[\mu,\Omega_-]}=\lim_{M\uparrow\infty}\theta_{[\mu,\Omega_-,M]}.$$

Moreover, for each $M>0$,

$$\theta_{[\lambda,\Omega_-,M]}\ll\theta_{[\mu,\Omega_-,M]}\quad\text{in }\Omega_-.$$

Therefore, passing to the limit as $M\uparrow\infty$ gives

$$L_{[\lambda,\Omega_-]}\leqslant L_{[\mu,\Omega_-]}\quad\text{in }\Omega_-.$$

Note that, necessarily,

$$L_{[\lambda,\Omega_-]}<L_{[\mu,\Omega_-]}\quad\text{in }\Omega_-\tag{7.50}$$

since $L_{[\lambda,\Omega_-]}=L_{[\mu,\Omega_-]}$ implies $\lambda=\mu$, which contradicts $\lambda<\mu$. Now, we proceed by contradiction. Suppose part (a) is not satisfied. Then, there exist $x_0\in\Omega_-$ and $R>0$ such that $\bar{B}_R(x_0)\subset\Omega_-$,

$$L_{[\lambda,\Omega_-]}(x)<L_{[\mu,\Omega_-]}(x)\quad\text{for each }x\in B_R(x_0)\tag{7.51}$$

and

$$L_{[\lambda,\Omega_-]}(y) = L_{[\mu,\Omega_-]}(y) \quad \text{for some } y \in \partial B_R(x_0). \tag{7.52}$$

Now, consider the auxiliary problem

$$\begin{cases} -\Delta u = \lambda u - au^p & \text{in } B_R(x_0), \\ u = L_{[\mu,\Omega_-]} & \text{on } \partial B_R(x_0). \end{cases} \tag{7.53}$$

If $L_{[\lambda,\Omega_-]} = L_{[\mu,\Omega_-]}$ on $\partial B_R(x_0)$, then, thanks to Theorem 3.5, $L_{[\lambda,\Omega_-]} = L_{[\mu,\Omega_-]}$ in $\bar{B}_R(x_0)$, which contradicts (7.51). Thus,

$$L_{[\lambda,\Omega_-]} < L_{[\mu,\Omega_-]} \quad \text{on } \partial B_R(x_0).$$

Hence, $L_{[\lambda,\Omega_-]}$ is a positive strict subsolution of (7.53), whose unique solution is $L_{[\mu,\Omega_-]}$. So, thanks to Lemma 3.6,

$$L_{[\mu,\Omega_-]} - L_{[\lambda,\Omega_-]} \gg 0 \quad \text{in } \bar{B}_R(x_0).$$

Consequently, it follows from (7.52) that

$$\frac{\partial (L_{[\mu,\Omega_-]} - L_{[\lambda,\Omega_-]})}{\partial n_y}(y) < 0, \quad n_y := \frac{y - x_0}{R},$$

and therefore, for each sufficiently small $t > 0$,

$$L_{[\mu,\Omega_-]}(y + tn_y) < L_{[\lambda,\Omega_-]}(y + tn_y),$$

which contradicts (7.50). Consequently, part (a) holds true. The previous proof adapts *mutatis mutantis* to prove part (b). □

The next result provides us with the continuity of the large solutions as functions of the parameter λ.

LEMMA 7.9. *For each $x \in \Omega_-$ the map*

$$\lambda \in \mathbb{R} \mapsto L_{[\lambda,\Omega_-]}(x) \tag{7.54}$$

is continuous. Similarly, for each $x \in \Omega \setminus \bar{\Omega}_{0,1}$, the map

$$\lambda \in (-\infty, \sigma_2) \mapsto L_{[\lambda,\Omega\setminus\bar{\Omega}_{0,1}]}(x) \tag{7.55}$$

is continuous.

PROOF. Thanks to Lemma 7.8(a), for each $\lambda \in \mathbb{R}$, the point-wise limit

$$L_\lambda := \lim_{\varepsilon \downarrow 0} L_{[\lambda-\varepsilon,\Omega_-]}$$

is well defined and finite, since $0 < \varepsilon_2 < \varepsilon_1$ implies

$$L_{[\lambda-\varepsilon_1,\Omega_-]}(x) < L_{[\lambda-\varepsilon_2,\Omega_-]}(x) < L_{[\lambda,\Omega_-]}(x), \quad x \in \Omega_-.$$

Moreover, by the compactness of $(-\Delta)^{-1}$, it provides us with a large positive solution of (1.9) in Ω_-; necessarily $L_{[\lambda,\Omega_-]}$, by uniqueness. Thus,

$$L_{[\lambda,\Omega_-]} = \lim_{\varepsilon \downarrow 0} L_{[\lambda-\varepsilon,\Omega_-]}.$$

Similarly, $0 < \varepsilon_2 < \varepsilon_1$ implies

$$L_{[\lambda+\varepsilon_1,\Omega_-]}(x) > L_{[\lambda+\varepsilon_2,\Omega_-]}(x) > L_{[\lambda,\Omega_-]}(x), \quad x \in \Omega_-,$$

and consequently,

$$\lim_{\varepsilon \downarrow 0} L_{[\lambda+\varepsilon,\Omega_-]} = L_{[\lambda,\Omega_-]},$$

which shows the continuity of (7.54). Adapting this argument, the continuity of (7.55) follows readily. □

Finally, the relations (2.16) follow from (2.5) and (2.7), by the uniqueness of the underlying large positive solutions, which is guaranteed by Theorems 2.6 and 2.7. This concludes the proof of Theorem 2.8.

8. Relevant bibliography and further results

The most pioneering result concerning the positive solutions of (1.8) is by Brezis and Oswald [8], Theorem 1, where it was shown, using variational methods, that (1.8) possesses a positive solution, necessarily unique, if and only if

$$\sigma\bigl[-\Delta - a_0(x), \Omega\bigr] < 0 < \sigma\bigl[-\Delta - a_\infty(x), \Omega\bigr], \tag{8.1}$$

where $a_0(x)$ and $a_\infty(x)$ are the functions defined through

$$a_0(x) := \lim_{u \downarrow 0}\bigl(\lambda - a(x) f(x,u)\bigr) = \lambda$$

and

$$a_\infty(x) := \lim_{u \uparrow \infty}\bigl(\lambda - a(x) f(x,u)\bigr) = \begin{cases} -\infty & \text{if } x \in \Omega_-, \\ \lambda & \text{if } x \in \Omega \setminus \Omega_-. \end{cases}$$

Although it is rather obvious that the first inequality of (8.1) becomes $\lambda > \sigma_0$, it is far from easy to realize why the second inequality should become

$$\lambda < \sigma_1 = \sigma[-\Delta, \Omega_{0,1}],$$

in order to provide us with the existence interval $\lambda \in (\sigma_0, \sigma_1)$ given by Theorem 2.3 here. Actually, this is an easy consequence from an extremely sharp result going back to [41], according with it one has that

$$\lim_{u \uparrow \infty} \sigma\big[-\Delta + af(\cdot, u), \Omega\big] = \sigma[-\Delta, \Omega_{0,1}] = \sigma_1. \tag{8.2}$$

Indeed, due to (8.2), (8.1) becomes into

$$\sigma_0 < \lambda < \sigma_1. \tag{8.3}$$

Another significant pioneering contribution was done by Ouyang [62], as a part of his Ph.D. Dissertation on Yamabe's problem [68] under the supervision of W.-M. Ni. Combining global continuation with the existence of uniform a priori bounds in compact subsets of $\lambda \in (-\infty, \sigma_1)$, Ouyang [62] established that, under assumption (2.9), condition (8.3) characterizes the existence of positive solutions of (1.8) and that

$$\lim_{\lambda \uparrow \sigma_1} \|\theta_{[\lambda,\Omega]}\|_{L^2(\Omega)} = \infty. \tag{8.4}$$

Later, Ambrosetti and Gámez [4] and del Pino [26] could improve (8.4) up to get

$$\lim_{\lambda \uparrow \sigma_1} \|\theta_{[\lambda,\Omega]}\|_{L^\infty(\Omega)} = \infty, \tag{8.5}$$

though in none of these works any explicit mention was made towards the problem of analyzing the different role played by each of the components of $a^{-1}(0)$, $\Omega_{0,1}$ and $\Omega_{0,2}$, in ascertaining the dynamics of (1.1).

The first general characterizations of the existence of positive solutions for a generalized class of logistic models including the basic prototype (1.8) were obtained by Fraile et al. [29] as a consequence from the method of sub- and supersolutions. In [29], instead of $-\Delta$, a general elliptic operator of second-order subject to rather general boundary conditions was considered. At the present stage of this monograph, the reader should be fully convinced that the method of sub- and supersolutions is the best available method to deal with generalized sublinear problems, since the supersolution itself provides us with the exact shape profile of the positive solutions of (1.8) for $\lambda > \sigma_0$ separated away from σ_0; for $\lambda \sim \sigma_0$, the solution looks like the subsolution of the problem. Besides this tremendous advantage, the method of sub- and supersolutions provides us simultaneously with the global attractive character of the maximal nonnegative solution of (1.8) within the range $\lambda \in (-\infty, \sigma_1)$. Undoubtedly, [29] has been a milestone for the further development of the theory that we have exposed in this monograph. Actually, the problem of the analysis of the asymptotic behavior of the solutions of (1.1) for $\lambda \geqslant \sigma_1$ was originally addressed in [29], where it was shown that the solutions of (1.1) must be unbounded, as $t \uparrow \infty$, within $\Omega_{0,1}$. Most of the results of [29] were obtained during the academic year 1993–1994, as a result of a question raised by P. Koch and S. Merino to the author during his first visit to Zürich University in June 1993. The paper was submitted for publication on November 1994.

According to the proof of Theorem 2.3 in Section 5, for each $\lambda \in (\sigma_0, \sigma_1)$, the positive supersolution of (1.8) has the form $\kappa\Phi$, where Φ is the function defined in (5.7) for a sufficiently large $\kappa > 1$ that blows up as $\lambda \uparrow \sigma_1$. Quite strikingly, Φ is the supersolution constructed in [41] to prove (8.2). This crucial *bisociation,* besides sharpening the paradigmatic theorem of Brezis and Oswald [8], enjoys a huge interest in its own right, because of its number of applications in Mathematical Biology (cf. [11,12,42], and the references therein), in the study of linear weighted boundary value problems (cf. [41] and [43]), and in the semiclassical analysis of linear oscillators at degenerate wells (cf. [22], where it was used to solve some classical open problems proposed by Simon [66]). *Bisociation...* a word coined by the political and scientific writer A. Koestler for designating unexpected – sharply hidden – connections between a priori unrelated separated fields (cf. [9]).

According to Theorem 2.3, the mapping $\lambda \mapsto \theta_{[\lambda,\Omega]}$ is differentiable and strictly pointwise increasing. Moreover, by (2.6), $\theta_{[\lambda,\Omega]}$ bifurcates from $u = 0$ at $\lambda = \sigma_0$ and, due to (2.7),

$$\lim_{\lambda \uparrow \sigma_1} \theta_{[\lambda,\Omega]}(x) = \infty \quad \text{for each } x \in \bar{\Omega}_{0,1} \setminus \partial\Omega. \tag{8.6}$$

The fact that (8.6) occurs in $\Omega_{0,1}$ goes back to [45], Theorem 2.4, which was submitted for publication on August 1996, thought it appeared in 2000. The proof given in Section 5 is the original one given in [45]. It should be noted that (8.6) extremely sharpens (8.5).

In Figure 10 we have represented $\theta_{[\lambda,\Omega]}(x)$ versus $\lambda \in (\sigma_0, \sigma_1)$ for a given $x \in \Omega_{0,1}$. Thanks to (8.6), the diagram exhibits a bifurcation from infinity at $\lambda = \sigma_1$. In Figure 10 stable solutions are indicated by solid lines, and unstable solutions by dashed lines. The

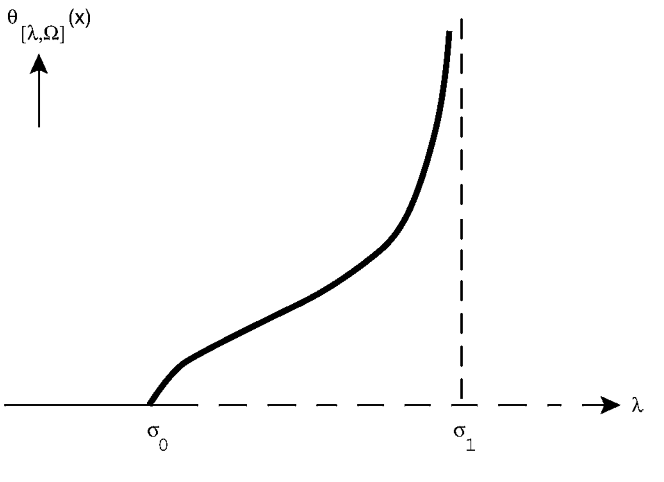

Fig. 10. The steady-states of (1.8).

state $u = 0$ loses stability as λ crosses σ_0, such stability being gained by $\theta_{[\lambda,\Omega]}$. By Theorem 2.4, local stability actually entails global attractiveness. It is remarkable the significant difference between the diagram shown by Figure 10 and the diagram of the classical positive solutions that has been already inserted in Figure 2, where the curve $\lambda \mapsto \theta_{[\lambda,\Omega]}(x)$ is bounded at $\lambda = \sigma_1$. The difference coming from the fact that, thanks to (2.7), for each $x \in \Omega \setminus \bar{\Omega}_{0,1}$ one has that

$$\lim_{\lambda \uparrow \sigma_1} \theta_{[\lambda,\Omega]}(x) = L^{\min}_{[\sigma_1,\Omega\setminus\bar{\Omega}_{0,1}]}(x) < \infty, \tag{8.7}$$

in severe contrast with (8.6). Bifurcation diagrams might drastically change according to the magnitude chosen to represent it, as it actually occurs here in, of course.

Subsequently, we consider the compact set K defined by

$$K := \left\{ x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) \geqslant \frac{1}{n} \right\}$$

for a sufficiently large $n \in \mathbb{N}$ (see Figure 1). Thanks to the Harnack inequality, for each $\lambda \in (\sigma_0, \sigma_1)$ there exists a constant $C_\lambda > 0$ such that

$$\max_K \theta_{[\lambda,\Omega]} \leqslant C_\lambda \min_K \theta_{[\lambda,\Omega]}. \tag{8.8}$$

Thus, if there are $\varepsilon > 0$ and $C > 0$ for which

$$\sup_{\sigma_1-\varepsilon<\lambda<\sigma_1} C_\lambda \leqslant C, \tag{8.9}$$

then, by (8.6) and (8.8), we would get

$$\lim_{\lambda \uparrow \sigma_1} \theta_{[\lambda,\Omega]} = \infty \quad \text{uniformly in } K \tag{8.10}$$

and therefore, $\theta_{[\lambda,\Omega]}$, $\sigma_0 < \lambda < \sigma_1$, would exhibit entire blow up in K as $\lambda \uparrow \sigma_1$, which contradicts (8.7). Consequently, (8.9) fails to be true. Hence,

$$\sup_{\sigma_1-\varepsilon<\lambda<\sigma_1} C_\lambda = \infty. \tag{8.11}$$

As a consequence, the Harnack inequality fails to be true uniformly in compact intervals of $\lambda \in \mathbb{R}$ for general nonlinear problems, which makes specially interesting (8.6) and (8.7). Actually, before developing the mathematical contains of this monograph, the validity of (8.9) was so much appealing that several experts in PDEs tried to convince the author of the complete blow up of $\theta_{[\lambda,\Omega]}$ as $\lambda \uparrow \sigma_1$. Finally, it was in [31], where (8.7) was shown to occur in the very special case when $\Omega_{0,2} = \emptyset$ and $x \in \Omega_-$, by using the minimal large solutions in small moving balls in order to get uniform a priori bounds within Ω_-. Although this – apparent – *paradox* was definitively solved in October 1996, [31] appeared in 1998.

Some time later we learned that a huge industry on large solutions was already available at our disposal in the literature (e.g., [6,67] and [57]). However, it should be emphasized that the original a priori bounds of Keller [38] and Osserman [61] do not apply straight away to (1.9), and that, occasionally, they have been used in the wrong way (cf. [44] for further technical details).

The interested readers are sent to [31], Figure 2.2, to see a numerical plot illustrating the stabilization of the positive solution $\theta_{[\lambda,\Omega]}$ to the minimal metasolution $L^{\min}_{[\lambda,\Omega\setminus\bar{\Omega}_{0,1}]}$ as $\lambda \uparrow \sigma_1$. Actually, that was the first metasolution constructed in the literature in the context of population dynamics.

On February 1997, combining some classical methods from the calculus of variations going back to [18] and [37] with the method of sub and supersolutions, López-Gómez and Sabina de Lis [56] could definitively prove the validity of (8.6) along Γ_1 – in the special case when $a \in \mathcal{C}^1$ and $\Omega_{0,2} = \emptyset$ – so getting (2.7). This result was later generalized by Gómez-Reñasco and López-Gómez [36], to cover the general radially symmetric case, and by Du and Huang [28], in order to eliminate the smoothness assumption on $a(x)$ from the theorem of [56]. Actually, the proof of this fact given in Section 5 (see the proof of (5.45)) follows [28], where the proof of the classical Hopf lemma given by Protter and Weinberger [63] was comfortably adapted.

The concept of *metasolution* and most of Theorems 2.1–2.3 go back to [36] and [34], which was Gómez-Reñasco Ph.D. Dissertation Thesis under the supervision of the author. The results of [36] sow light during the summer of 1998. Then, the manuscript was sent to H. Brezis on September 1998 for submission to *Archive for Rational Mechanics and Analysis*, though, quite unfortunately, on March 4th 1999, it was formally rejected. Further, it was sent to P.H. Rabinowitz for submission to *Nonlinear Analysis. Theory Methods and Applications*, where it finally appeared in 2002, after three years time. Fixing dates might be imperative to avoid any eventual priority dispute.

The original theory developed in [36] was extraordinarily generalized and refined in [44]. Actually, Theorems 2.1–2.3 and most of Theorem 2.4, in their full generality, go back to [36] and [44]. Unfortunately, the proof of Theorem 2.4(c) and (d) given in [44] has a serious gap that has been filled in [51]. The author was aware of that gap from a beautiful question raised by H. Amann during a seminar delivered by the author in the Institute of Mathematics of the University of Zürich on June 2002. Although Theorem 2.4(d)(ii) had been already obtained by Du and Huang [28] in the very special case when $\Omega_{0,2} = \emptyset$, parts (c), (d)(iii) and, actually, the whole theorem in its full generality go back to [44] and [51]. Nevertheless, since $\Omega_{0,2} \neq \emptyset$, obtaining the a priori bounds in $\Omega \setminus \bar{\Omega}_{0,1}$ is extraordinarily more involved than in the very special case when $\Omega_{0,2} = \emptyset$. The main problem coming from the fact that one has to adapt the available proof of part (d), obtained for the special case when $\Omega_{0,2} = \emptyset$, to study the singular counterpart of the problem when $u|_{\partial(\Omega\setminus\bar{\Omega}_{0,1})} = \infty$ and $\Omega_{0,2} \neq \emptyset$, this being an extremely delicate matter from the technical point of view. Actually, the level of difficulty is much higher than passing from the classical logistic equation, where $a(x)$ is bounded away from zero, to the degenerate situation when $a(x)$ vanishes somewhere, as one must deal with a degenerate problem for a singular boundary value problem when $\Omega_{0,2} \neq \emptyset$.

Actually, during the celebration of the referred seminar in Zürich, though H. Amann realized the fact that the limiting behavior of the increasing solutions is governed by the minimal metasolution, he could not see at first glance the controlled behavior of the non-increasing solutions of (1.1) in between the minimal and the maximal metasolutions, which forced the author to check all technical details of the proofs given in [44] finding a mistake in the original proof (cf. [51] for further details). The proof of Theorem 2.4 given here goes back to [51], and, consequently, it is extremely indebted with the extraordinary intuition of H. Amann. It should be emphasized that Theorem 2.4 in its full generality was implicitly announced in the general discussion carried out in [36] four years before.

More recently, Cirstea and Radulescu [17] have considered the following singular problem

$$\begin{cases} \Delta u + au = b(x) f(u) & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega, \end{cases} \tag{8.12}$$

where Ω is a smooth domain of $\mathbb{R}^N$, $a \in \mathbb{R}$, $b \geqslant 0$, $b \neq 0$, is continuous and $f \in \mathcal{C}^1$ is a positive function satisfying the Keller–Osserman condition and such that $f(u)/u$ is increasing in $(0, \infty)$. If we denote by σ_1 the principal eigenvalue of $-\Delta$ in the region where $b^{-1}(0)$, then the main theorem of [17] reads as follows.

THEOREM. *Problem* (8.12) *has a solution if and only if* $a \in (-\infty, \sigma_1)$. *Moreover, in this case, the solution is positive.*

Incidentally, Cirstea and Radulescu [17] knowledge the readers that

> We point out that our framework in the above result includes the case when b vanishes at some points of $\partial\Omega$, or even if $b = 0$ on $\partial\Omega$. In this sense, our result responds to a question raised to one of us by Professor Haim Brezis in Paris, May 2001.

The previous theorem follows as a direct consequence from the general theory developed in [36] and [44], and considerably tided up in this monograph. Actually, the problem studied in [17] had already been considered in [36].

Theorems 2.6 and 2.7 go back to [50]. Theorem 2.7 is a substantial improvement of [28], Theorem 2.1 and [32], Theorem 1, where it was imposed the substantially strongest assumption that

$$a(x) = \beta\big[d(x)\big]^{\gamma}\big[1 + \rho d(x) + \mathrm{o}\big(d(x)\big)\big] \quad \text{as } d(x) \downarrow 0, \tag{8.13}$$

where $d(x) := \mathrm{dist}(x, \partial\Omega_-)$ and $\beta > 0$, $\gamma \geqslant 0$, $\rho \in \mathbb{R}$ are constant. So, in such case,

$$\lim_{x \to x_1} \frac{a(x)}{\beta[d(x)]^{\gamma}} = 1 \quad \text{uniformly in } x_1 \in \partial\Omega_-,$$

while in Theorem 2.7 the weight function $a(x)$ is allowed to decay to zero on $\partial\Omega_-$ at different rates according to each particular point $x_1 \in \partial\Omega_-$. Theorem 2.6 is completely new even in the simplest case when (8.13) is satisfied, since $a(x)$ vanishes within $\Omega_{0,2}$, which is a situation not previously treated in the literature.

It should be mentioned that all these results are substantial extensions of some pioneering contributions of Loewner and Nirenberg [40], Kondratiev and Nikishin [39], Bandle and Marcus [5–7] and Véron [67], who exclusively studied the simplest case when $a(x)$ is bounded away from zero in $\bar{\Omega}$. As we already know, in such case, large positive constants provide us with a priori bounds for the positive solutions of the underlying homogeneous Dirichlet boundary value problem, while if $a(x)$ decays to zero somewhere on $\partial\Omega_-$, as it occurs in the case we are dealing with, live is much harder as large positive constants are not supersolutions anymore, and, in general, the a priori bounds of the positive solutions of those underlying problems are lost. Actually, explaining the huge differences between these two cases has been the main issue addressed in this monograph.

As asserted by Theorem 2.7, the blow-up rates of the large solutions of (1.9) in Ω_- depend upon the precise decay of $a(x)$ at each of the points of $\partial\Omega_-$. Thus, to prove Theorem 2.7 we cannot adapt the proof of the corresponding counterparts of Du and Huang [28] and García-Melián et al. [32], based on the construction of some appropriate global sub- and supersolutions. Instead of that, in Section 7 we have used a new *localization method*, which consists in constructing a local subsolution and a local supersolution supported at each particular point of the boundary of the domain. Although this method goes back to [50], it should be noted that the proof given in [50] exclusively shows the validity of Propositions 7.1 and 7.2 here, since the proof of the fact that the quotient function q defined in (7.39) is continuous contains an important gap. Nevertheless, the main theorem is correct and all necessary technical details to complete it have been clarified in Section 7 here. The same gap occurred in [25], where, besides characterizing the existence of large solutions for a general class of porous media equations of logistic type, the blow-up rates of these solutions in terms of the decay rates of $a(x)$ and of the nonlinear diffusion rate were ascertained.

Several other recent works dealing with similar but distinct problems are [14–16], and [58]. Their main goal is obtaining the blow-up rates of the large solutions of (1.9) in Ω_- for large classes of f's keeping $a(x)$ bounded away from zero.

Incidentally, [50] was submitted to the *Journal of Differential Equations* on January 18th 2002 and the referee's report, recommending rejection, was received on May 23rd 2003. During that period, appeared [14–16] and [58], whose publication was the main argument exhibited by the referee to recommend rejection of [50]. At the end of the day, the editors of the *Journal of Differential Equations* decided to publish it.

Further, it turns out that the minimal metasolutions of (1.1) that are stable can be also obtained by adopting the approach of [47], where (1.8) versus the perturbed problem,

$$\begin{cases} -\Delta u = \lambda u - (a+\varepsilon) f(\cdot, u)u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{8.14}$$

were analyzed. In (8.14), $\varepsilon > 0$ should be regarded as a perturbing parameter in order to transform (1.8) into a classical logistic problem where the weight function in front of the nonlinearity is positive and bounded away from zero. Thanks to Theorem 4.2 (with $D = \Omega_- = \Omega$), we already know that (8.14) possesses a positive solution if and only if $\lambda > \sigma_0$, which is unique if it exists. Subsequently, we shall denote it by $\theta_{[\lambda,\Omega,\varepsilon]}$. Then, though the proof will not be included here, the main theorem of [47] can be stated as follows.

THEOREM 8.1. *The following assertions are true*:

1. $\lim_{\varepsilon\downarrow 0}\theta_{[\lambda,\Omega,\varepsilon]}=\theta_{[\lambda,\Omega]}$ *if* $\sigma_0<\lambda<\sigma_1$.
2. $\lim_{\varepsilon\downarrow 0}\theta_{[\lambda,\Omega,\varepsilon]}=\mathfrak{M}^{\min}_{[\lambda,\Omega\setminus\bar{\Omega}_{0,1}]}$ *if* $\sigma_1\leqslant\lambda<\sigma_2$.
3. $\lim_{\varepsilon\downarrow 0}\theta_{[\lambda,\Omega,\varepsilon]}=\mathfrak{M}^{\min}_{[\lambda,\Omega_-]}$ *if* $\lambda\geqslant\sigma_2$.

Consequently, there is a continuous transition between the dynamics of (1.1) and the trivial dynamics exhibited by the parabolic counterpart of (8.14). This should be a crucial property from the point of view of the eventual applications of the underlying theory to Population Dynamics, as it permits designing appropriate environments where the population exhibits a *differentiated logistic behavior* according to the several patches that conform the habitat. Also, it should be extremely relevant from the point of view of the numerical approximation of the dynamics of (1.8), as it permits to compute the minimal metasolutions that are stable through the following scheme:

1. Computing the global curve of positive solutions of (8.14) by performing a global numerical continuation in λ.
2. Pick any of these positive solutions and then perform a global continuation in ε, to approximate the corresponding metasolution.

Such strategies have already been shown to be pivotal to compute isolated components of positive solutions in wide classes of reaction diffusion systems and weighted nonlinear boundary value problems (e.g., [13,46] and [54]).

Another mechanism to approximate the dynamics of (1.1) through a model exhibiting an asymptotically bounded population profile, whose level might be designed according to the region of the inhabiting area, was given in [23] by means of the classical porous medium equation. Most precisely, in [23] we considered the following differential equation with nonlinear diffusion

$$\begin{cases} -\Delta w^m=\lambda w-aw^p & \text{in } \Omega,\\ w=0 & \text{on } \partial\Omega,\end{cases} \tag{8.15}$$

where $m>1$, in order to obtain the following result, whose proof is omitted here.

THEOREM 8.2. *Suppose* $1<m<p$. *Then* (8.15) *possesses a positive solution* (*unique*) *if and only if* $\lambda>0$. *Moreover, if we denote it by* $W_{[\lambda,\Omega,m]}$, *then, setting*

$$\theta_{[\lambda,\Omega,m]}:=W^m_{[\lambda,\Omega,m]},$$

the following properties are satisfied:

1. $\lim_{m\downarrow 1}\theta_{[\lambda,\Omega,m]}=0$ *if* $0<\lambda\leqslant\sigma_0$.
2. $\lim_{m\downarrow 1}\theta_{[\lambda,\Omega,m]}=\theta_{[\lambda,\Omega]}$ *if* $\sigma_0<\lambda<\sigma_1$.
3. $\lim_{m\downarrow 1}\theta_{[\lambda,\Omega,m]}=\mathfrak{M}^{\min}_{[\lambda,\Omega\setminus\bar{\Omega}_{0,1}]}$ *if* $\sigma_1\leqslant\lambda<\sigma_2$.
4. $\lim_{m\downarrow 1}\theta_{[\lambda,\Omega,m]}=\mathfrak{M}^{\min}_{[\lambda,\Omega_-]}$ *if* $\lambda\geqslant\sigma_2$.

Metasolutions have also shown to be extremely relevant in analyzing the dynamics of very large classes of reaction diffusion systems (cf. [27,49] and [21]), and of very general classes of indefinite superlinear parabolic problems, where the weight function $a(x)$ is allowed to change sign within Ω.

In order to sketch how metasolutions arise in the context of indefinite superlinear problems, suppose a satisfies the following

$$a^{-1}(0) = \bar{\Omega}_{0,1} \cup \partial\Omega_{0,2}, \qquad a^{-1}\big((0,\infty)\big) = \Omega_{-}, \qquad a^{-1}\big((-\infty,0)\big) = \Omega_{0,2}, \tag{8.16}$$

and denote by a^{+} and a^{-} the positive and negative parts of a,

$$a = a^{+} - a^{-},$$

whose supports are Ω_{-} and $\Omega_{0,2}$, respectively. Then, by bringing together the theory developed in this monograph with the techniques and results of López-Gómez and Quittner [55] and Quittner and Simondon [64], one can get the following result, which is a sort of summary of the main findings of [52].

THEOREM 8.3. *Suppose* (8.16), (2.9), $\lambda > \sigma_1$, *and* $u_0 > 0$ *satisfies*

$$-\Delta u_0 \leqslant \lambda u_0 - a^{+}u^{p}.$$

Then, the following properties are satisfied:

1. *For each* $\lambda \in (\sigma_0, \sigma_1)$, *there exists* $\varepsilon(\lambda) > 0$ *such that* $\|a^{-}\|_{\mathcal{C}(\bar{\Omega}_{0,2})} < \varepsilon(\lambda)$ *implies*
$$\lim_{t\uparrow\infty} u_{[\lambda,\Omega]}(\cdot,t;u_0) = \theta_{[\lambda,\Omega]},$$
where $\theta_{[\lambda,\Omega]}$ *stands for the minimal positive solution of* (1.8), *whose existence is guaranteed by Amann and López-Gómez* [3].
2. *For each* $\lambda \in (\sigma_0, \sigma_1)$, *there exists* $\kappa(\lambda) > \varepsilon(\lambda)$ *such that* $u_{[\lambda,\Omega]}(\cdot,t;u_0)$ *blows up in a finite time if* $\|a^{-}\|_{\mathcal{C}(\bar{\Omega}_{0,2})} > \kappa(\lambda)$. *Moreover, the blow up is complete in* $\bar{\Omega}_{0,2}$ *if* $p-1>0$ *is sufficiently small. Furthermore, in this case,* $u_{[\lambda,\Omega]}(\cdot,t;u_0)$ *approximates* $\mathfrak{M}^{\min}_{[\lambda,\Omega\setminus\bar{\Omega}_{0,2}]}$ *as* $t\uparrow\infty$, *though, in* $\bar{\Omega}_{0,2}$, *infinity is reached in a finite time.*
3. *For each* $\lambda \in [\sigma_1, \sigma_2)$, *there exists* $\varepsilon(\lambda) > 0$ *such that* $\|a^{-}\|_{\mathcal{C}(\bar{\Omega}_{0,2})} < \varepsilon(\lambda)$ *implies*
$$\lim_{t\uparrow\infty} u_{[\lambda,\Omega]}(\cdot,t;u_0) = \mathfrak{M}^{\min}_{[\lambda,\Omega\setminus\bar{\Omega}_{0,1}]},$$
where $\mathfrak{M}^{\min}_{[\lambda,\Omega\setminus\bar{\Omega}_{0,1}]}$ *stands for the minimal metasolution of* (1.9) *supported in* $\Omega\setminus\bar{\Omega}_{0,1}$. *Note that* $a(x)$ *changes sign in* $\Omega\setminus\bar{\Omega}_{0,1}$.
4. *For each* $\lambda \in [\sigma_1, \sigma_2)$, *there exists* $\kappa(\lambda) > \varepsilon(\lambda)$ *such that* $u_{[\lambda,\Omega]}(\cdot,t;u_0)$ *blows up in a finite time if* $\|a^{-}\|_{\mathcal{C}(\bar{\Omega}_{0,2})} > \kappa(\lambda)$. *Moreover, the blow up is complete in* $\bar{\Omega}_{0,2}$ *if* $p-1>0$ *is sufficiently small. Furthermore, in this case,* $u_{[\lambda,\Omega]}(\cdot,t;u_0)$ *approximates* $\mathfrak{M}^{\min}_{[\lambda,\Omega_{-}]}$ *as* $t\uparrow\infty$, *though, in* $\bar{\Omega}_{0,2}$, *infinity is reached in a finite time.*

5. *For each* $\lambda \geqslant \sigma_2$, $u_{[\lambda,\Omega]}(\cdot, t; u_0)$ *blows up in a finite time. Moreover, the blow up is complete in* $\bar{\Omega}_{0,2}$ *if* $p - 1 > 0$ *is sufficiently small. Furthermore, in this case,* $u_{[\lambda,\Omega]}(\cdot, t; u_0)$ *approximates* $\mathfrak{M}^{\min}_{[\lambda,\Omega_-]}$ *as* $t \uparrow \infty$, *though, in* $\bar{\Omega}_{0,2}$, *infinity is reached in a finite time.*

Consequently, though metasolutions arose in analyzing the most paradigmatic model of Population Dynamics, they seem to be crucial to understand the role of spatial heterogeneities in wide areas of Science and Technology. Undoubtedly, within the next few years metasolutions will play a significant role in the modern theory of Partial Differential Equations, where using spatial heterogeneities is imperative in order to get more realistic models.

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CHAPTER 5

Elliptic Problems with Nonlinear Boundary Conditions and the Sobolev Trace Theorem

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Contents

Abstract

This work constitutes a short survey of the subject of elliptic partial differential equations with nonlinear boundary conditions. We will focus especially on the relevance of the Sobolev trace theorem in the analysis of this kind of problems. We will also describe some of the techniques employed when dealing with such a kind of problems.

Keywords: Elliptic problems, Nonlinear boundary conditions, Sobolev inequalities

MSC: 35J65, 35J50, 35J55

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1. Introduction

In this chapter we provide a survey of mathematical results concerning solutions of elliptic problems with nonlinear boundary conditions. We mean solutions of problems that can be written in the general form

$$\begin{cases} Lu = f(u) & \text{in } \Omega, \\ Bu = g(u) & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here, and through this work, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, Lu is a second-order elliptic operator and Bu is a boundary condition that involves $\frac{\partial}{\partial\nu}$, the outer normal derivative.

Of importance in the study of boundary value problems for differential operators in Ω are the Sobolev spaces and inequalities. Hence, Sobolev inequalities and their optimal constants is a subject of interest in the analysis of PDEs and related topics. It has been widely studied in the past by many authors and is still an area of intensive research. See, for instance, the book [15] and the survey [50] for recent developments in this field.

When analyzing problems with nonlinear boundary conditions like (1.1) it turns out that among the Sobolev embeddings the Sobolev trace theorem plays a fundamental role. Also one is lead to the study of nonlinear boundary conditions when one tries to find out properties of the Sobolev trace best constant.

Our main interest here is to look closely at this relation between nonlinear boundary conditions and the Sobolev trace theorem.

Another motivation to study problems with nonlinear boundary conditions comes from geometry. One is lead to nonlinear boundary conditions when one performs a description of conformal deformations on Riemannian manifolds with boundary. Look at the results of Cherrier and Escobar [29,54,55]. In [53] and [105] a geometric problem in the half-space $\mathbb{R}^N_+$ with nonlinear boundary conditions is studied.

Also, nonlinear boundary conditions appear in a rather natural way in some physical models. For example, problem (1.1) can be thought of as a model for heat propagation. In this case u stands for the temperature and the normal derivative $\frac{\partial u}{\partial\nu}$ that appears in the boundary condition $B(u)$ represents the heat flux. Hence the boundary condition represents a nonlinear radiation law at the boundary. This kind of boundary conditions appear also in combustion problems when the reaction happens only at the boundary of the container, for example, because of the presence of a solid catalyzer, see [99] for a justification. Eigenvalue problems with the eigenvalue placed at the boundary condition, $\frac{\partial u}{\partial\nu} = \lambda u$, are studied since the pioneering work of Steklov, see [115].

There are works that consider fully nonlinear equations like $F(x, u, \nabla u, D^2u) = 0$ in Ω with nonlinear boundary conditions, $H(x, u, \nabla u) = 0$ on $\partial\Omega$, where H is assumed to be strictly increasing with respect to ∇u in the normal direction to $\partial\Omega$ at x. See, for example, [16] and [89] where viscosity solutions are considered. There are also papers that deal with higher-order equations, for example in [72] a fourth-order problem is considered. However, to simplify the exposition, we will be only concerned with second-order problems like (1.1).

Elliptic problems, $Lu = f(u)$, with Dirichlet boundary conditions, $u|_{\partial\Omega} = 0$, have been widely treated in the literature, see the survey [44] and references therein. Many of the known results that hold for Dirichlet boundary conditions have an analogous counterpart when dealing with nonlinear boundary conditions. However many times the proofs are different. We will try to emphasize the differences and similitudes between both types of boundary conditions. Therefore, a strong motivation to study elliptic problems with nonlinear boundary conditions is to see how the well-established theory for the Dirichlet problem extends to other situations (as nonlinear boundary conditions) and to develop new ideas and methods when the available theory is not applicable. These developments may have consequences for other related problems that do not involve necessarily nonlinear boundary conditions.

We have to mention that there is a large amount of literature dealing with parabolic problems with nonlinear boundary conditions. In the last years there has been an increasing interest in the study of blow-up due to reaction at the boundary, both for scalar problems and for systems, see, for example, the surveys [34,77] and references therein. Often parabolic results are related to elliptic results. For example, when every positive solution of a parabolic problem blows up there cannot exist any positive stationary solution. Hence we have a nonexistence result of positive solutions for the elliptic problem in this case, see [30], [31] and [81] for such type of results. The stability properties of a stationary solution is also a problem to deal with, see, for example, [31] and [39]. Moreover, many times regularity results for elliptic and parabolic problems are related, see [4].

The References do not escape the usual rule of being incomplete. In general, we have listed those papers which are more close to the topics discussed here. But, even for those papers, the list is far from being exhaustive and we apologize for omissions.

Organization of the chapter. The rest of the chapter contains eleven sections. They are organized by subject, however many times there are relations between them that we will try to outline. In some cases we will provide full proofs (or at least sketch the main arguments) in order to give the reader an idea of the involved techniques. In each section a change of subject or problem will be marked with ✓.

Section 2. In this section we state some preliminaries relating the best Sobolev trace constant with problems with nonlinear boundary conditions and give some ideas about regularity results from J. García-Azorero, I. Peral and the author that can be found in [81].

Section 3. In this section we see how to adapt usual variational techniques to deal with nonlinear boundary conditions. We follow ideas from M. Chipot, M. Fila, P. Quittner, J. García-Azorero, I. Peral, K. Umezu, J. Fernández Bonder and the author, see [31], [32], [71], [81] and [121].

Section 4. We collect some results in the half-space, $\mathbb{R}^N_+$, proved by B. Hu, X. Cabre, M. Chipot, M. Chlebik, I. Shafrir, M. Fila, Y. Park, W. Reichel, J. Sola-Morales, S. Terracini, Y. Li and M. Zhu, see [24], [30], [33], [87], [93], [106] and [117].

Section 5. Here we discuss about the results of A. Ambrosetti, Y. Li, A. Malchiodi, V. Felli, M. Ahmedou and J.F. Escobar, which have a strong geometrical motivation, the so-called Yamabe problem for manifolds with boundary. In this geometrical problem the critical exponent for the Sobolev trace embedding appears. See [5], [53], [54], [55] and [60].

Section 6. In this section we deal with the dependence of the best Sobolev trace constant on the domain for a subcritical exponent. We focus on families of domains obtained by expanding or contracting a fixed domain and obtain the asymptotic behavior of the best Sobolev trace constant. Also we prove that the first eigenvalue of an associated nonlinear Steklov-like eigenvalue problem is isolated and simple. We collect results of C. Flores, M. del Pino, S. Martinez, J. Fernández Bonder and the author, [46], [66], [74] and [100].

Section 7. We look at symmetry of the extremals for the Sobolev trace embedding. In particular, we study when the extremals are radial if Ω is the ball of radius R, $B(0, R)$. We report on the results of O. Torne, E. Lami-Dozo, J. Fernández Bonder and the author, [65] and [92].

Section 8. In this section we consider the trace of functions that belong to a Sobolev space and vanish over some subset of Ω. We look at the problem of optimizing the best Sobolev trace constant when varying the subset where the involved functions vanish keeping its area fixed. We follow ideas of J. Fernández Bonder, N. Wolanski and the author, [76].

Section 9. Here we look at the Sobolev trace embedding with the critical exponent. The results presented in this section are mainly due to E. Abreu, P. Carriao, O. Miyagaki, D. Pierotti, S. Terracini, F. Demengel, M. Motron, M. Chlebik, M. Fila, W. Reichel, F. Andreu, J. Mazon, J. Fernández Bonder and the author, [1], [11], [35], [47], [75], [104], [108] and [109]. See also Section 5 for other results that involve the critical exponent.

Section 10. Now we study the dependence of the Sobolev trace constant on the exponents involved. We rely on results of R. Ferreira, J. Fernández Bonder and the author, [64].

Section 11. In this section we collect results concerning elliptic systems with nonlinear boundary conditions, from M. Schechter, W. Zou, S. Li, J. Fernández Bonder, S. Martinez and the author, see [67], [68], [69], [70], [112] and [125].

Section 12. Finally we collect other results for problems with nonlinear boundary conditions, concerning maximum/antimaximum principle, isoperimetric inequalities, self-similar profiles for parabolic problems with blow-up, free boundaries, equations involving maximal monotone graphs and their relation with semigroup theory, resonance problems, the Fučik spectrum at the boundary, etc. In this section we do not provide any proofs and refer for details to the papers of F. Andreu, D. Arcoya, Ph. Benilan, F. Brock, M. Crandall, J. Davila, S. Martinez, J.M. Mazon, M. Montenegro, P. Sacks, S. Segura de Leon, J. Toledo and the author. See [12], [14], [19], [23], [41], [42], [101] and [102].

Notations. We end the Introduction fixing some of the notation that will be used in the following sections.

Along this chapter there are two measures involved, the usual Lebesgue measure in $\Omega \subset \mathbb{R}^N$ and the surface measure on $\partial\Omega$. With $\mathrm{d}x$ and $\mathrm{d}\sigma$ we denote the corresponding N- and $(N-1)$-dimensional measures. Also we will use the notation $|A|$ for the measure of the set A in its corresponding dimension, that is, if A is a set of dimension r, $|A|$ stands for the r-dimensional measure of A. We will call the characteristic function of the set A as χ_A.

With $p_* = p(N-1)/(N-p)$ and $p^* = pN/(N-p)$ we denote the critical exponents for the Sobolev trace embedding $W^{1,p}(\Omega) \mapsto L^q(\partial\Omega)$ and the Sobolev embedding $W^{1,p}(\Omega) \mapsto L^r(\Omega)$. $S(\Omega, p, q)$ stands for the best Sobolev trace constant, see Section 2. Remark that we make explicit the dependence of the constant on the domain and on the involved exponents. This dependence will be analyzed throughout this work.

2. Preliminaries

Let us look for a simple example where problems with nonlinear boundary conditions appear in a very natural way. To this end, let us first remark that in the study of elliptic and parabolic partial differential equations the Sobolev spaces are a very useful and versatile tool for their analysis. For general references on Sobolev spaces we cite [2].

Recall that the Sobolev space $H^1(\Omega)$ is defined as the space of $L^2(\Omega)$ functions with weak first derivatives that also belong to $L^2(\Omega)$. In $H^1(\Omega)$ we have a Hilbert space structure. We consider the usual norm that comes from the inner product,

$$\langle u, v\rangle = \int_\Omega \nabla u \nabla v \,\mathrm{d}x + \int_\Omega uv \,\mathrm{d}x,$$

that is,

$$\|u\|_{H^1(\Omega)} = \left(\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \int_\Omega |u|^2 \,\mathrm{d}x\right)^{1/2}.$$

Given a smooth function (e.g., $u \in C^1(\overline{\Omega}) \subset H^1(\Omega)$) we can define the restriction to the boundary $u|_{\partial\Omega}$. It turns out that this restriction operator can be extended from smooth functions to $H^1(\Omega)$ giving a linear continuous operator from $H^1(\Omega)$ to $L^r(\partial\Omega)$, if $1 \leqslant r \leqslant 2_* = 2(N-1)/(N-2)$,

$$T : H^1(\Omega) \to L^r(\partial\Omega).$$

This result is the very well-known Sobolev trace theorem. See, for example, [2]. The norm of this operator is given by

$$\begin{aligned} S(\Omega, 2, r) &= \inf\bigl\{\|u\|^2_{H^1(\Omega)};\, u \in H^1(\Omega) \text{ with } \|u\|^2_{L^r(\partial\Omega)} = 1\bigr\} \\ &= \inf_{u \in H^1(\Omega)\setminus H^1_0(\Omega)} \frac{\int_\Omega |\nabla u|^2 + |u|^2 \,\mathrm{d}x}{(\int_{\partial\Omega} |u|^r \,\mathrm{d}\sigma)^{2/r}}. \end{aligned} \tag{2.1}$$

This value $S(\Omega, 2, r)$ is known as the best Sobolev trace constant.

For values of $r < 2_*$ (subcritical values) we have that the trace operator is a compact operator, therefore an easy compactness argument proves that there exist extremals, that is, functions in $H^1(\Omega)$ where the norm is attained. These extremals turn out to be weak solutions of

$$\begin{cases} \Delta u = u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda |u|^{r-2} u & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

where λ is a Lagrange multiplier.

For a weak solution of (2.2) we understand a function $u \in H^1(\Omega)$ that verifies

$$\int_\Omega \nabla u \nabla v \,\mathrm{d}x + \int_\Omega uv \,\mathrm{d}x = \int_{\partial\Omega} \lambda |u|^{r-2} uv \,\mathrm{d}\sigma \tag{2.3}$$

for every test function $v \in H^1(\Omega)$. Remark that in (2.3) two measures (dx, the volume measure and $d\sigma$, the surface measure) are involved. If we assume the normalization of the extremal given by $\|u\|_{L^r(\partial\Omega)} = 1$, taking $v = u$ in (2.3), we get that the Lagrange multiplier is related to the best Sobolev trace constant given by (2.1). It holds, $\lambda = S(\Omega, 2, r)$. Problem (2.2) has associated an energy functional

$$\mathcal{F}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + |u|^2 \, dx - \int_{\partial\Omega} \lambda \frac{|u|^r}{r} \, d\sigma. \tag{2.4}$$

In fact, critical points of (2.4) in $H^1(\Omega)$ are weak solutions of the problem (2.2) in the weak sense (2.3).

An important case is when $r = 2$. In this case (2.2) becomes the linear eigenvalue problem

$$\begin{cases} \Delta u = u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda u & \text{on } \partial\Omega. \end{cases}$$

Using the compactness of the embedding $H^1(\Omega) \hookrightarrow L^2(\partial\Omega)$ it is obtained that there exists a sequence of eigenvalues $\lambda_n \to \infty$ and the first one corresponds to the best constant that we are looking for, see [71]. These eigenvalues can be regarded as the eigenvalues of the Dirichlet to Neumann map for the operator $-\Delta + I$ acting in the space $H^1(\Omega)$.

At this point, we have to mention the results of Escobar [57–59] for the Steklov eigenvalue problem

$$\begin{cases} \Delta \varphi = 0 & \text{in } \Omega, \\ \frac{\partial \varphi}{\partial \nu} = \lambda \varphi & \text{on } \partial\Omega. \end{cases}$$

This problem was introduced by Steklov [115] and was initially studied by Calderon [25], because the set of eigenvalues for this problem coincides with the eigenvalues for the Dirichlet to Neumann map. Escobar proves some estimates and isoperimetric results for this problem on manifolds.

Going back to the Sobolev trace embedding in full generality, one can consider

$$W^{1,p}(\Omega) \mapsto L^q(\partial\Omega),$$

where $W^{1,p}(\Omega)$ is the usual Sobolev space of functions $u \in L^p(\Omega)$ with $\nabla u \in (L^p(\Omega))^N$ endowed with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \left(\int_\Omega |\nabla u|^p \, dx + \int_\Omega |u|^p \, dx\right)^{1/p}.$$

Then, problems with nonlinear boundary conditions also appear in a natural way when one considers the Sobolev trace inequality in $W^{1,p}(\Omega)$ and $L^q(\partial\Omega)$,

$$S\|u\|^p_{L^q(\partial\Omega)} \leqslant \|u\|^p_{W^{1,p}(\Omega)}, \quad 1 \leqslant q \leqslant p_* = \frac{p(N-1)}{N-p}.$$

The best constant S is given by

$$S(\Omega, p, q) = \inf_{u \in W^{1,p}(\Omega) \setminus W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p + |u|^p \, dx}{(\int_{\partial\Omega} |u|^q \, d\sigma)^{p/q}}.$$

The extremals (if there exist) are weak solutions of

$$\begin{cases} \Delta_p u = |u|^{p-2} u & \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \lambda |u|^{q-2} u & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

Here, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the well-known operator called the p-Laplacian. Remark that (2.5) is a quasilinear elliptic problem.

Also in this case we find an eigenvalue problem. When $q = p$, problem (2.5) becomes

$$\begin{cases} \Delta_p u = |u|^{p-2} u & \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \lambda |u|^{p-2} u & \text{on } \partial\Omega. \end{cases}$$

This is a nonlinear eigenvalue problem. Also in this case it can be proved that there exists a sequence of variational eigenvalues $\lambda_n \to \infty$, see [71]. As for $p = 2$, the first eigenvalue, which is isolated and simple (see [100]), is related to the best constant for the Sobolev trace embedding. However, as happens for the eigenvalue problem for the p-Laplacian with Dirichlet boundary conditions, it is not known if the obtained sequence constitutes the whole spectrum.

Now, for the sake of completeness, we will provide an answer for the question: Among the functions $f : \partial\Omega \to \mathbb{R}$, which are the trace of a function of $W^{1,p}(\Omega)$?

To answer this question we have to introduce the fractional Sobolev spaces. We follow the presentation made by Bourgain, Brezis and Mironescu in [21].

DEFINITION 2.1. For $0 < s < 1$, we define

$$W^{s,p}(A) = \left\{ u \in L^p(A) \colon [u]_{s,p}^p := \int_A \int_A \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy < \infty \right\}$$

with the norm $\|u\|_{W^{s,p}(A)} = \|u\|_{L^p(A)} + (p(1-s))^{1/p} [u]_{s,p}$.

The quantity $[\cdot]_{s,p}$ is known as the Gagliardo seminorm. The factor $(p(1-s))^{1/p}$ that appears in the definition of the norm guarantees that $\lim_{s \to 1} \|u\|_{W^{s,p}(A)} = \|u\|_{W^{1,p}(A)}$, see [21].

Let us consider the quotient space $W^{1,p}(\Omega)/W_0^{1,p}(\Omega)$ with the quotient norm $\|u\| = \inf_{u|_{\partial\Omega} = v|_{\partial\Omega}} \|v\|_{W^{1,p}(\Omega)}$. We have the following theorem.

THEOREM 2.1. *For* $1 < p < \infty$, *the trace operator*

$$T : \frac{W^{1,p}(\Omega)}{W_0^{1,p}(\Omega)} \mapsto W^{1-1/p,p}(\partial\Omega)$$

is a linear homeomorphism.

✓ *Some remarks on regularity.* Now we want to state some regularity results for elliptic problems with nonlinear boundary conditions. We are not going to provide the more general result nor enter into the details of the proofs. However we give a sketch of the arguments just to look at the ideas involved. Also we have to mention here that we are not facing any regularity constraints coming from domain regularity, since we are assuming that Ω is a smooth domain. See [51] for regularity results in polyhedra.

Let us deal with a solution of

$$\begin{cases} -\Delta u + u = |u|^{p-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda |u|^{q-2}u & \text{on } \partial\Omega, \end{cases} \tag{2.6}$$

with p and q critical or subcritical. Now we prove $C^\alpha(\overline{\Omega})$ estimates for the solutions of (2.6), see [81]. We include some details for the sake of completeness.

THEOREM 2.2. *Every weak solution of* (2.6) *with* $1 \leqslant q \leqslant 2(N-1)/(N-2)$, $1 \leqslant p \leqslant 2N/(N-2)$ *belongs to* $C^\alpha(\overline{\Omega})$.

First, we deal with the subcritical case. Namely, $1 < q < 2(N-1)/(N-2)$, $1 < p < 2N/(N-2)$. The idea is to adapt the classical bootstrapping argument, taking into account the nonlinear boundary condition. We start by recalling some linear results.

PROPOSITION 2.1.

(I) *Assume that* $g \in L^r(\Omega)$ *with* $r > 2N/(N+2)$ *and let* $\phi \in H^1(\Omega)$ *be the weak solution to*

$$\begin{cases} -\Delta\phi + \phi = g & \textit{in } \Omega, \\ \frac{\partial \phi}{\partial \nu} = 0 & \textit{on } \partial\Omega, \end{cases} \tag{2.7}$$

then $\|\phi\|_{W^{1,\beta}(\Omega)} \leqslant C\|g\|_{L^r(\Omega)}$ *with* $\beta = Nr/(N-r) > 2$.

(II) *Assume that* $h \in L^s(\partial\Omega)$ *with* $s > 2(N-1)/N$, *and let* ψ *be the weak solution to problem*

$$\begin{cases} -\Delta\psi + \psi = 0 & \textit{in } \Omega, \\ \frac{\partial \psi}{\partial \nu} = h & \textit{on } \partial\Omega. \end{cases} \tag{2.8}$$

Then $\|\psi\|_{W^{1,\gamma}(\Omega)} \leqslant C\|h\|_{L^s(\partial\Omega)}$ *with* $\gamma = Ns/(N-1) > 2$.

PROOF. Part (I) can be considered as the simplest case of the results in [114]. In this case the proof is easier: just integrating by parts we find

$$\left| \int_\Omega \nabla\phi \nabla\rho \,\mathrm{d}x + \int_\Omega \phi\rho \,\mathrm{d}x \right| \leqslant \|g\|_{L^r(\Omega)} \|\rho\|_{L^{r'}(\Omega)},$$

with $\frac{1}{r}+\frac{1}{r'}=1$, and by Sobolev embedding we can take a test function $\rho \in W^{1,\beta'}(\Omega)$ with $\beta' = Nr'/(N+r')$. As a consequence, using Proposition 1 of [29] we get $\phi \in W^{1,\beta}(\Omega)$ and $\|\phi\|_{W^{1,\beta}(\Omega)} \leqslant C\|g\|_{L^r(\Omega)}$, where $\beta = Nr/(N-r)$, and since $r > 2N/(N+2)$, it follows $\beta > 2$.

As for part (II), if ψ is the weak solution, multiplying by a regular test function $\eta \in C^1(\Omega)$ we get

$$\left|\int_\Omega \nabla\psi\nabla\eta\,\mathrm{d}x + \int_\Omega \psi\eta\,\mathrm{d}x\right| \leqslant \|h\|_{L^s(\partial\Omega)}\|\eta\|_{L^{s'}(\partial\Omega)},$$

where $\frac{1}{s}+\frac{1}{s'}=1$. Then by density we can take $\eta \in W^{1,\gamma}(\Omega)$ and therefore, by the trace theorem, $\eta|_{\partial\Omega} \in W^{1-(1/\gamma'),\gamma'}(\partial\Omega) \subset L^{\gamma'(N-1)/(N-\gamma')}(\partial\Omega)$, where $s' = \gamma'(N-1)/(N-\gamma')$, which implies that $\gamma = Ns/(N-1)$. Hence, by Proposition 1 of [29], we get that $\psi \in W^{1,\gamma}(\Omega)$ and $\|\psi\|_{W^{1,\gamma}(\Omega)} \leqslant C\|h\|_{L^s(\partial\Omega)}$. This finishes the proof. □

PROOF OF THEOREM 2.2. We decompose our original problem, taking $g = |u|^{p-2}u$ and $h = \lambda|u|^{q-2}u$, in such a way that $u = \phi + \psi$, where ϕ and ψ are the corresponding solutions to the linear problems (2.7) and (2.8).

The idea to prove regularity for solutions of (2.6) is that we can iterate the estimates in Proposition 2.1, improving from step to step the regularity of u. The argument is as follows: We start assuming $g \in L^{r_0}(\Omega)$ and $h \in L^{s_0}(\partial\Omega)$, where

$$r_0 = \frac{2N}{(N-2)(p-1)} \quad \text{and} \quad s_0 = \frac{2(N-1)}{(N-2)(q-1)}.$$

In particular, if $r_0 > N/2$ (that is, $p < (N+2)/(N-2)$) we get an exponent $\beta_0 > N$ such that $\phi \in W^{1,\beta_0}(\Omega) \subset C^\alpha(\overline{\Omega})$. On the other hand, if $q < N/(N-2)$, we get that $\psi \in W^{1,\gamma_0}(\Omega)$ with $\gamma_0 > N$ and in this case $\psi \in C^\alpha(\overline{\Omega})$. As a consequence, the C^α regularity for u is proved in the case $q < N/(N-2)$, $p < (N+2)/(N-2)$.

If not, in any case we have proved that $u \in W^{1,\tau_0}(\Omega)$ with $\tau_0 = \min\{\beta_0,\gamma_0\} > 2$. Then we can iterate exactly the same calculation as before, starting with $g \in L^{r_1}(\Omega)$ and $h \in L^{s_1}(\partial\Omega)$, where

$$r_1 = \frac{N\tau_0}{(N-\tau_0)(p-1)} \quad \text{and} \quad s_1 = \frac{(N-1)\tau_0}{(N-\tau_0)(q-1)}.$$

If r_1 and s_1 were both large enough (namely, $r_1 > N/2$ and $s_1 > N-1$), then we have finished. If not, we get that $u \in W^{1,\tau_1}(\Omega)$, where

$$\tau_1 = \begin{cases} \frac{Ns_1}{N-1} & \text{if } r_1 > \frac{N}{2} \text{ and } s_1 \leqslant N-1, \\ \min\left\{\frac{Ns_1}{N-1}, \frac{Nr_1}{N-r_1}\right\} & \text{if } r_1 \leqslant \frac{N}{2}. \end{cases}$$

Let us estimate these quantities in terms of the starting exponent τ_0. Since $\tau_0 > 2$, we have

$$\frac{Ns_1}{N-1} = \frac{N}{(N-\tau_0)(q-1)}\tau_0 \geqslant \frac{N}{(N-2)(q-1)}\tau_0.$$

And, on the other hand, it is easy to see that

$$\frac{Nr_1}{N-r_1} = \frac{N}{p(N-\tau_0)-N}\tau_0 > \frac{N}{p(N-2)-N}\tau_0.$$

Therefore (taking into account that p and q are subcritical) we have proved that there exists a constant $C = C(N, p, q) > 1$ such that $\tau_1 \geqslant C\tau_0$, and, in general, $\tau_k \geqslant C^k\tau_0$. This implies that in a finite number of steps we reach that $u \in W^{1,\tau^*}(\Omega)$ with $\tau^* > N$, and hence $u \in C^\alpha(\overline{\Omega})$.

Next, we will sketch briefly the arguments in the critical case, $p = 2^*$. In this case, the problem comes from the first iteration, since there is no margin to improve directly the initial exponent, getting $W^{1,\beta_0}(\Omega)$ regularity for some $\beta_0 > 2$. To overcome this difficulty we can use a truncation argument by Trudinger (see [120]) which proves that $\|u\|_{L^\tau(\Omega)} \leqslant C(|\Omega|, \|u\|_{L^{p^*}(\Omega)})$, where $\tau > p^*$. The sketch of the argument is as follows: consider the problem

$$\begin{cases} -\Delta u + u = \lambda |u|^{2^*-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = |u|^{q-2}u & \text{on } \partial\Omega, \end{cases}$$

where q is subcritical. Assume that $u \in H^1(\Omega)$, $u > 0$, is a solution and let us prove that $u \in L^\tau(\Omega)$ for some $\tau > 2^*$. The main idea is to choose a suitable truncation of u^β as test function with β greater but close to one. After some manipulations, that in our case involve the Sobolev trace inequality to handle the integrals over the boundary that appear, we arrive to $u \in L^{\beta 2^*}(\Omega)$. As β is greater than one this estimate gives the required starting point, after which the argument follows as in the previous case, getting finally $u \in C^\alpha(\overline{\Omega})$.

The case $q = 2_*$ with p subcritical can be handled in a similar way. With the argument given by Trudinger [120], we can begin the iterative procedure and also in this case we get $u \in C^\alpha(\overline{\Omega})$. □

From the results of Cherrier [29], we have that weak positive solutions of (2.6) are $C^\infty(\overline{\Omega})$. See also [4], [84] and [119] for regularity results. Also one can extend the $C^\alpha(\overline{\Omega})$-regularity of the solutions using arguments based in [114] for more general elliptic problems like

$$\begin{cases} -\operatorname{div}\big(a(x)\nabla v\big) + v = h & \text{in } \Omega, \\ a(x)\frac{\partial v}{\partial \nu} = g & \text{on } \partial\Omega. \end{cases}$$

See [14] for the details.

3. Existence results for an elliptic problem with nonlinear boundary conditions. A variational approach

To begin the study of existence of solutions to problems with nonlinear boundary conditions let us see how the usual variational techniques can be adapted.

First, we follow the ideas from J. Fernández Bonder and the author, see [71]. We study the existence of nontrivial solutions for the following problem

$$\begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = f(u) & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

As we explained in the Introduction (see also Section 2), problems of the form (3.1) appears in a natural way when one considers the Sobolev trace inequality

$$S^{1/p}\|u\|_{L^q(\partial\Omega)} \leqslant \|u\|_{W^{1,p}(\Omega)}, \quad 1 \leqslant q \leqslant p_* = \frac{p(N-1)}{N-p}.$$

In fact, the extremals (if there exists) are solutions of (3.1) for $f(u) = \lambda|u|^{q-2}u$.

For weak solutions of (3.1) we understand critical points of the associated energy functional

$$\mathcal{F}(u) = \frac{1}{p}\int_\Omega |\nabla u|^p + |u|^p \,\mathrm{d}x - \int_{\partial\Omega} F(u)\,\mathrm{d}\sigma, \tag{3.2}$$

where $F'(u) = f(u)$.

We fix $1 < p < N$ and look for conditions on the nonlinear term $f(u)$ that provide us with the existence of nontrivial solutions of (3.1). This functional $\mathcal{F}$ is well defined and C^1 in $W^{1,p}(\Omega)$ if f has a critical or subcritical growth, namely $|f(u)| \leqslant C(1+|u|^q)$ with $1 \leqslant q \leqslant p_* = p(N-1)/(N-p)$. Moreover, in the subcritical case $1 < q < p^*$, the immersion $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$ is compact while in the critical case $q = p_*$ is only continuous, see Section 2.

First, we deal with a superlinear and subcritical nonlinearity. For simplicity we will consider

$$f(u) = \lambda|u|^{q-2}u, \tag{3.3}$$

where q verifies $1 < q < p_* = p(N-1)/(N-p)$. We prove the following theorems using standard variational arguments together with the Sobolev trace immersion that provide the necessary compactness. See [79] for similar results for the p-Laplacian with Dirichlet boundary conditions.

We divide the presentation in three cases according to $p < q$, $p = q$ and $p > q$.

First, for $p < q < p_*$ we have the following theorem.

THEOREM 3.1. *Let f satisfy* (3.3) *with $p < q < p_*$, then there exists infinitely many nontrivial solutions of* (3.1) *which are unbounded in $W^{1,p}(\Omega)$.*

Now we proceed with a sketch of the proof of Theorem 3.1. We believe that the ideas involved are illustrative on how to deal with nonlinear boundary conditions from a variational point of view.

Let us begin with the following lemma that will be helpful in order to prove the Palais–Smale condition for the functional (3.2).

LEMMA 3.1. *Let $\phi \in W^{1,p}(\Omega)'$. Then there exists a unique weak solution $u \in W^{1,p}(\Omega)$ of*

$$-\Delta_p u + |u|^{p-2}u = \phi. \tag{3.4}$$

Moreover, the operator $A_p : \phi \mapsto u$ is continuous.

PROOF. Let us observe that weak solutions $u \in W^{1,p}(\Omega)$ of (3.4) are critical points of the functional

$$I(u) = \frac{1}{p}\int_\Omega |\nabla u|^p + |u|^p \,\mathrm{d}x - \langle \phi, u\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality paring in $W^{1,p}(\Omega)$. Hence, existence and uniqueness are a consequence of the fact that I is a weakly lower semicontinuous, strictly convex and bounded below functional. For the continuous dependence, let us first recall the following inequality (cf. [113])

$$\left(|x|^{p-2}x - |y|^{p-2}y, x-y\right) \geqslant \begin{cases} C_p|x-y|^p & \text{if } p \geqslant 2,\\ C_p\frac{|x-y|^2}{(|x|+|y|)^{2-p}} & \text{if } p \leqslant 2,\end{cases} \tag{3.5}$$

where $(\cdot,\cdot)$ denotes the usual scalar product in $\mathbb{R}^m$.

Now, given $\phi_1, \phi_2 \in W^{1,p}(\Omega)'$ let us consider $u_1, u_2 \in W^{1,p}(\Omega)$ the corresponding solutions of problem (3.4). Then, for $i = 1, 2$ we have

$$\int_\Omega |\nabla u_i|^{p-2}\nabla u_i(\nabla u_1 - \nabla u_2)\,\mathrm{d}x + \int_\Omega |u_i|^{p-2}u_i(u_1-u_2) - \phi_i(u_1-u_2)\,\mathrm{d}x = 0.$$

Hence, substracting and using inequality (3.5) we obtain, for $p \geqslant 2$,

$$\begin{aligned} C_p\int_\Omega |\nabla u_1 - \nabla u_2|^p + |u_1-u_2|^p\,\mathrm{d}x &\leqslant \big\langle(\phi_1-\phi_2),(u_1-u_2)\big\rangle \\ &\leqslant \|\phi_1-\phi_2\|_{W^{1,p}(\Omega)'}\|u_1-u_2\|_{W^{1,p}(\Omega)}. \end{aligned}$$

Therefore, $\|A_p(\phi_1) - A_p(\phi_2)\|_{W^{1,p}(\Omega)} \leqslant C(\|\phi_1 - \phi_2\|_{W^{1,p}(\Omega)'})^{1/(p-1)}$. Now, for the case $p \leqslant 2$, we first observe that

$$\begin{aligned}&\int_\Omega \left|\nabla(u_1-u_2)\right|^p \mathrm{d}x\\&\quad\leqslant \left(\int_\Omega \frac{|\nabla(u_1-u_2)|^2}{(|\nabla u_1|+|\nabla u_2|)^{2-p}}\,\mathrm{d}x\right)^{p/2}\left(\int_\Omega \left(|\nabla u_1|+|\nabla u_2|\right)^p \mathrm{d}x\right)^{(2-p)/2}\end{aligned}$$

and

$$\begin{aligned}&\int_\Omega |u_1-u_2|^p \,\mathrm{d}x\\&\quad\leqslant \left(\int_\Omega \frac{|u_1-u_2|^2}{(|u_1|+|u_2|)^{2-p}}\,\mathrm{d}x\right)^{p/2}\left(\int_\Omega \left(|u_1|+|u_2|\right)^p \mathrm{d}x\right)^{(2-p)/2}.\end{aligned}$$

As in the previous case, we get

$$\frac{\|u_1-u_2\|_{W^{1,p}(\Omega)}}{(\|u_1\|_{W^{1,p}(\Omega)}+\|u_2\|_{W^{1,p}(\Omega)})^{2-p}} \leqslant C\|\phi_1-\phi_2\|_{W^{1,p}(\Omega)'}. \tag{3.6}$$

Now we observe that $\|u_i\|^p_{W^{1,p}(\Omega)} \leqslant \|\phi_i\|_{W^{1,p}(\Omega)'}\|u_i\|_{W^{1,p}(\Omega)}$. Hence, (3.6) becomes

$$\begin{aligned}&\left\|A_p(\phi_1)-A_p(\phi_2)\right\|_{W^{1,p}(\Omega)}\\&\quad\leqslant C\left(\|\phi_1\|^{1/(p-1)}_{W^{1,p}(\Omega)'}+\|\phi_2\|^{1/(p-1)}_{W^{1,p}(\Omega)'}\right)^{2-p}\|\phi_1-\phi_2\|_{W^{1,p}(\Omega)'}\end{aligned}$$

and the proof is finished. □

With this lemma we can verify the Palais–Smale condition for $\mathcal{F}$.

LEMMA 3.2. *The functional $\mathcal{F}$ satisfies the Palais–Smale condition.*

PROOF. Let $(u_k)_{k\geqslant 1} \subset W^{1,p}(\Omega)$ be a Palais–Smale sequence, that is a sequence such that

$$\mathcal{F}(u_k) \to c \quad \text{and} \quad \mathcal{F}'(u_k) \to 0. \tag{3.7}$$

Let us first prove that (3.7) implies that (u_k) is bounded. From (3.7) it follows that there exists a sequence $\varepsilon_k \to 0$ such that $|\mathcal{F}'(u_k)w| \leqslant \varepsilon_k\|w\|_{W^{1,p}(\Omega)}$ for all $w \in W^{1,p}(\Omega)$. Now we have

$$\begin{aligned}c+1 &\geqslant \mathcal{F}(u_k) - \frac{1}{q}\mathcal{F}'(u_k)u_k + \frac{1}{q}\mathcal{F}'(u_k)u_k\\&= \left(\frac{1}{p}-\frac{1}{q}\right)\|u_k\|^p_{W^{1,p}(\Omega)} + \frac{1}{q}\mathcal{F}'(u_k)u_k\end{aligned}$$

$$\geqslant \left(\frac{1}{p} - \frac{1}{q}\right)\|u_k\|^p_{W^{1,p}(\Omega)} - \frac{1}{q}\|u_k\|_{W^{1,p}(\Omega)}\varepsilon_k$$
$$\geqslant \left(\frac{1}{p} - \frac{1}{q}\right)\|u_k\|^p_{W^{1,p}(\Omega)} - \frac{1}{q}\|u_k\|_{W^{1,p}(\Omega)}.$$

Hence, u_k is bounded in $W^{1,p}(\Omega)$. By compactness we can assume that $u_k \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$ and $u_k \to u$ strongly in $L^q(\partial\Omega)$ and a.e. in $\partial\Omega$. Then, as $p < q < p_*$, it follows that $|u_k|^{q-2}u_k \to |u|^{q-2}u$ in $L^{p'_*}(\partial\Omega)$ and hence in $W^{1,p}(\Omega)'$. Therefore, according to Lemma 3.1, $u_k \to A_p(|u|^{q-2}u)$ in $W^{1,p}(\Omega)$. This completes the proof. □

Now we introduce a topological tool, the *genus*, see [91].

DEFINITION 3.1. Given a Banach space X, we consider the class $\Sigma = \{A \subset X\colon\ A$ is closed, $A = -A\}$. Over this class we define the genus, $\gamma : \Sigma \to \mathbb{N} \cup \{\infty\}$, as

$$\gamma(A) = \min\bigl\{k \in \mathbb{N}\colon \text{ there exists } \varphi \in C\bigl(A, \mathbb{R}^k \setminus \{0\}\bigr),\ \varphi(x) = -\varphi(-x)\bigr\}.$$

We will use the following proposition whose proof can be found in [6].

PROPOSITION 3.1 ([6], Theorem 2.23). *Let $\mathcal{F} : X \to \mathbb{R}$ verifying*:

(1) *$\mathcal{F} \in C^1(X)$ and even.*
(2) *$\mathcal{F}$ verifies the Palais–Smale condition.*
(3) *There exists a constant $r > 0$ such that $\mathcal{F}(u) > 0$ in $0 < \|u\|_X < r$ and $\mathcal{F}(u) \geqslant c > 0$ if $\|u\|_X = r$.*
(4) *There exists a closed subspace $E_m \subset X$ of dimension m, and a compact set $A_m \subset E_m$ such that $\mathcal{F} < 0$ on A_m and 0 lies in a bounded component of $E_m - A_m$ in E_m.*

Let B be the unit ball in X, we define

$$\Gamma = \bigl\{h \in C(X, X)\colon\ h(0) = 0, h \text{ is an odd homeomorphism and } \mathcal{F}\bigl(h(B)\bigr) \geqslant 0\bigr\}$$

and

$$\mathcal{K}_m = \bigl\{K \subset X\colon\ K = -K,\ K \text{ is compact}, \\ \text{and } \gamma\bigl(K \cap h(\partial B)\bigr) \geqslant m \text{ for all } h \in \Gamma\bigr\}.$$

Then

$$c_m = \inf_{K \in \mathcal{K}_m} \max_{u \in K} \mathcal{F}(u)$$

is a critical value of $\mathcal{F}$, with $0 < c \leqslant c_m \leqslant c_{m+1} < \infty$. Moreover, if $c_m = c_{m+1} = \cdots = c_{m+r}$ then $\gamma(K_{c_m}) \geqslant r + 1$ where $K_{c_m} = \{u \in X\colon\ \mathcal{F}'(u) = 0, \mathcal{F}(u) = c_m\}$.

PROOF OF THEOREM 3.1. We need to check the hypotheses of Proposition 3.1. The fact that $\mathcal{F}$ is C^1 is a straightforward adaptation of the results in [110]. The Palais–Smale condition was already checked in Lemma 3.2.

Let us check (3). From the Sobolev trace theorem, we obtain

$$\begin{aligned}\mathcal{F}(u) &= \frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)} - \frac{\lambda}{q}\|u\|^q_{L^q(\partial\Omega)} \geqslant \frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)} - C\frac{\lambda}{q}\|u\|^q_{W^{1,p}(\Omega)} \\ &= g\big(\|u\|_{W^{1,p}(\Omega)}\big),\end{aligned}$$

where $g(t) = \frac{1}{p}t^p - C\frac{\lambda}{q}t^q$. As $q > p$, (3) follows for $r = r(C, \lambda, p, q)$ small.

Finally, to verify (4), let us consider a sequence of subspaces $E_m \subset W^{1,p}(\Omega)$ of dimension m such that $E_m \subset E_{m+1}$ and $u|_{\partial\Omega} \not\equiv 0$ for $u \neq 0$, $u \in E_m$. Hence,

$$\min_{u\in B_m} \int_{\partial\Omega} |u|^q \,\mathrm{d}\sigma > 0,$$

where $B_m = \{u \in E_m\colon \|u\|_{W^{1,p}(\Omega)} = 1\}$. Now we observe that

$$\mathcal{F}(tu) \leqslant \frac{t^p}{p}\|u\|_{W^{1,p}(\Omega)} - \frac{\lambda t^q}{q}\min_{u\in B_m}\int_{\partial\Omega}|u|^q\,\mathrm{d}\sigma < 0$$

for all $u \in B_m$ and $t \geqslant t_0$. Therefore, (4) follows by taking $A_m = t_0 B_m$. □

In order to see that the sequence of critical points of $\mathcal{F}$ that we have found is unbounded in $W^{1,p}(\Omega)$, we need the following result.

LEMMA 3.3. *Let $(c_m) \subset \mathbb{R}$ be the sequence of critical values given by Theorem* 3.1. *Then* $\lim_{m\to\infty} c_m = \infty$.

PROOF. Let $M = \{u \in W^{1,p}(\Omega) \setminus \{0\}\colon \frac{1}{\lambda p}\|u\|^p_{W^{1,p}(\Omega)} \leqslant \|u\|^q_{L^q(\partial\Omega)}\}$. By the Sobolev trace theorem, there exists a constant $r > 0$ such that

$$r < \|u\|^q_{L^q(\partial\Omega)} \quad \forall u \in M. \tag{3.8}$$

Let us define

$$b_m = \sup_{h\in\Gamma} \inf_{\{u\in\partial B\cap E^c_{m-1}\}} \mathcal{F}\big(h(u)\big).$$

It is proved in [6] that $b_m \leqslant c_m$, hence to prove our result it is enough to show that $b_m \to \infty$.

Now, $b_{m+1} \geqslant \inf_{u\in\partial B\cap E^c_m} \mathcal{F}(h(u))$ for all $h \in \Gamma$. We will construct $\tilde{h}_m \in \Gamma$ such that

$$\lim_{m\to\infty} \inf_{u\in\partial B\cap E^c_m} \mathcal{F}\big(\tilde{h}_m(u)\big) = \infty.$$

First, let us define the following sequence $d_m = \inf\{\|u\|_{W^{1,p}(\Omega)}\colon u \in M \cap E_m^c\}$ and observe that $d_m \to \infty$. In fact, if not, there exists a sequence $u_m \in M \cap E_m^c$ such that $u_m \rightharpoonup 0$ weakly in $W^{1,p}(\Omega)$ and therefore $u_m \to 0$ in $L^q(\partial\Omega)$, a contradiction with (3.8). Next, let us consider $h_m(u) = R^{-1} d_m u$ where $R > 1$ is to be fixed. From h_m we will construct $\tilde{h}_m$. Given $u \in W^{1,p}(\Omega)$ such that $u|_{\partial\Omega} \not\equiv 0$, pick $\beta = \beta(u)$ such that

$$\frac{1}{\lambda p}\|\beta u\|^p_{W^{1,p}(\Omega)} = \|\beta u\|^q_{L^q(\partial\Omega)},$$

so $\beta u \in M$. If we consider $g(t) = \mathcal{F}(tu)$ with $u|_{\partial\Omega} \not\equiv 0$, it is easy to see that g is increasing in $[0, \beta(u)]$ so g achieves its maximum on that interval for $t = \beta(u)$. Take $u_0 \in E_m^c \cap B$ such that $u_0|_{\partial\Omega} \not\equiv 0$, then for $R > 1$,

$$R^{-1} d_m \leqslant d_m \leqslant \|\beta u_0\|_{W^{1,p}(\Omega)} = \beta(u_0).$$

This inequality implies that, for every $R > 1$ and for every $u_0 \in E_m^c \cap B$ such that $u_0|_{\partial\Omega} \not\equiv 0$, it holds $\mathcal{F}(h_m(u_0)) = \mathcal{F}(R^{-1} d_m u_0) \geqslant 0$. As $h_m(0) = 0$, it follows that $h_m(E_m^c \cap B) \subset \{u \in W^{1,p}(\Omega)\colon \mathcal{F}(u) \geqslant 0\}$. Therefore, $h_m|_{E_m^c}$ satisfies the requirements needed in order to belong to Γ so it comes natural try to extend h_m to $W^{1,p}(\Omega)$ so it belongs to Γ. Given $\varepsilon > 0$, consider $Z_\varepsilon = d_m R^{-1}(E_m^c \cap B) + \varepsilon(E_m \cap B)$. Let us see that for ε small, $Z_\varepsilon \subset M^c$. If not, there exists a sequence $\varepsilon_j \to 0$ and a sequence $(u_j) \subset M$ such that $u_j \in Z_{\varepsilon_j}$. In particular, u_j is bounded in $W^{1,p}(\Omega)$ so we can assume that

$$\begin{aligned} &u_j \rightharpoonup u \quad \text{weakly in } W^{1,p}(\Omega),\\ &u_j \to u \quad \text{in } L^q(\partial\Omega). \end{aligned}$$

Moreover, as $u_j \in M$ it follows that $u|_{\partial\Omega} \not\equiv 0$. On the other hand, as $\|\cdot\|_{W^{1,p}(\Omega)}$ is weakly lower semicontinuous, we have that $u \in M$ and, as $\varepsilon_j \to 0$, $u \in d_m R^{-1}(E_m^c \cap B)$, a contradiction. So we have proved that there exists $\varepsilon_0 > 0$ such that $Z_{\varepsilon_0} \subset M^c$. This fact allows us to define

$$\tilde{h}_m(u) = \begin{cases} h_m(u) = d_m R^{-1} u & \text{if } u \in E_m^c,\\ \varepsilon_0 u & \text{if } u \in E_m. \end{cases}$$

Now, if $u \in E_m \cap B$ we have $\tilde{h}_m(u) = \varepsilon_0 u \in Z_{\varepsilon_0} \subset M^c$, then

$$\begin{aligned} \mathcal{F}\big(\tilde{h}_m(u)\big) &= \mathcal{F}(\varepsilon_0 u) = \frac{1}{p}\|\varepsilon_0 u\|^p_{W^{1,p}(\Omega)} - \frac{\lambda}{q}\|\varepsilon_0 u\|^q_{L^q(\partial\Omega)}\\ &= \frac{\lambda}{q}\left(\frac{q-1}{\lambda p}\|\varepsilon_0 u\|^p_{W^{1,p}(\Omega)} + \left(\frac{1}{\lambda p}\|\varepsilon_0 u\|^p_{W^{1,p}(\Omega)} - \|\varepsilon_0 u\|^q_{L^q(\partial\Omega)}\right)\right)\\ &\geqslant 0, \end{aligned}$$

that is, given $u \in B$ if we decompose $u = u_1 + u_2$ with $u_1 \in E_m^c$ and $u_2 \in E_m \cap B$, we obtain $\tilde{h}_m(u) = \tilde{h}_m(u_1) + \tilde{h}_m(u_2) = d_m R^{-1} u_1 + \varepsilon_0 u_2 \in Z_{\varepsilon_0} \subset M^c$ from where it follows that $\mathcal{F}(\tilde{h}_m(u)) \geqslant 0$ and hence $\tilde{h}_m \in \Gamma$.

Finally, we need to prove that $\mathcal{F}(\tilde{h}_m(u)) \to \infty$ as $m \to \infty$ for $u \in \partial B \cap E_m^c$, but this follows from the facts that $d_m \to \infty$, that $d_m \leqslant \beta(u)$ for $u \in B \cap E_m^c$ and that we can choose R large enough. If $u \in \partial B \cap E_m^c$, $\tilde{h}_m(u) = d_m R^{-1} u$ and

$$\begin{aligned}
\mathcal{F}\bigl(\tilde{h}_m(u)\bigr) &= \frac{(d_m R^{-1})^p}{p} \|u\|^p_{W^{1,p}(\Omega)} - \frac{\lambda (d_m R^{-1})^q}{q} \|u\|^q_{L^q(\partial\Omega)} \\
&= \bigl(d_m R^{-1}\bigr)^p \biggl(\frac{1}{p} - \frac{\lambda}{q} \bigl(d_m R^{-1}\bigr)^{q-p} \|u\|^q_{L^q(\partial\Omega)}\biggr) \\
&\geqslant \bigl(d_m R^{-1}\bigr)^p \biggl(\frac{1}{p} - \frac{\lambda}{q} \bigl(\beta(u) R^{-1}\bigr)^{q-p} \|u\|^q_{L^q(\partial\Omega)}\biggr) \\
&= \bigl(d_m R^{-1}\bigr)^p \biggl(\frac{1}{p} - \frac{R^{p-q}}{pq}\biggr).
\end{aligned}$$

As $q > p$ we conclude that if R is large enough, then $\mathcal{F}(\tilde{h}_m(u)) \to +\infty$. □

Now we consider the case $1 < q < p$. Using the genus and that the functional $\mathcal{F}$ verifies a Palais–Smale condition, we have,

THEOREM 3.2. *Let f satisfy* (3.3) *with $1 < q < p$, then there exists infinitely many nontrivial solutions of* (3.1) *which form a compact set in $W^{1,p}(\Omega)$.*

The proof of Theorem 3.2 follows from a series of lemmas, the proofs will be omitted or sketched, see [71] for details.

LEMMA 3.4. *For every $n \in \mathbb{N}$ there exists a constant $\varepsilon > 0$ such that*

$$\gamma\bigl(\mathcal{F}^{-\varepsilon}\bigr) \geqslant n,$$

where $\mathcal{F}^c = \{u \in W^{1,p}(\Omega)\colon \mathcal{F}(u) \leqslant c\}$.

LEMMA 3.5. *The functional $\mathcal{F}$ is bounded below and verifies the Palais–Smale condition.*

The following two propositions give us the proof of Theorem 3.2.

PROPOSITION 3.2. *Let*

$$\Sigma_k = \bigl\{A \subset W^{1,p}(\Omega) \setminus \{0\}\colon\ A \text{ is closed, } A = -A, \text{ and } \gamma(A) \geqslant k\bigr\},$$

where γ stands for the genus. Then

$$c_k = \inf_{A \in \Sigma_k} \sup_{u \in A} \mathcal{F}(u)$$

is a negative critical value of $\mathcal{F}$ *and moreover, if* $c = c_k = \cdots = c_{k+r}$*, then* $\gamma(K_c) \geqslant r + 1$*, where* $K_c = \{u \in W^{1,p}(\Omega)\colon\ \mathcal{F}(u) = c, \mathcal{F}'(u) = 0\}$.

PROOF. According to Lemma 3.4 for every $k \in \mathbb{N}$ there exists $\varepsilon > 0$ such that $\gamma(\mathcal{F}^{-\varepsilon}) \geqslant k$. As $\mathcal{F}$ is even and continuous it follows that $\mathcal{F}^{-\varepsilon} \in \Sigma_k$ therefore $c_k \leqslant -\varepsilon < 0$. Moreover, by Lemma 3.5, $\mathcal{F}$ is bounded below so $c_k > -\infty$. One can see that c_k is in fact a critical value for $\mathcal{F}$. To this end let us suppose that $c = c_k = \cdots = c_{k+r}$. As $\mathcal{F}$ is even it follows that K_c is symmetric. The Palais–Smale condition implies that K_c is compact, therefore if $\gamma(K_c) \leqslant r$ by the continuity property of the genus (see [110]) there exists a neighborhood of K_c, $N_\delta(K_c) = \{v \in W^{1,p}(\Omega)\colon\ d(v, K_c) \leqslant \delta\}$ such that $\gamma(N_\delta(K_c)) = \gamma(K_c) \leqslant r$.

By the usual deformation argument, we get $\eta(1, \mathcal{F}^{c+\varepsilon/2} - N_\delta(K_c)) \subset \mathcal{F}^{c-\varepsilon/2}$. On the other hand, by the definition of c_{k+r}, there exists $A \subset \Sigma_{k+r}$ such that $A \subset \mathcal{F}^{c+\varepsilon/2}$, hence

$$\eta\big(1, A - N_\delta(K_c)\big) \subset \mathcal{F}^{c-\varepsilon/2}. \tag{3.9}$$

Now by the monotonicity of the genus (see [110]), we have $\gamma(\overline{A - N_\delta(K_c)}) \geqslant \gamma(A) - \gamma(N_\delta(K_c)) \geqslant k$. As $\eta(1,\cdot)$ is an odd homeomorphism, it follows that (see [110]) $\gamma(\eta(1, \overline{A - N_\delta(K_c)})) \geqslant \gamma(\overline{A - N_\delta(K_c)}) \geqslant k$. But as $\eta(1, \overline{A - N_\delta(K_c)}) \in \Sigma_k$ then

$$\sup_{u \in \eta(1, \overline{A - N_\delta(K_c)})} \mathcal{F}(u) \geqslant c = c_k,$$

a contradiction with (3.9). □

Now we show that the critical points of $\mathcal{F}$ are a compact set of $W^{1,p}(\Omega)$.

PROPOSITION 3.3. *The set* $K = \{u \in W^{1,p}(\Omega)\colon\ \mathcal{F}'(u) = 0\}$ *is compact in* $W^{1,p}(\Omega)$.

PROOF. As $\mathcal{F}$ is C^1 it is immediate that K is closed. Let u_j be a sequence in K. We have that

$$\begin{aligned} 0 = F'(u_j)u_j &= \|u_j\|^p_{W^{1,p}(\Omega)} - \lambda \int_{\partial\Omega} |u_j|^q \,\mathrm{d}\sigma \\ &\geqslant \|u_j\|^p_{W^{1,p}(\Omega)} - C\lambda \|u_j\|^q_{W^{1,p}(\Omega)}. \end{aligned}$$

As $1 < q < p$, we conclude that u_j is bounded in $W^{1,p}(\Omega)$. Now we can use Palais–Smale condition to extract a convergent subsequence. □

In the case $p = q$, the equation and the boundary condition are homogeneous of the same degree, so we are dealing with a nonlinear eigenvalue problem. In the linear case, that is, for $p = 2$, this eigenvalue problem is known as the Steklov problem [115]. We have the following result whose proof can be found in [71], we do not provide the details in this case.

THEOREM 3.3. *Let f satisfy* (3.3) *with $p = q$, then there exists a sequence of eigenvalues λ_n of* (3.1) *such that $\lambda_n \to +\infty$ as $n \to +\infty$.*

The variational eigenvalues λ_k can be characterized by

$$\frac{1}{\lambda_k} = \sup_{C \in C_k} \min_{u \in C} \frac{\|u\|^p_{L^p(\partial\Omega)}}{\|u\|^p_{W^{1,p}(\Omega)}}, \tag{3.10}$$

where $C_k = \{C \subset W^{1,p}(\Omega); C$ is compact, symmetric and $\gamma(C) \geqslant k\}$ and γ is the genus (see [91] and Definition 3.1). It is shown in [73] that there exists a second eigenvalue for (6.4) and that it coincides with the second variational eigenvalue λ_2. Moreover, the following characterization of the second eigenvalue λ_2 holds

$$\lambda_2 = \inf_{u \in A} \left\{ \int_\Omega |\nabla u|^p + |u|^p \, dx \right\},$$

where $A = \{u \in W^{1,p}(\Omega); \|u\|_{L^p(\partial\Omega)} = 1$ and $|\partial\Omega^\pm| > 0\}$, with $\partial\Omega^+ = \{x \in \partial\Omega; u(x) > 0\}$ and $\partial\Omega^-$ defined analogously.

Next we consider the critical growth on f. In this case the compactness of the immersion $W^{1,p}(\Omega) \hookrightarrow L^{p_*}(\partial\Omega)$ fails, so in order to recover some sort of compactness, in the same spirit of [22], we consider a perturbation of the critical power, that is,

$$f(u) = |u|^{p_*-2}u + \lambda|u|^{r-2}u = |u|^{p(N-1)/(N-p)-2}u + \lambda|u|^{r-2}u. \tag{3.11}$$

Here we use the concentration–compactness method introduced in [96,97] and follow ideas from [80]. First, we have the following theorem.

THEOREM 3.4. *Let f satisfy* (3.11) *with $p < r < p_*$, then there exists a constant $\lambda_0 > 0$ depending on p, r, N and Ω, such that if $\lambda > \lambda_0$, problem* (3.1) *has at least a nontrivial solution in $W^{1,p}(\Omega)$.*

To prove this existence result, since we have lost the compactness in the inclusion $W^{1,p}(\Omega) \hookrightarrow L^{p_*}(\partial\Omega)$, we can no longer expect the Palais–Smale condition to hold. Anyway we can prove a *local Palais–Smale condition* that will hold for $\mathcal{F}(u)$ below a certain value of energy.

Let u_j be a bounded sequence in $W^{1,p}(\Omega)$ then there exists a subsequence that we still denote u_j, such that

$$u_j \rightharpoonup u \quad \text{weakly in } W^{1,p}(\Omega),$$
$$u_j \to u \quad \text{strongly in } L^r(\partial\Omega), 1 \leqslant r < p_*,$$
$$|\nabla u_j|^p \rightharpoonup d\mu, \quad \left|u_j|_{\partial\Omega}\right|^{p_*} \rightharpoonup d\eta,$$

weakly-* in the sense of measures. Observe that $d\eta$ is a measure supported on $\partial\Omega$.

If we consider $\phi \in C^\infty(\overline{\Omega})$, from the Sobolev trace inequality we obtain, passing to the limit,

$$\left(\int_{\partial\Omega} |\phi|^{p_*}\,\mathrm{d}\eta\right)^{1/p_*} S^{1/p} \leqslant \left(\int_\Omega |\phi|^p\,\mathrm{d}\mu + \int_\Omega |u|^p|\nabla\phi|^p\,\mathrm{d}x + \int_\Omega |\phi u|^p\,\mathrm{d}x\right)^{1/p}, \tag{3.12}$$

where S is the best constant in the Sobolev trace embedding theorem. From (3.12) we observe that, if $u = 0$ we get a reverse Hölder-type inequality (but it involves one integral over $\partial\Omega$ and one over Ω) between the two measures μ and η.

Now we state the following lemma due to Lions [96,97].

LEMMA 3.6. *Let u_j be a weakly convergent sequence in $W^{1,p}(\Omega)$ with weak limit u such that*

$$|\nabla u_j|^p \rightharpoonup \mathrm{d}\mu \quad \textit{and} \quad \left|u_j|_{\partial\Omega}\right|^{p_*} \rightharpoonup \mathrm{d}\eta,$$

weakly- in the sense of measures. Then there exists $x_1, \dots, x_l \in \partial\Omega$ such that*

(1) $\mathrm{d}\eta = |u|^{p_*} + \sum_{j=1}^{l} \eta_j \delta_{x_j}$, $\eta_j > 0$,
(2) $\mathrm{d}\mu \geqslant |\nabla u|^p + \sum_{j=1}^{l} \mu_j \delta_{x_j}$, $\mu_j > 0$,
(3) $(\eta_j)^{p/p_*} \leqslant \mu_j/S$.

Next, we use Lemma 3.6 to prove a local Palais–Smale condition.

LEMMA 3.7. *Let $u_j \subset W^{1,p}(\Omega)$ be a Palais–Smale sequence for $\mathcal{F}$, with energy level c. If $c < (\frac{1}{p} - \frac{1}{p_*})S^{p_*/(p_*-p)}$, where S is the best constant in the Sobolev trace inequality, then there exists a subsequence u_{j_k} that converges strongly in $W^{1,p}(\Omega)$.*

PROOF. From the fact that u_j is a Palais–Smale sequence it follows that u_j is bounded in $W^{1,p}(\Omega)$ (see Lemma 3.2). By Lemma 3.6 there exists a subsequence, that we still denote u_j, such that

$$\begin{aligned}
&u_j \rightharpoonup u \quad \text{weakly in } W^{1,p}(\Omega),\\
&u_j \to u \quad \text{in } L^r(\partial\Omega), 1 < r < p_*, \text{ and a.e. in } \partial\Omega,\\
&|\nabla u_j|^p \rightharpoonup \mathrm{d}\mu \geqslant |\nabla u|^p + \sum_{k=1}^{l} \mu_k \delta_{x_k},\\
&\left|u_j|_{\partial\Omega}\right|^{p_*} \rightharpoonup \mathrm{d}\eta = \left|u|_{\partial\Omega}\right|^{p_*} + \sum_{k=1}^{l} \eta_k \delta_{x_k}.
\end{aligned} \tag{3.13}$$

Let $\phi \in C^\infty(\mathbb{R}^N)$ such that $\phi \equiv 1$ in $B(x_k, \varepsilon)$, $\phi \equiv 0$ in $B(x_k, 2\varepsilon)^c$ and $|\nabla\phi| \leqslant \frac{2}{\varepsilon}$, where x_k belongs to the support of $\mathrm{d}\eta$. Consider $\{u_j\phi\}$. Obviously this sequence is bounded in $W^{1,p}(\Omega)$. As $\mathcal{F}'(u_j) \to 0$ in $W^{1,p}(\Omega)'$, we obtain that $\langle \mathcal{F}'(u_j); \phi u_j\rangle \to 0$ as $j \to \infty$. By (3.13), we obtain

$$\begin{aligned}
&\lim_{j\to\infty} \int_\Omega |\nabla u_j|^{p-2}\nabla u_j \nabla\phi u_j \,\mathrm{d}x \\
&\quad = \int_{\partial\Omega} \phi \,\mathrm{d}\eta + \lambda \int_{\partial\Omega} |u|^r \phi \,\mathrm{d}\sigma - \int_\Omega \phi \,\mathrm{d}\mu - \int_\Omega |u|^p \phi \,\mathrm{d}x.
\end{aligned}$$

Now, by Hölder inequality and weak convergence, we obtain

$$\begin{aligned}
0 &\leqslant \lim_{j\to\infty} \left| \int_\Omega |\nabla u_j|^{p-2}\nabla u_j \nabla\phi u_j \,\mathrm{d}x \right| \\
&\leqslant \lim_{j\to\infty} \left(\int_\Omega |\nabla u_j|^p \,\mathrm{d}x \right)^{(p-1)/p} \left(\int_\Omega |\nabla\phi|^p |u_j|^p \,\mathrm{d}x \right)^{1/p} \\
&\leqslant C \left(\int_{B(x_k,2\varepsilon)\cap\Omega} |\nabla\phi|^N \,\mathrm{d}x \right)^{1/N} \left(\int_{B(x_k,2\varepsilon)\cap\Omega} |u|^{pN/(N-p)} \,\mathrm{d}x \right)^{(N-p)/(pN)} \\
&\leqslant C \left(\int_{B(x_k,2\varepsilon)\cap\Omega} |u|^{pN/(N-p)} \,\mathrm{d}x \right)^{(N-p)/(pN)} \to 0 \quad \text{as } \varepsilon \to 0.
\end{aligned}$$

Then

$$\lim_{\varepsilon\to 0} \left[\int_{\partial\Omega} \phi \,\mathrm{d}\eta + \lambda \int_{\partial\Omega} |u|^r \phi \,\mathrm{d}\sigma - \int_\Omega \phi \,\mathrm{d}\mu - \int_\Omega |u|^p \phi \,\mathrm{d}x \right] = \eta_k - \mu_k = 0. \tag{3.14}$$

By Lemma 3.6 we have that $(\eta_k)^{p/p_*} S \leqslant \mu_k$, therefore by (3.14) we get $(\eta_k)^{p/p_*} S \leqslant \eta_k$. Then, either $\eta_k = 0$ or

$$\eta_k \geqslant S^{p_*/(p_*-p)}. \tag{3.15}$$

If (3.15) does indeed occur for some k_0 then, from the fact that u_j is a Palais–Smale sequence, we obtain

$$\begin{aligned}
c &= \lim_{j\to\infty} \mathcal{F}(u_j) \\
&= \lim_{j\to\infty} \mathcal{F}(u_j) - \frac{1}{p}\langle \mathcal{F}'(u_j); u_j \rangle \\
&\geqslant \left(\frac{1}{p} - \frac{1}{p_*} \right) \int_{\partial\Omega} |u|^{p_*} \,\mathrm{d}\sigma + \left(\frac{1}{p} - \frac{1}{p_*} \right) S^{p_*/(p_*-p)}
\end{aligned}$$

$$+\lambda\left(\frac{1}{p}-\frac{1}{r}\right)\int_{\partial\Omega}|u|^r\,\mathrm{d}\sigma$$
$$\geqslant\left(\frac{1}{p}-\frac{1}{p_*}\right)S^{p_*/(p_*-p)}. \tag{3.16}$$

As $c<(\frac{1}{p}-\frac{1}{p_*})S^{p_*/(p_*-p)}$, it follows that $\int_{\partial\Omega}|u_j|^{p_*}\,\mathrm{d}\sigma\to\int_{\partial\Omega}|u|^{p_*}\,\mathrm{d}\sigma$ and therefore $u_j\to u$ in $L^{p_*}(\partial\Omega)$. Now the proof finishes using the continuity of A_p. □

PROOF OF THEOREM 3.4. In view of the previous result, we seek for critical values below level c. For that purpose, we want to use the mountain pass lemma. Hence we have to check the following conditions:

(1) There exist constants $R,r>0$ such that if $\|u\|_{W^{1,p}(\Omega)}=R$, then $\mathcal{F}(u)>r$.

(2) There exists $v_0\in W^{1,p}(\Omega)$ such that $\|v_0\|_{W^{1,p}(\Omega)}>R$ and $\mathcal{F}(v_0)<r$.

Let us first check (1). By the Sobolev trace theorem, we have

$$\mathcal{F}(u)=\frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)}-\frac{1}{p_*}\int_{\partial\Omega}|u|^{p_*}\,\mathrm{d}\sigma-\frac{\lambda}{r}\int_{\partial\Omega}|u|^r\,\mathrm{d}\sigma$$
$$\geqslant\frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)}-\frac{1}{p_*}S^{p_*}\|u\|^{p_*}_{W^{1,p}(\Omega)}-\frac{\lambda}{r}C\|u\|^r_{W^{1,p}(\Omega)}.$$

Let

$$g(t)=\frac{1}{p}t^p-\frac{1}{p_*}S^{p_*}t^{p_*}-\frac{\lambda}{r}Ct^r.$$

It is easy to check that $g(R)>r$ for some $R,r>0$.

(2) is immediate as for a fixed $w\in W^{1,p}(\Omega)$ with $w|_{\partial\Omega}\not\equiv 0$, we have

$$\lim_{t\to\infty}\mathcal{F}(tw)=-\infty.$$

Now, the candidate for critical value according to the mountain pass theorem is

$$c=\inf_{\phi\in\mathcal{C}}\sup_{t\in[0,1]}\mathcal{F}\big(\phi(t)\big),$$

where $\mathcal{C}=\{\phi:[0,1]\to W^{1,p}(\Omega);\ \text{continuous and }\phi(0)=0,\phi(1)=v_0\}$. The problem is to show that $c<(\frac{1}{p}-\frac{1}{p_*})S^{p_*/(p_*-p)}$ in order to apply the local Palais–Smale condition.

We fix $w\in W^{1,p}(\Omega)$ with $\|w\|_{L^{p_*}(\partial\Omega)}=1$, and define $h(t)=\mathcal{F}(tw)$. We want to study the maximum of h. As $\lim_{t\to\infty}h(t)=-\infty$ it follows that there exists a $t_\lambda>0$ such that $\sup_{t>0}\mathcal{F}(tw)=h(t_\lambda)$. Differentiating we obtain

$$0=h'(t_\lambda)=t_\lambda^{p-1}\|w\|^p_{W^{1,p}(\Omega)}-t_\lambda^{p_*-1}-t_\lambda^{r-1}\lambda\|w\|^r_{L^r(\partial\Omega)}, \tag{3.17}$$

from where it follows that $\|w\|^p_{W^{1,p}(\Omega)} = t_\lambda^{p_*-p} + t_\lambda^{r-p}\lambda\|w\|^r_{L^r(\partial\Omega)}$. Hence $t_\lambda \leqslant \|w\|^{p/(p_*-p)}_{W^{1,p}(\Omega)}$. From (3.17), as $t_\lambda^{p_*-r} + \lambda\|w\|^r_{L^r(\partial\Omega)} \to \infty$ when $\lambda \to \infty$, we obtain that

$$\lim_{\lambda\to\infty} t_\lambda = 0. \tag{3.18}$$

On the other hand, it is easy to check that if $\lambda > \tilde{\lambda}$ it must be $\mathcal{F}(t_{\tilde{\lambda}} w) \geqslant \mathcal{F}(t_\lambda w)$, so by (3.18) we get $\lim_{\lambda\to\infty} \mathcal{F}(t_\lambda w) = 0$. But this identity means that there exists a constant $\lambda_0 > 0$ such that if $\lambda \geqslant \lambda_0$, then

$$\sup_{t\geqslant 0} \mathcal{F}(tw) < \left(\frac{1}{p} - \frac{1}{p_*}\right) S^{p_*/(p_*-p)},$$

and the proof is finished if we choose $v_0 = t_0 w$ with t_0 large in order to have $\mathcal{F}(t_0 w) < 0$. □

Now we prove a second result for the critical case.

THEOREM 3.5. *Let f satisfy* (3.11) *with* $1 < r < p$, *then there exists a constant* $\lambda_1 > 0$ *depending on* p, r, N *and* Ω *such that if* $0 < \lambda < \lambda_1$, *problem* (3.1) *has infinitely many nontrivial solutions in* $W^{1,p}(\Omega)$.

We begin, as we have done previously, proving a local Palais–Smale condition using Lemma 3.6.

LEMMA 3.8. *Let $(u_j) \subset W^{1,p}(\Omega)$ be a Palais–Smale sequence for $\mathcal{F}$, with energy level c. If $c < (\frac{1}{p} - \frac{1}{p_*})S^{p_*/(p_*-p)} - K\lambda^{p_*/(p_*-r)}$, where K depends only on p, r, N, and $|\partial\Omega|$, then there exists a subsequence (u_{j_k}) that converges strongly in $W^{1,p}(\Omega)$.*

PROOF. From the fact that u_j is a Palais–Smale sequence it follows that u_j is bounded in $W^{1,p}(\Omega)$ (see Lemmas 3.2 and 3.7).

Now the proof follows exactly as in Lemma 3.7 until we get to

$$\begin{aligned} c \geqslant{}& \left(\frac{1}{p} - \frac{1}{p_*}\right)\int_{\partial\Omega} |u|^{p_*}\,\mathrm{d}\sigma + \left(\frac{1}{p} - \frac{1}{p_*}\right)S^{p_*/(p_*-p)} \\ &+ \lambda\left(\frac{1}{p} - \frac{1}{r}\right)\int_{\partial\Omega} |u|^r\,\mathrm{d}\sigma, \end{aligned}$$

where u is the weak limit of u_j in $W^{1,p}(\Omega)$. Applying Hölder inequality we find

$$\begin{aligned} c \geqslant{}& \left(\frac{1}{p} - \frac{1}{p_*}\right)S^{p_*/(p_*-p)} + \left(\frac{1}{p} - \frac{1}{p_*}\right)\|u\|^{p_*}_{L^{p_*}(\partial\Omega)} \\ &+ \lambda\left(\frac{1}{p} - \frac{1}{r}\right)|\partial\Omega|^{1-r/p_*}\|u\|^r_{L^{p_*}(\partial\Omega)}. \end{aligned}$$

Now, let $f(x) = c_1 x^{p_*} - \lambda c_2 x^r$. This function reaches its absolute minimum at $x_0 = (\frac{\lambda c_2 r}{p_* c_1})^{1/(p_*-r)}$, that is, $f(x) \geqslant f(x_0) = -K\lambda^{p_*/(p_*-r)}$, where $K = K(p, q, N, |\partial\Omega|)$. Hence $c \geqslant (\frac{1}{p} - \frac{1}{p_*})S^{p_*/(p_*-p)} - K\lambda^{p_*/(p_*-r)}$, which contradicts our hypothesis. Therefore,

$$\lim_{j\to\infty} \int_{\partial\Omega} |u_j|^{p_*} \,\mathrm{d}\sigma = \int_{\partial\Omega} |u|^{p_*} \,\mathrm{d}\sigma,$$

and the rest of the proof is as that of Lemma 3.7. □

We observe, using the Sobolev trace theorem, that

$$\mathcal{F}(u) \geqslant \frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)} - c_1\|u\|^{p_*}_{W^{1,p}(\Omega)} - \lambda c_2\|u\|^r_{W^{1,p}(\Omega)} = j\big(\|u\|_{W^{1,p}(\Omega)}\big),$$

where $j(x) = \frac{1}{p}x^p - c_1 x^{p_*} - \lambda c_2 x^r$. As j attains a local but not a global minimum (j is not bounded below), we have to perform some sort of truncation. To this end, let x_0, x_1 be such that $m < x_0 < M < x_1$ where m is the local minimum of j and M is the local maximum and $j(x_1) > j(m)$. For these values x_0 and x_1 we can choose a smooth function $\tau(x)$ such that $\tau(x) = 1$ if $x \leqslant x_0$, $\tau(x) = 0$ if $x \geqslant x_1$ and $0 \leqslant \tau(x) \leqslant 1$. Finally, let $\varphi(u) = \tau(\|u\|_{W^{1,p}(\Omega)})$ and define the truncated functional as follows

$$\widetilde{\mathcal{F}}(u) = \frac{1}{p}\int_\Omega |\nabla u|^p + |u|^p \,\mathrm{d}x - \frac{1}{p_*}\int_{\partial\Omega} |u|^{p_*}\varphi(u)\,\mathrm{d}\sigma - \frac{\lambda}{r}\int_{\partial\Omega} |u|^r \,\mathrm{d}\sigma.$$

As above, $\widetilde{\mathcal{F}}(u) \geqslant \tilde{j}(\|u\|_{W^{1,p}(\Omega)})$, where $\tilde{j}(x) = \frac{1}{p}x^p - c_1 x^{p_*}\tau(x) - \lambda c_2 x^r$. We observe that if $x \leqslant x_0$ then $\tilde{j}(x) = j(x)$ and if $x \geqslant x_1$ then $\tilde{j}(x) = \frac{1}{p}x^p - \lambda c_2 x^r$.

Now we state a lemma that contains the main properties of $\widetilde{\mathcal{F}}$.

LEMMA 3.9. *$\widetilde{\mathcal{F}}$ is C^1, if $\widetilde{\mathcal{F}}(u) \leqslant 0$ then $\|u\|_{W^{1,p}(\Omega)} < x_0$ and $\mathcal{F}(v) = \widetilde{\mathcal{F}}(v)$ for every v close enough to u. Moreover, there exists $\lambda_1 > 0$ such that, if $0 < \lambda < \lambda_1$ then $\widetilde{\mathcal{F}}$ satisfies a local Palais–Smale condition for $c \leqslant 0$.*

PROOF. We only have to check the local Palais–Smale condition. Observe that every Palais–Smale sequence for $\widetilde{\mathcal{F}}$ with energy level $c \leqslant 0$ must be bounded, therefore by Lemma 3.8, if λ verifies $0 < (\frac{1}{p} - \frac{1}{p_*})S^{p_*/(p_*-p)} - K\lambda^{p_*/(p_*-r)}$ then there exists a convergent subsequence. □

LEMMA 3.10. *For every $n \in \mathbb{N}$ there exists $\varepsilon > 0$ such that $\gamma(\widetilde{\mathcal{F}}^{-\varepsilon}) \geqslant n$, where $\widetilde{\mathcal{F}}^{-\varepsilon} = \{u, \widetilde{\mathcal{F}}(u) \leqslant -\varepsilon\}$.*

PROOF. The proof is analogous to that of Lemma 3.4. □

PROOF OF THEOREM 3.5. The proof is analogous to that of Theorem 3.2, here we use Lemmas 3.8 and 3.10 instead of Lemmas 3.5 and 3.4, respectively, to work with the functional $\widetilde{\mathcal{F}}$ and Lemma 3.9 to conclude on $\mathcal{F}$. □

Next, we deal with supercritical growth on f. More precisely, we study a subcritical perturbation of the supercritical power, that is, we consider

$$f(u) = \lambda |u|^{q-2}u + |u|^{r-2}u, \tag{3.19}$$

with $q \geqslant p_* > r > p$. In this case, not only the compactness fails but also the functional $\mathcal{F}$ is not well defined in $W^{1,p}(\Omega)$, so we have to perform a truncation in the nonlinear term $\lambda|u|^{q-2}u$ following ideas from [27]. For this case we have the following theorem.

THEOREM 3.6. *Let f satisfy* (3.19) *with $q \geqslant p_* > r > p$, then there exists a constant λ_2 depending on p, q, r, N and Ω such that if* $0 < \lambda < \lambda_2$, *problem* (3.1) *has a nontrivial positive solution in $W^{1,p}(\Omega) \cap L^\infty(\partial\Omega)$.*

PROOF OF THEOREM 3.6. Let us consider the following truncation of $|u|^{q-2}u$

$$h(u) = \begin{cases} 0, & u < 0, \\ u^{q-1}, & 0 \leqslant u < K, \\ K^{q-r}u^{r-1}, & u \geqslant K. \end{cases}$$

Then h verifies $h(u) \leqslant K^{q-r}u^{r-1}$. So we consider the truncated problem

$$\begin{cases} \Delta_p u = u^{p-1} & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda h(u) + u^{r-1} & \text{on } \partial\Omega, \end{cases} \tag{3.20}$$

and we look a positive nontrivial solution of (3.20) that satisfies $u \leqslant K$. Such a solution will be a nontrivial positive solution of (3.1).

To this end, we consider the truncated functional

$$\mathcal{F}_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p + |u|^p \, dx - \lambda \int_{\partial\Omega} H(u)\, d\sigma - \int_{\partial\Omega} \frac{|u|^r}{r}\, d\sigma, \tag{3.21}$$

where $H(u)$ verifies $H'(u) = h(u)$.

One can check that here exists a mountain pass solution $u = u_\lambda$ for (3.20), that is, a critical point of $\mathcal{F}_\lambda$ with energy level c_λ. One can easily check that this least energy solution u is positive. Moreover, the energy level c_λ is a decreasing function of λ, so we have that $\mathcal{F}_\lambda(u) = c_\lambda \leqslant c_0$. Now, using (3.21), (3.20) and that $H(u) \leqslant \frac{1}{r}h(u)u$ we have that

$$\begin{aligned} c_0 \geqslant \mathcal{F}_\lambda(u) &= \frac{1}{p}\int_\Omega |\nabla u|^p + |u|^p \, dx - \lambda \int_{\partial\Omega} H(u)\, d\sigma - \int_{\partial\Omega} \frac{|u|^r}{r}\, d\sigma \\ &\geqslant \frac{1}{p}\int_\Omega |\nabla u|^p + |u|^p \, dx - \frac{1}{r}\left(\lambda \int_{\partial\Omega} h(u)u\, d\sigma + \int_{\partial\Omega} |u|^r\, d\sigma\right) \\ &= \left(\frac{1}{p} - \frac{1}{r}\right)\int_\Omega |\nabla u|^p + |u|^p \, dx. \end{aligned}$$

So, as $r > p$, we obtain $\|u\|_{W^{1,p}(\Omega)} \leqslant C = C(c_0, p, r)$. Now by the Sobolev trace inequality, we get

$$\|u\|_{L^s(\partial\Omega)} \leqslant S^{-1/p}\|u\|_{W^{1,p}(\Omega)} \leqslant C = C(c_0, p, r, s, \Omega). \tag{3.22}$$

Let us define

$$u_L(x) = \begin{cases} u(x), & u(x) \leqslant L, \\ L, & u(x) > L. \end{cases}$$

Multiplying (3.20) by $u_L^{p\beta}u$ we get

$$\begin{aligned} &\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\left(u_L^{p\beta}u\right) \mathrm{d}x + \int_\Omega u^p u_L^{p\beta}\,\mathrm{d}x \\ &\quad = \lambda \int_{\partial\Omega} h(u)uu_L^{p\beta}\,\mathrm{d}\sigma + \int_{\partial\Omega} u^r u_L^{p\beta}\,\mathrm{d}\sigma. \end{aligned}$$

Therefore, using that $h(u)u \leqslant K^{q-r}u^r$ and the definition of u_L, we obtain

$$\int_\Omega |\nabla u|^p u_L^{p\beta}\,\mathrm{d}x + \int_\Omega u^p u_L^{p\beta}\,\mathrm{d}x \leqslant \left(\lambda K^{q-r} + 1\right) \int_{\partial\Omega} u^r u_L^{p\beta}\,\mathrm{d}\sigma.$$

Now we set $w_L = uu_L^\beta$. Then, we obtain

$$\begin{aligned} &\|w_L\|_{W^{1,p}(\Omega)}^p \\ &\quad = \int_\Omega |\nabla w_L|^p + |w_L|^p\,\mathrm{d}x \\ &\quad \leqslant C\left(\int_\Omega |\nabla u|^p u_L^{p\beta}\,\mathrm{d}x + \int_\Omega u^p \beta^p u_L^{p(\beta-1)}|\nabla u_L|^p\,\mathrm{d}x + \int_\Omega u^p u_L^{p\beta}\,\mathrm{d}x\right) \\ &\quad \leqslant C\left(\int_\Omega |\nabla u|^p u_L^{p\beta}\,\mathrm{d}x + \int_\Omega u^p u_L^{p\beta}\,\mathrm{d}x\right) \\ &\quad \leqslant C\left(\lambda K^{q-r} + 1\right) \int_{\partial\Omega} u^r u_L^{p\beta}\,\mathrm{d}\sigma. \end{aligned}$$

Therefore, by Hölder and Sobolev trace inequalities, we get

$$\begin{aligned} \|w_L\|_{L^{p^*}(\partial\Omega)}^p &\leqslant S^{-1}\|w_L\|_{W^{1,p}(\Omega)}^p \leqslant C\left(\lambda K^{q-r} + 1\right) \int_{\partial\Omega} u^r u_L^{p\beta}\,\mathrm{d}\sigma \\ &\leqslant C\left(\lambda K^{q-r} + 1\right)\left(\int_{\partial\Omega} u^{p_*}\,\mathrm{d}\sigma\right)^{(r-p)/p_*}\left(\int_{\partial\Omega} w_L^{\alpha^*}\,\mathrm{d}\sigma\right)^{p/\alpha^*}, \end{aligned}$$

where $\alpha^* = \frac{pp_*}{p_*-r+p} < p_*$. So by (3.22),

$$\|w_L\|^p_{L^{p^*}(\partial\Omega)} \leqslant C\big(\lambda K^{q-r}+1\big)\|u\|^{r-p}_{L^{p_*}(\partial\Omega)}\|w_L\|^p_{L^{\alpha^*}(\partial\Omega)}$$
$$\leqslant C\big(\lambda K^{q-r}+1\big)\|w_L\|^p_{L^{\alpha^*}(\partial\Omega)}.$$

Now if $u^{\beta+1} \in L^{\alpha^*}(\partial\Omega)$, by the dominated convergence theorem and Fatou's lemma, we get $\|u^{\beta+1}\|^p_{L^{p_*}(\partial\Omega)} \leqslant C(\lambda K^{q-r}+1)\|u^{\beta+1}\|^p_{L^{\alpha^*}(\partial\Omega)}$, that is,

$$\|u\|_{L^{p_*(\beta+1)}(\partial\Omega)} \leqslant C\big(\lambda K^{q-r}+1\big)^{(\beta+1)/p}\|u\|_{L^{\alpha^*(\beta+1)}(\partial\Omega)}.$$

Let $\kappa = \frac{p_*}{\alpha^*}$. Iterating the last inequality we have

$$\|u\|_{L^{\kappa^j\alpha^*}(\partial\Omega)} \leqslant C\big(\lambda K^{q-r}+1\big)^{\theta}\|u\|_{L^{\alpha^*}(\partial\Omega)}.$$

Using again (3.22) we get $\|u\|_{L^\infty(\partial\Omega)} \leqslant C(\lambda K^{q-r}+1)^\theta$. Hence, if $K_0 > C$, for every $K \geqslant K_0$, there exists $\lambda(K)$ such that if $\lambda < \lambda(K)$ then $\|u\|_{L^\infty(\partial\Omega)} \leqslant K$. This result finishes the proof. □

Now, we give a nonexistence result for (3.1) in the half-space $\mathbb{R}^N_+ = \{x_1 > 0\}$ (see also Section 4 for more results in $\mathbb{R}^N_+$) that shows that existence may fail when one considers critical or subcritical growth in an unbounded domain. This nonexistence result is a consequence of a Pohozaev-type identity.

THEOREM 3.7. *Let f satisfy* (3.3) *with $q \leqslant p_*$. Let $u \in W^{1,p}(\mathbb{R}^N_+) \cap C^2(\overline{\mathbb{R}^N_+}) \cap L^q(\partial\mathbb{R}^N_+)$ be a nonnegative solution of* (3.1) *such that*

$$\big|\nabla u(x)\big||x|^{N/p} \to 0 \quad \textit{as } |x| \to +\infty.$$

Then $u \equiv 0$.

We remark that the decay hypothesis at infinity is necessary, because for $p = 2$ $u(x) = \mathrm{e}^{-x_1}$ is a solution of (3.1) for every q.

PROOF OF THEOREM 3.7. First, we multiply the equation by u and integrate by parts to obtain

$$\int_{\mathbb{R}^N_+} |\nabla u|^p + u^p\,\mathrm{d}x - \int_{\partial\mathbb{R}^N_+} u^q\,\mathrm{d}x' = 0. \tag{3.23}$$

Note that our decaying and integrability assumptions on u justify all the integrations by parts made along this proof.

Now we multiply by $x\nabla u$ and integrate by parts to obtain

$$-\int_{\mathbb{R}^N_+} |\nabla u|^{p-2}\nabla u \nabla(x\nabla u)\,\mathrm{d}x + \int_{\partial\mathbb{R}^N_+} u^{q-1} x\nabla u\,\mathrm{d}x' = \frac{1}{p}\int_{\mathbb{R}^N_+} x\nabla u^p\,\mathrm{d}x.$$

Hence, further integrations by parts gives us

$$\left(-1+\frac{N}{p}\right)\int_{\mathbb{R}^N_+} |\nabla u|^p\,\mathrm{d}x - \frac{N-1}{q}\int_{\partial\mathbb{R}^N_+} u^q\,\mathrm{d}x' = \frac{N}{p}\int_{\mathbb{R}^N_+} u^p\,\mathrm{d}x.$$

Using (3.23) we arrive at

$$\left(-1+\frac{N}{p}-\frac{N-1}{q}\right)\int_{\partial\mathbb{R}^N_+} u^q\,\mathrm{d}x' = \left(-1+\frac{2N}{p}\right)\int_{\mathbb{R}^N_+} u^p\,\mathrm{d}x > 0.$$

Therefore, if u is not identically zero, we must have $q > p_* = p(N-1)/(N-p)$ as we wanted to show. □

✓ Now we study a convex–concave problem with a nonlinear boundary condition. We follow [81] and refer to that paper for the proofs, see also [107]. In [108] a similar problem is studied.

We study the existence of nontrivial solutions for the following problem

$$\begin{cases} -\Delta u + u = |u|^{p-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda |u|^{q-2}u & \text{on } \partial\Omega. \end{cases} \tag{3.24}$$

The study of existence when the nonlinear term is placed in the equation, that is if one considers a problem of the form $-\Delta u = f(u)$ with Dirichlet boundary conditions, has received considerable attention, see, for example, [22,80] etc.

We want to remark that we are facing two nonlinear terms in problem (3.24), one in the equation $|u|^{p-2}u$, and one in the boundary condition $|u|^{q-2}u$. Our interest now is to analyze the interplay between both.

By solutions to (3.24) we understand critical points of the associated energy functional (defined on $H^1(\Omega)$)

$$\mathcal{F}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + |u|^2\,\mathrm{d}x - \frac{1}{p}\int_\Omega |u|^p\,\mathrm{d}x - \frac{\lambda}{q}\int_{\partial\Omega} |u|^q\,\mathrm{d}\sigma.$$

This functional $\mathcal{F}$ is well defined and C^1 in $H^1(\Omega)$ if p and q verify

$$1 < q \leqslant 2_* = \frac{2(N-1)}{N-2} \quad \text{and} \quad 1 < p \leqslant 2^* = \frac{2N}{N-2}.$$

We look for conditions that ensure the existence of nontrivial solutions of (3.24), focusing our attention on the existence of positive ones. We distinguish several cases.

Convex–concave subcritical case. We suppose that

$$1 < q < 2 < p. \tag{3.25}$$

We want to remark that the new feature of these problems is that we are facing a convex–concave problem where the convex nonlinearity appears in the equation and the concave one at the boundary condition. Notice that if we look at the positive solutions of these problems as the stationary states of the corresponding evolution equation since the right-hand side of the equation represents a positive reaction term, and the boundary condition means a positive flux at the boundary, then some absorption is required to reach a nontrivial equilibrium. In our equation, this is the linear term $+u$.

First, we assume that the exponents involved are subcritical, that is,

$$1 < q < 2_* = \frac{2(N-1)}{N-2} \quad \text{and} \quad 1 < p < 2^* = \frac{2N}{N-2}. \tag{3.26}$$

We have the following theorems that can be proved using standard variational arguments together with the Sobolev trace immersion that provides the necessary compactness.

THEOREM 3.8. *Let p and q satisfy* (3.25) *and* (3.26). *Then there exists $\lambda_0 > 0$ such that if $0 < \lambda < \lambda_0$ then problem* (3.24) *has infinitely many nontrivial solutions.*

Now we concentrate on positive solutions for (3.24).

THEOREM 3.9. *Let p and q satisfy* (3.25) *and* (3.26). *Then there exists $\Lambda > 0$ such that there exist at least two positive solutions of* (3.24) *for every $\lambda < \Lambda$, at least one positive solution for $\lambda = \Lambda$, and there is no positive solution of* (3.24) *for $\lambda > \Lambda$. Moreover, there exists a constant C such that $\|u\|_{L^\infty(\Omega)} \leqslant C$ for every positive solution.*

Critical case. Next we analyze the existence of solution when we have a critical exponent $p = 2^*$. Here we use again the concentration–compactness method introduced in [96,97] and follow some ideas from [80]. In these kind of problems the concentration is a priori possible on the boundary. This difficulty leads us to use technical estimates that are implicit in [98].

For $2 < q < 2_*$ (notice that in this case q means a *convex* term) we have the following theorem.

THEOREM 3.10. *Let $p = 2^*$ with $2 < q < 2_*$, then problem* (3.24) *has at least a positive nontrivial solution for every $\lambda > 0$.*

And for $1 < q < 2$, the next one.

THEOREM 3.11. *If $p = 2^*$ with $1 < q < 2$, then there exists Λ such that problem* (3.24) *has at least two positive solutions for $\lambda < \Lambda$, at least one positive solution for $\lambda = \Lambda$ and no positive solution for $\lambda > \Lambda$.*

Further results. Moreover, to obtain existence of solutions we can apply the implicit function theorem near $\lambda_0 = 0$, $u_0 = 1$ to get existence of solutions for any p and q, but imposing a restriction on the domain. We have the following result.

THEOREM 3.12. *Given $1 < p < \infty$, let Ω be a domain such that $(p-1) \notin \sigma_{\text{Neu}}(-\Delta + I)$. Then, for any $q \in (1, \infty)$, there exists $\lambda_0 > 0$ such that, for every $\lambda \in (0, \lambda_0)$, there exists a positive solution $u_\lambda \in C^\alpha$ of* (3.24) *with $u_\lambda \to 1$ in C^α as $\lambda \to 0$. Here $\sigma_{\text{Neu}}(-\Delta + I)$ stands for the spectrum of $-\Delta + I$ with homogeneous Neumann boundary conditions.*

Finally let us state a result for the remaining case, $q = 2$. In this case we have a bifurcation problem from the first eigenvalue of a related problem. Let λ_1 be the first eigenvalue of

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda u & \text{on } \partial\Omega. \end{cases}$$

Notice that λ_1 is just the best constant in the Sobolev trace embedding $H^1(\Omega) \hookrightarrow L^2(\partial\Omega)$ in the sense that $\lambda_1 \|u\|^2_{L^2(\partial\Omega)} \leqslant \|u\|^2_{H^1(\Omega)}$. Then we have the following theorem.

THEOREM 3.13. *Let $q = 2$ with $2 < p < 2^*$. Then there exists a positive solution of* (3.24) *if and only if $0 < \lambda < \lambda_1$.*

These ideas can also be applied to

$$\begin{cases} -\Delta u + u = \lambda |u|^{q-2} u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = |u|^{p-2} u & \text{on } \partial\Omega. \end{cases} \tag{3.27}$$

For this problem we assume that $1 < q < 2 < p$, that is, p stands for the convex term, and q for the concave one, and p is subcritical (notice that in this case this means $p < 2(N-1)/(N-2)$). The results presented here for (3.24) have analogous statements for (3.27).

✓ In [52] the system

$$\begin{cases} -\Delta_p u = \lambda |u|^{q-2} u + |u|^{r-2} u & \text{in } \Omega, \\ \delta |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} + a(x) |u|^{p-2} u = 0 & \text{on } \partial\Omega \end{cases}$$

is considered. Here $\delta \in \{0, 1\}$, the function a is strictly positive and the exponents verify $1 < q < p < r < pN/(N-p)$. The author find the existence of two values $\lambda_1 < \lambda_2$ such that two branches of nonnegative solutions exist for $\lambda \in (0, \lambda_2)$ with the energy of one of them changing sign at λ_1.

✓ In [31] Chipot, Fila and Quittner studied the problem

$$\begin{cases} \Delta u = a u^p & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = u^q & \text{on } \partial\Omega, \end{cases} \tag{3.28}$$

when $p, q > 1$, $a > 0$. The authors consider the existence of positive solutions, multiplicity, its symmetry, bifurcations (depending on the parameter a) and stability properties. They give a very complete picture in the one-dimensional case and extend some of the results to the general case, see also [32] for a more detailed study of the multidimensional case. Let us present the results for the one-dimensional problem. As it can be noticed the existence and symmetry of the solutions depend strongly on p and q.

THEOREM 3.14. *Denote by E the set of solutions and E_s the symmetric solutions of* (3.28). *For $N = 1$ ($\Omega = (-l, l)$) we have five cases*:

(1) $p > 2q - 1$. *Then* $\operatorname{card}(E) = 1$, $E = E_s$, *for any* $a > 0$.

(2) $p = 2q - 1$. *Then* $E = \emptyset$ *for* $0 < a \leqslant q$ *and* $\operatorname{card}(E) = 1$, $E = E_s$, *for* $a > q$.

(3) $q < p < 2q - 1$. *Then there exist* $0 < a_0 < a_1$ *such that*: $E = \emptyset$ *for* $a < a_0$; $\operatorname{card}(E) = 1$, $E = E_s$, *for* $a = a_0$; $\operatorname{card}(E) = 2$, $E = E_s$, *for* $a_0 < a \leqslant a_1$; $\operatorname{card}(E) \geqslant 4$, *even*, $\operatorname{card}(E_s) = 2$ *for* $a > a_1$.

(4) $p = q$. *Then there exists* $0 < a_1$ *such that*: $E = \emptyset$ *for* $0 < a \leqslant 1/l$; $\operatorname{card}(E) = 1$, $E = E_s$, *for* $1/l < a \leqslant a_1$; $\operatorname{card}(E) = 3$, $\operatorname{card}(E_s) = 1$ *for* $a > a_1$.

(5) $p < q$. *Then there exists* $0 < a_1$ *such that*: $\operatorname{card}(E) = 1$, $E = E_s$ *for* $0 < a \leqslant a_1$; $\operatorname{card}(E) = 3$, $\operatorname{card}(E_s) = 1$ *for* $a > a_1$.

✓ In [121] and [122] Umezu studied the problem

$$\begin{cases} -\Delta u + c(x)u = \lambda f(u) & \text{in } \Omega, \\ a(x)\frac{\partial u}{\partial \nu} + b(x)g(u) = 0 & \text{on } \partial\Omega. \end{cases}$$

It is discussed there the existence and uniqueness of a branch of positive solutions. The main tools used here are not variational, the results are obtained via the implicit function theorem and ideas relying on super- and subsolutions.

✓ In [36,37,103] the problem,

$$\begin{cases} -\operatorname{div}\big(a(x)|\nabla u|^{p-2}\nabla u\big) + h(x)u^{r-1} = f(\lambda, x, u) & \text{in } \Omega, \\ a(x)|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} + b(x)u^{p-1} = \theta g(x, u) & \text{on } \partial\Omega, \end{cases}$$

is studied using a bifurcation approach with parameters λ and θ. The authors consider the subcritical case, $p < r < pN/(N - p)$. They also consider the case of an unbounded domain using appropriate weighted Sobolev spaces.

4. Problems in $\mathbb{R}^N_+$

In this section we consider solutions of elliptic problems with nonlinear boundary conditions in $\mathbb{R}^N_+$.

We have already find a nonexistence result in $\mathbb{R}^N_+$ for subcritical exponents, Theorem 3.7.

Now, let us look for positive solutions of the problem

$$\begin{cases} -\Delta u = au^p & \text{in } \mathbb{R}^N_+, \\ -\frac{\partial u}{x_N} = u^q & \text{on } \{x_N = 0\}. \end{cases} \tag{4.1}$$

Here $a \geqslant 0$ and $p, q > 1$. It is easy to see that for $N = 1$ solutions do not exist, also for $N = 2$ there is no positive solution of (4.1).

When $N \geqslant 3$ solutions exist for the critical exponents $p = (N+2)/(N-2)$ and $q = N/(N-2)$. It was proved in [33] and [93] using the method of moving spheres (instead of moving planes) that in this case any positive solution has the form

$$u(x) = \frac{\alpha}{(|x-\tilde{x}|^2+\beta)^{(N-2)/2}},$$

where $\alpha > 0$, $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_N) \in \mathbb{R}^N$ with $\tilde{x}_N = -\alpha^{2/(N-2)}/(N-2)$ and $\beta = a\alpha^{4/(N-2)}/(N(N-2))$.

When $a = 0$ positive solutions do not exist if $q < N/(N-2)$, see [87]. Here the moving planes technique is applied with a plane parallel to the x_N direction to prove that the solution has to depend only in the x_N variable and therefore is the zero solution. See also [117] for symmetry properties of problems with nonlinear boundary conditions in $\mathbb{R}^N_+$ obtained via the moving planes device.

In [30] it is proved that positive solutions do exist when $p \geqslant (N+2)/(N-2)$ and $q \geqslant N/(N-2)$. Also it is proved there that solutions do not exist in any of the following cases:

(i) $p \leqslant (N+2)/(N-2)$, $q \leqslant N/(N-2)$ with at least one strict inequality,
(ii) $p < N/(N-2)$,
(iii) $q < N/(N-1)$.

Moreover, explicit solutions exist when $p = q > N/(N-2)$ and $a > 0$. They have the form

$$u(x) = \frac{\alpha}{(|x-\tilde{x}|^2)^{2/(p-1)}},$$

where $\tilde{x}_N = -\frac{1}{a}(N - 2p/(p-1))$ and $\alpha = (-2\tilde{x}_N/(p-1))^{1/(p-1)}$.

The proof of nonexistence in case (i) follows by an application of the moving planes method, in cases (ii) and (iii) it is a consequence of some blow-up results for parabolic problems.

The idea of existence is as follows, consider the auxiliary nonlinear eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u + a|u|^{p-1}u & \text{in } B^+(0,R) = \big\{|x| < R\big\} \cap \mathbb{R}^N_+, \\ -\frac{\partial u}{x_N} = |u|^{q-1}u & \text{on } \big\{|x| < R, x_N = 0\big\}, \\ u = 0 & \text{on } \big\{|x| = R, x_N > 0\big\}. \end{cases}$$

The proof comes from the use of Rabinowitz's theorem that shows that there exists a branch of solutions (u^R, λ^R) emanating from $(0, \lambda_1)$, $0 < \lambda < \lambda_1$, λ_1 the first eigenvalue of the corresponding linear problem. The fact that λ^R is positive is proved by a Pohozaev identity and used to show that $\lambda^R \to 0$ as $R \to \infty$. It is the Pohozaev identity where $p \geqslant (N+2)/(N-2)$ and $q \geqslant N/(N-2)$ are needed. Then, Schauder estimates show that u^R converges along a subsequence to a solution of (4.1) as $R \to \infty$. The proof of existence finishes by showing some symmetry and monotonicity results that allow us to conclude that the limit u is nontrivial.

✓ Escobar [53] and Beckner [18] proved the following sharp trace inequality in the half-space for the critical exponent,

$$\left(\int_{\partial\mathbb{R}^N_+} |u|^{2(N-1)/(N-2)}\,\mathrm{d}\sigma\right)^{(N-2)/(N-1)} \leqslant \frac{1}{\sqrt{\pi}(N-2)}\left[\frac{\Gamma(N-1)}{\Gamma((N-2)/2)}\right]^{1/(N-1)}\left(\int_{\mathbb{R}^N_+} |\nabla u|^2\,\mathrm{d}x\right).$$

This result has applications in geometry to the Yamabe problem on manifolds with boundary, see Section 5.

✓ In [106] Park proved the following logarithmic Sobolev trace inequality

$$\int_{\partial\mathbb{R}^N_+} |u|^2 \ln|u|\,\mathrm{d}\sigma \leqslant \frac{N}{2}\ln\left(A_N \int_{\mathbb{R}^N_+} |\nabla u|^2\,\mathrm{d}x\right).$$

The result is obtained as a limit case for the best Sobolev trace inequality proved by Escobar in [53], see Section 5. Also bounds for the best constant in this logarithmic inequality are proved in [106].

✓ See also Section 12 and [42] for existence and symmetry results for a problem with a non-Lipschitz nonlinearity in the half-space.

✓ Finally, in [24] Cabre and Sola-Morales studied existence and uniqueness of layer solutions for the problem,

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^N_+, \\ -\frac{\partial u}{x_N} = f(u) & \text{on } \{x_N = 0\}. \end{cases} \tag{4.2}$$

For layer solutions we mean solutions of the form $u(x_1, \dots, x_N)$ that are monotone increasing from -1 to 1 in the x_i variable with $i \neq N$.

5. Yamabe problem on manifolds with boundary

In this section we describe some results obtained by Escobar [56] concerning the problem of prescribing the mean curvature of the boundary for a Riemannian manifold. See also [3]. This problem leads naturally to an elliptic problem with nonlinear boundary conditions. We will only present the geometrical motivation and some results, refereing to [3,5, 53–56,60,85,86] for further references.

Let (M^N, g) be a compact Riemannian manifold with boundary, $N \geqslant 3$. Let $\tilde{g} = u^{4/(N-2)}g$ be a conformally related metric to g. Now we address the following problem: Given a function f on the boundary of M, can you find a conformally related metric $\tilde{g}$ such that the scalar curvature of $\tilde{g}$ is zero on M and the mean curvature of ∂M with respect to the metric $\tilde{g}$ is the function f?

This problem is equivalent to finding a smooth positive function u defined on M such that

$$\begin{cases} \Delta u - \frac{N-2}{4(N-1)} Ru = 0 & \text{on } M, \\ \frac{\partial u}{\partial \nu} + \frac{N-2}{2} hu = \frac{N-2}{2} f u^{N/(N-2)} & \text{on } \partial M, \end{cases} \tag{5.1}$$

where R is the scalar curvature of M, h is the mean curvature of ∂M, and ν is the outer normal vector with respect to the metric g. If we choose $\tilde{g} = u^{4/(N-2)}g$, the first equation in (5.1) says that the metric $\tilde{g}$ has zero scalar curvature and the second equation says that the boundary has mean curvature f with respect to the metric $\tilde{g}$.

For a function $v \in C^1(\overline{M})$ we define the energy associated to our problem as

$$E(v) = \int_M \left(|\nabla v|^2 + \frac{N-2}{4(N-1)} R v^2 \right) + \frac{N-2}{2} \int_{\partial M} h v^2,$$

and the Sobolev quotient $Q(M, \partial M)$ by

$$Q(M, \partial M) = \inf\bigl\{ Q(v)\colon\ v \in C^1\bigl(\overline{M}\bigr), v \not\equiv 0 \text{ on } \partial M \bigr\},$$

where

$$Q(v) = \frac{E(v)}{(\int_{\partial M} |v|^{2(N-1)/(N-2)})^{(N-2)/(N-1)}}.$$

The constraint set $C(M)$ is defined as

$$C(M) = \left\{ v \in C^1\bigl(\overline{M}\bigr)\colon \int_{\partial M} f |v|^{2(N-1)/(N-2)} = 1 \right\}.$$

We have the following proposition.

PROPOSITION 5.1. *There is a function u that realizes the minimum energy in $C(M)$ if*

$$(\max f)^{(N-2)/(N-1)} \inf_{v \in C} E(v) < \frac{N-2}{2} \bigl(\mathrm{Vol}\bigl(S^{N-1}\bigr)\bigr)^{1/(N-1)}.$$

The constant that appear in the right-hand side is the Sobolev constant of the ball in the Euclidean space.

Now we assume the generic condition that there exists a point $x_0 \in \partial M$ where the eigenvalues of the second fundamental form at x_0 are not the same. That is to say that x_0 is a nonumbilic point.

THEOREM 5.1. *Let M be an N-dimensional compact Riemannian manifold with boundary, $N > 5$. If M has a nonumbilic point on ∂M then*

$$Q(M, \partial M) < Q\big(B(0,1), \partial B(0,1)\big),$$

where $B(0,1)$ is the ball in the Euclidean space.

Let f verifies that is somewhere positive and achieves a global maximum at a nonumbilic point x_0 where $\Delta f(x_0) \leqslant C(N)\|\pi - hg\|^2(x_0)$ (π stands for the second fundamental form), then problem (5.1) *has a solution.*

For manifolds with positive Sobolev quotient and ∂M umbilic we have the following theorem.

THEOREM 5.2. *Let M be an N-dimensional compact Riemannian manifold with boundary, $N \geqslant 3$ and $Q(M, \partial M) > 0$. Assume that M is locally conformally flat and ∂M umbilic. If M is not conformally diffeomorphic to the ball and f verifies that it is somewhere positive and achieves a global maximum at $x_0 \in \partial M$ with $\nabla^k f(x_0) = 0$ for $k = 1, \dots, N-2$, then problem* (5.1) *has a positive smooth solution.*

Finally, we state a nonexistence result.

THEOREM 5.3. *Let B be the N-dimensional Euclidean ball. Let X be the conformal vector field on ∂B. For a function f such that*

$$\int_{\partial B} \nabla f \dot{X} \neq 0,$$

problem (5.1) *has no positive solution.*

See also [85] and [86] for other existence results assuming that $(B(0,1), g)$ is of positive type.

✓ In [5] it is shown that when the metric g is close to the standard metric in the ball then there exists a positive solution of (5.1).

✓ In [60] and [61] some a priori estimates on the solutions are given.

✓ See [124] for existence results for general elliptic operators, may be in nondivergence form.

6. Dependence of the best Sobolev trace constant on the domain

In this section we deal with the dependence of the best Sobolev trace constant on the domain.

First, we consider the family of domains given by contraction or expanding a fixed domain, that is, for $\mu > 0$ we consider the family of domains

$$\Omega_\mu = \mu\Omega = \{\mu x; x \in \Omega\}.$$

The main purpose of this section is to describe the asymptotic behavior of the best Sobolev trace constants $S(\Omega_\mu, p, q)$ as $\mu \to 0+$ and $\mu \to +\infty$.

As we have mentioned in the preliminaries, for any $1 < p < N$ and $1 < q \leqslant p_* = p(N-1)/(N-p)$, we have that $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$ and hence the following inequality holds

$$S\|u\|^p_{L^q(\partial\Omega)} \leqslant \|u\|^p_{W^{1,p}(\Omega)}$$

for all $u \in W^{1,p}(\Omega)$. This is known as the Sobolev trace embedding theorem. The best constant for this embedding is the largest S such that the above inequality holds, that is,

$$S(\Omega, p, q) = \inf_{u \in W^{1,p}(\Omega)\setminus W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p + |u|^p \,\mathrm{d}x}{(\int_{\partial\Omega} |u|^q \,\mathrm{d}\sigma)^{p/q}}.$$

Moreover, if $1 < q < p_*$ the embedding is compact and as a consequence we have the existence of extremals, that is, functions where the infimum is attained, see [71]. These extremals are weak solutions of the following problem

$$\begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda |u|^{q-2}u & \text{on } \partial\Omega. \end{cases}$$

Standard regularity theory, like the one sketched in Section 2, see also [119], and the strong maximum principle [123], show that any extremal u belongs to the class $C^{1,\alpha}_{\mathrm{loc}}(\Omega) \cap C^\alpha(\overline{\Omega})$ and that is strictly one signed in Ω, so we can assume that $u > 0$ in Ω.

In [46] Flores and del Pino, performed a detailed analysis of the behavior of extremals and best Sobolev constants in expanding domains for $p = 2$ and $q > 2$. Their first result says that the best Sobolev trace constant in an expanding domain approaches the one in the half-space and gives an estimate of the error.

THEOREM 6.1. *There exists a constant $\gamma = \gamma(q, N) > 0$ such that the following expansion holds*

$$S(\mu\Omega, 2, q) = S(\mathbb{R}^N, 2, q) - \frac{1}{\mu}\gamma \max_{x \in \partial\Omega} H(x) + \mathrm{o}\left(\frac{1}{\mu}\right)$$

as $\mu \to +\infty$. Here $H(x)$ denotes the mean curvature of the boundary at x.

The second result proved in that paper says that the extremals constitute a single bump at the boundary, whose shape is asymptotically that of an extremal for the half-space trace theorem. This bump is centered around a point of maximum mean curvature.

THEOREM 6.2. *Let y_μ be a maximum point of u_μ. Then $x_\mu = y_\mu/\mu \in \partial\Omega$ verifies*

$$H(x_\mu) \to \max_{x \in \partial\Omega} H(x)$$

as $\mu \to +\infty$. *Also, there exist constants* α, $\beta > 0$ *such that* $u_\mu(y) \leqslant \alpha \mathrm{e}^{-\beta|y-y_\mu|}$ *for all* $y \in \mu\Omega$. *Besides, given a sequence* $\mu_n \to \infty$ *there exists a subsequence, an extremal* w *of* $S(\mathbb{R}^N, 2, q)$ *and a rotation* Q *such that*

$$\sup_{y\in\mu_{n_k}\Omega} \left|u_{n_k}(y) - w\big(Q(y - y_{\mu_{n_k}})\big)\right| \to 0$$

as $k \to \infty$.

The main ingredient of the proof is to obtain some information on the limit problem

$$\begin{cases} \Delta w = w & \text{in } \mathbb{R}^N_+, \\ \frac{\partial w}{\partial \nu} = w^p & \text{on } \partial\mathbb{R}^N_+. \end{cases}$$

For this problem we have the existence of a least energy solution that verifies the decay estimate $|w(x)| + |\nabla w(x)| \leqslant C_1 \mathrm{e}^{-C_2|x|}$. See also Section 4 for more results for problems with nonlinear boundary conditions in $\mathbb{R}^N_+$.

Let us go back to our general problem $W^{1,p}(\Omega_\mu) \mapsto L^q(\partial\Omega_\mu)$. Now we deal with the case of contractions, $\mu \to 0+$. As we will see the behavior of the Sobolev constant and extremals is very different when the domain is contracted than when it is expanded.

Let us call u_μ an extremal corresponding to Ω_μ. Making a change of variables, we go back to the original domain Ω. If we define $v_\mu(x) = u_\mu(\mu x)$, we have that $v_\mu \in W^{1,p}(\Omega)$ and

$$S(\Omega_\mu, p, q) = \mu^{(Nq-Np+p)/q} \frac{\int_\Omega \mu^{-p}|\nabla v_\mu|^p + |v_\mu|^p \,\mathrm{d}x}{(\int_{\partial\Omega} |v_\mu|^q \,\mathrm{d}\sigma)^{p/q}}. \tag{6.1}$$

We can assume, and we do so, that the functions u_μ are normalized so that

$$\int_{\partial\Omega} |v_\mu|^q \,\mathrm{d}\sigma = 1.$$

We remark that the quantity (6.1) is not homogeneous under dilations or contractions of the domain. This is a remarkable difference with the study of the Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$.

The first result of Fernández Bonder and the author in [74] is the following theorem.

THEOREM 6.3. *Let* $1 < q < p_*$. *Then*

$$\lim_{\mu\to 0+} \frac{S(\Omega_\mu, p, q)}{\mu^{(Nq-Np+p)/q}} = \frac{|\Omega|}{|\partial\Omega|^{p/q}}, \tag{6.2}$$

and if we scale the extremals u_μ *to the original domain* Ω *as* $v_\mu(x) = u_\mu(\mu x)$, $x \in \Omega$, *with* $\|v_\mu\|_{L^q(\partial\Omega)} = 1$, *then* v_μ *is nearly constant in the sense that* $v_\mu \to |\partial\Omega|^{-1/q}$ *in* $W^{1,p}(\Omega)$.

Observe that the behavior of the Sobolev trace constant, strongly depends on p and q. If we call $\beta_{pq} = (Nq - Np + p)/q$ then we have that, as $\mu \to 0+$,

$$S \to \begin{cases} 0 & \text{if } \beta_{pq} > 0, \\ +\infty & \text{if } \beta_{pq} < 0, \\ C \neq 0 & \text{if } \beta_{pq} = 0. \end{cases}$$

Let us remark that the influence of the geometry of the domain appears in (6.2).

An idea of the proof of Theorem 6.3 runs as follows. Let us begin by the simple observation that, taking $u \equiv 1$ as a test function in (6.1), it follows that

$$S(\Omega_\mu, p, q) \leqslant \mu^{(Nq-Np+p)/q} \frac{|\Omega|}{|\partial\Omega|^{p/q}}. \tag{6.3}$$

This shows that the ratio $S(\Omega_\mu, p, q)/\mu^{(Nq-Np+p)/q}$ is bounded. So a natural question will be to determine if it converges to some value. This is answered in Theorem 6.3 that we prove next.

PROOF OF THEOREM 6.3. Let $u_\mu \in W^{1,p}(\Omega_\mu)$ be a extremal for $S(\Omega_\mu, p, q)$ and define $v_\mu(x) = u_\mu(\mu x)$, we have that $v_\mu \in W^{1,p}(\Omega)$. We can assume that the functions u_μ are chosen so that

$$\int_{\partial\Omega} |v_\mu|^q \,\mathrm{d}\sigma = 1.$$

Equations (6.1) and (6.3) give, for $\mu < 1$,

$$\|v_\mu\|^p_{W^{1,p}(\Omega)} \leqslant \int_\Omega \mu^{-p}|\nabla v_\mu|^p + |v_\mu|^p \,\mathrm{d}x \leqslant \frac{|\Omega|}{|\partial\Omega|^{p/q}},$$

so there exists a function $v \in W^{1,p}(\Omega)$ and a sequence $\mu_j \to 0+$ such that

$$\begin{aligned} v_{\mu_j} &\rightharpoonup v \quad \text{weakly in } W^{1,p}(\Omega), \\ v_{\mu_j} &\to v \quad \text{in } L^p(\Omega), \\ v_{\mu_j} &\to v \quad \text{in } L^q(\partial\Omega). \end{aligned}$$

Moreover,

$$\int_\Omega |\nabla v_\mu|^p \,\mathrm{d}x \leqslant \frac{|\Omega|}{|\partial\Omega|^{p/q}} \mu^p.$$

Hence $\nabla v_\mu \to 0$ in $L^p(\Omega)$. It follows that the limit v is a constant and must verify $\int_{\partial\Omega} |v|^q = 1$, hence $v = \text{const} = |\partial\Omega|^{-1/q}$ and so the full sequence v_μ converges weakly in $W^{1,p}(\Omega)$ to v. From our previous bounds we have

$$v_\mu \to \frac{1}{|\partial\Omega|^{1/q}} \quad \text{in } L^p(\Omega) \quad \text{and} \quad \int_\Omega |\nabla v_\mu|^p \, dx \to 0.$$

Therefore, we have strong convergence, $v_\mu \to |\partial\Omega|^{-1/q}$ in $W^{1,p}(\Omega)$. The proof is finished. □

In the special case $p = q$, the problem,

$$\begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda |u|^{q-2}u & \text{on } \partial\Omega, \end{cases} \tag{6.4}$$

becomes a nonlinear eigenvalue problem. For $p = 2$, this eigenvalue problem is known as the Steklov problem [115]. In [71] it is proved, applying the Ljusternik–Schnirelman critical point theory on C^1 manifolds, that there exists a sequence of variational eigenvalues $\lambda_k \nearrow +\infty$, see Section 3. It is easy to see that the first eigenvalue $\lambda_1(\Omega)$ verifies $\lambda_1(\Omega) = S(\Omega, p, p)$. So Theorem 6.3 shows a difference in the behavior of the first eigenvalue of (6.4) with $p = q$ with respect to the domain with the behavior of the first eigenvalue of the following Dirichlet problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where it is a well-known fact that λ_1 increases as the domain decreases, see [79]. Recall from Section 3 that variational eigenvalues λ_k of (6.4) are characterized by

$$\frac{1}{\lambda_k} = \sup_{C \in C_k} \min_{u \in C} \frac{\|u\|^p_{L^p(\partial\Omega)}}{\|u\|^p_{W^{1,p}(\Omega)}},$$

where $C_k = \{C \subset W^{1,p}(\Omega);\, C$ is compact, symmetric and $\gamma(C) \geqslant k\}$ and γ is the genus. It is shown in [72] that there exists a second eigenvalue for (6.4) and that it coincides with the second variational eigenvalue λ_2. Moreover, the following characterization of the second eigenvalue λ_2 holds

$$\lambda_2 = \inf_{u \in A} \left\{ \int_\Omega |\nabla u|^p + |u|^p \, dx \right\},$$

where $A = \{u \in W^{1,p}(\Omega);\, \|u\|_{L^p(\partial\Omega)} = 1$ and $|\partial\Omega^\pm| \geqslant c\}$, with $\partial\Omega^+ = \{x \in \partial\Omega;\, u(x) > 0\}$ and $\partial\Omega^-$ is defined analogously. Concerning the eigenvalue problem, we have the following result.

THEOREM 6.4. *There exists a constant $\tilde{\lambda}_2$ such that*

$$\lim_{\mu \to 0+} \mu^{p-1} \lambda_2(\Omega_\mu) = \tilde{\lambda}_2.$$

This constant $\tilde{\lambda}_2$ is the first nonzero eigenvalue of the following problem

$$\begin{cases} \Delta_p u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \tilde{\lambda} |u|^{p-2} u & \text{on } \partial\Omega. \end{cases} \tag{6.5}$$

Moreover, if we take an eigenfunction $u_{2,\mu}$ associated to $\lambda_2(\Omega_\mu)$ and scale it to Ω as in Theorem 6.3, *we obtain that $v_{2,\mu} \to \tilde{v}_2$ in $W^{1,p}(\Omega)$, where $\tilde{v}_2$ is an eigenfunction of* (6.5) *associated to $\tilde{\lambda}_2$. Also, every eigenvalue $\lambda_2(\Omega_\mu) \leqslant \lambda(\Omega_\mu) \leqslant \lambda_k(\Omega_\mu)$ of* (6.4) (*variational or not*) *behaves as $\lambda(\Omega_\mu) \sim \mu^{1-p}$ as $\mu \to 0+$. Finally, if $\mu_j \to 0$ and $\lambda_j = \lambda(\Omega_{\mu_j})$ is a sequence of eigenvalues such that there exists λ with*

$$\lim_{j \to \infty} \mu_j^{p-1} \lambda_j = \lambda.$$

Let (v_j) be the sequence of associated eigenfunctions rescaled as in Theorem 6.3, *then (v_j) has a convergent subsequence (v_{j_k}) and a limit v, that is, an eigenfunction of* (6.5) *with eigenvalue λ.*

Observe that the first eigenvalue of (6.5) is zero with associated eigenfunction a constant. Hence, Theorem 6.3 says that the first eigenvalue and the first eigenfunction of our problem (6.4) converges to the ones of (6.5). Theorem 6.4 says that $\lambda(\Omega_\mu) \to +\infty$ as $\mu \to 0+$ for the remaining eigenvalues and that problem (6.5) is a limit problem for (6.4) when $\mu \to 0+$.

Since we are dealing with the eigenvalue problem, let us prove the isolation and simplicity for the first eigenvalue of the p-Laplacian with a nonlinear boundary condition. This result was proved by Martinez and the author and is contained in [100]. So, let us study the first eigenvalue for the following problem

$$\begin{cases} \Delta_p u = |u|^{p-2} u & \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \lambda |u|^{p-2} u & \text{on } \partial\Omega. \end{cases} \tag{6.6}$$

We have the following result, similar to the one known for Dirichlet boundary conditions [7].

THEOREM 6.5. *λ_1 is isolated and simple.*

We remark that this theorem says that the extremals of the Sobolev trace inequality are unique up to multiplication by a real number. In the special case of a ball, $\Omega = B(0, R)$, our result implies that the first eigenfunction is radial. In fact, if $u_1(x)$ is an eigenfunction associated to λ_1 and $\theta(x)$ is any rotation then $u_1(\theta(x))$ is also an eigenfunction, by our result we have that $u_1(x) = u_1(\theta(x))$. We conclude that u_1 must be radial. Also from our results it follows that any other eigenvalue has nonradial eigenfunctions as they have to change sign on the boundary (see Lemma 6.4).

Now we prove Theorem 6.5. To clarify the exposition, we will divide the proof in several lemmas.

LEMMA 6.1. *Let u_1 be an eigenfunction with eigenvalue λ_1, then u_1 does not change sign on Ω. Moreover, if u_1 is $C^{1,\alpha}(\Omega)$, it does not vanish on $\overline{\Omega}$.*

PROOF. Recall that

$$\lambda_1 = \inf_{u \in W^{1,p}(\Omega)} \left\{ \int_\Omega |\nabla u|^p + |u|^p \,\mathrm{d}x,\ \int_{\partial\Omega} |u|^p \,\mathrm{d}\sigma = 1 \right\}. \tag{6.7}$$

Hence if u_1 is a minimizer, we have that $|u_1|$ is also a minimizer of (6.7). By the maximum principle (see [123]), we have that $|u_1| > 0$ in Ω. Assume that u_1 is regular and that there exists $x_0 \in \partial\Omega$ such that $u_1(x_0) = 0$. By the Hopf lemma (see [123]) we have that the normal derivative has strict sign, $\frac{\partial |u_1|}{\partial \nu}(x_0) < 0$, but the boundary condition imposes $\frac{\partial |u_1|}{\partial \nu}(x_0) = 0$, a contradiction, that proves that $|u_1| > 0$ in $\overline{\Omega}$. The result follows. □

Now we state an auxiliary lemma, whose proof can be found in [95].

LEMMA 6.2. (a) *Let $p \geqslant 2$. Then for all $\xi_1, \xi_2 \in \mathbb{R}^N$,*

$$|\xi_2|^p \geqslant |\xi_1|^p + p|\xi_1|^{p-2}\langle \xi_1, \xi_2 - \xi_1\rangle + C(p)|\xi_1 - \xi_2|^p.$$

(b) *Let $p < 2$. Then for all $\xi_1, \xi_2 \in \mathbb{R}^N$,*

$$|\xi_2|^p \geqslant |\xi_1|^p + p|\xi_1|^{p-2}\langle \xi_1, \xi_2 - \xi_1\rangle + C(p)\frac{|\xi_1 - \xi_2|^p}{(|\xi_2| + |\xi_1|)^{2-p}},$$

where $C(p)$ is a constant depending only on p.

LEMMA 6.3. *λ_1 is simple. Let u, v be two eigenfunctions associated with λ_1, then there exists c such that $u = cv$.*

PROOF. By Lemma 6.1 we can assume that u, v are positive in Ω. We perform the following calculations assuming that u, v are strictly positive in $\overline{\Omega}$, to obtain our result we can consider $u + \varepsilon$ and $v + \varepsilon$ and let $\varepsilon \to 0$ at the end as in [95]. Therefore we can take $\eta_1 = (u^p - v^p)/u^{p-1}$ and $\eta_2 = (v^p - u^p)/v^{p-1}$ as test functions in the weak form of (6.6) satisfied by u and v, respectively. We have

$$\begin{aligned}
&\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\left(\frac{u^p - v^p}{u^{p-1}}\right) \mathrm{d}x \\
&\quad = \lambda \int_{\partial\Omega} |u|^{p-2} u \left(\frac{u^p - v^p}{u^{p-1}}\right) \mathrm{d}\sigma - \int_\Omega |u|^{p-2} u \left(\frac{u^p - v^p}{u^{p-1}}\right) \mathrm{d}x
\end{aligned}$$

and

$$\int_{\Omega} |\nabla v|^{p-2} \nabla v \nabla \left(\frac{v^p - u^p}{v^{p-1}} \right) \mathrm{d}x$$
$$= \lambda \int_{\partial\Omega} |v|^{p-2} v \left(\frac{v^p - u^p}{v^{p-1}} \right) \mathrm{d}\sigma - \int_{\Omega} |v|^{p-2} v \left(\frac{v^p - u^p}{v^{p-1}} \right) \mathrm{d}x.$$

Adding both equations we get

$$0 = \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \left(\frac{u^p - v^p}{u^{p-1}} \right) \mathrm{d}x + \int_{\Omega} |\nabla v|^{p-2} \nabla v \nabla \left(\frac{v^p - u^p}{v^{p-1}} \right) \mathrm{d}x. \tag{6.8}$$

Using that

$$\nabla \left(\frac{u^p - v^p}{u^{p-1}} \right) = \nabla u - p \frac{v^{p-1}}{u^{p-1}} \nabla v + (p-1) \frac{v^p}{u^p} \nabla u,$$

we obtain that the first term of (6.8) is

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \left(\frac{u^p - v^p}{u^{p-1}} \right) \mathrm{d}x$$
$$= \int_{\Omega} |\nabla u|^p - p \int_{\Omega} \frac{v^{p-1}}{u^{p-1}} |\nabla u|^{p-2} \nabla v \nabla u \, \mathrm{d}x + \int_{\Omega} (p-1) \frac{v^p}{u^p} |\nabla u|^p \, \mathrm{d}x$$
$$= \int_{\Omega} |\nabla \ln u|^p u^p \, \mathrm{d}x - p \int_{\Omega} v^p |\nabla \ln u|^{p-2} \langle \nabla \ln u, \nabla \ln v \rangle u v \, \mathrm{d}x$$
$$+ \int_{\Omega} (p-1) |\nabla \ln u|^p v^p \, \mathrm{d}x.$$

We also have an analogous expression for the second term of (6.8). Using both expressions we get that (6.8) becomes

$$0 = \int_{\Omega} \left(u^p - v^p\right)\left(|\nabla \ln u|^p - |\nabla \ln v|^p\right) \mathrm{d}x$$
$$- p \int_{\Omega} v^p |\nabla \ln u|^{p-2} \langle \nabla \ln u, \nabla \ln v - \nabla \ln u \rangle \, \mathrm{d}x$$
$$- p \int_{\Omega} u^p |\nabla \ln v|^{p-2} \langle \nabla \ln v, \nabla \ln u - \nabla \ln v \rangle \, \mathrm{d}x.$$

Taking $\xi_1 = \nabla \ln u$ and $\xi_2 = \nabla \ln v$ and using Lemma 6.2 we get, for $p \geqslant 2$,

$$0 \geqslant \int_{\Omega} C(p) |\nabla \ln u - \nabla \ln v|^p \left(u^p + v^p\right) \mathrm{d}x.$$

Hence, $0 = |\nabla \ln u - \nabla \ln v|$. This implies that $u = kv$ as we wanted to prove. For $p < 2$ we use the second part of Lemma 6.2 as above. □

Now we turn our attention to the proof of the isolation of the first eigenvalue. In order to prove this we need the following nodal result.

LEMMA 6.4. *Let w be an eigenfunction corresponding to $\lambda \neq \lambda_1$. Then w changes sign on $\partial\Omega$, that is, $w^+|_{\partial\Omega} \neq 0$ and $w^-|_{\partial\Omega} \neq 0$. Moreover, there exists a constant C such that*

$$\left|\partial\Omega^+\right| \geqslant C\lambda^{-\beta}, \qquad \left|\partial\Omega^-\right| \geqslant C\lambda^{-\beta}, \tag{6.9}$$

where $\partial\Omega^+ = \partial\Omega \cap \{w > 0\}$, $\partial\Omega^- = \partial\Omega \cap \{w < 0\}$, $\beta = (N-1)/(p-1)$ if $1 < p < N$ and $\beta = 2$ if $p \geqslant N$. Here $|A|$ denotes the $(N-1)$-dimensional measure of a subset A of the boundary.

PROOF. Assume that w does not change sign in Ω, then we can assume that $w > 0$ in Ω using ideas similar to those of Lemma 6.1. Let u_1 be a positive eigenfunction associated to λ_1. Making similar computations as the ones performed in the proof of Lemma 6.3 we arrive at

$$(\lambda_1 - \lambda) \int_{\partial\Omega} \left(u_1^p - w^p\right) \mathrm{d}\sigma \geqslant C \int_{\Omega} |\nabla \ln w - \nabla \ln u_1|^p \left(u_1^p + w^p\right) \mathrm{d}x \geqslant 0.$$

Therefore if we take kw instead of w we get that, for every $k > 0$, we have

$$\int_{\partial\Omega} \left(u_1^p - k^p w^p\right) \mathrm{d}\sigma \leqslant 0,$$

a contradiction if we take

$$k^p \left(\int_{\partial\Omega} w^p \, \mathrm{d}\sigma \right) < \left(\int_{\partial\Omega} u_1^p \, \mathrm{d}\sigma \right).$$

Therefore w changes sign in Ω and by the maximum principle [123], also w changes sign in $\partial\Omega$. Let us use w^- as test function in the weak form of (6.6) satisfied by w to obtain

$$\int_{\Omega} |\nabla w^-|^p \, \mathrm{d}x + \int_{\Omega} |w^-|^p \, \mathrm{d}x = \lambda \int_{\partial\Omega \cap \{w<0\}} |w^-|^p \, \mathrm{d}\sigma.$$

Hence,

$$\|w^-\|^p_{W^{1,p}(\Omega)} \leqslant \lambda \left(\int_{\partial\Omega} |w^-|^{p\alpha} \, \mathrm{d}\sigma \right)^{1/\alpha} \left|\partial\Omega^-\right|^{1/\beta}.$$

If $1 < p < N$ we choose $\alpha = (N-1)/(N-p)$ and $\beta = (N-1)/(p-1)$. Now, we use the trace theorem to get that there exists a constant C such that $\|w^-\|^p_{L^{p\alpha}(\partial\Omega)} \leqslant$

$C\|w^-\|^p_{W^{1,p}(\Omega)}$. If $p \geqslant N$ we choose $\alpha = \beta = 2$ and we argue as before using that $W^{1,p}(\Omega) \hookrightarrow L^{2p}(\partial\Omega)$. A similar argument works for w^+. □

With these lemmas we can prove the isolation of λ_1.

LEMMA 6.5. *λ_1 is isolated, that is, there exists $a > \lambda_1$ such that λ_1 is the unique eigenvalue in* $[0, a]$.

PROOF. From the characterization of λ_1 it is easy to see that $\lambda_1 \leqslant \lambda$ for every eigenvalue λ. Assume that λ_1 is not isolated, then there exists a sequence λ_k with $\lambda_k > \lambda_1$, $\lambda_k \searrow \lambda_1$. Let w_k be an eigenfunction associated to λ_k, we can assume that $\|w_k\|_{W^{1,p}(\Omega)} = 1$. Therefore we can extract a subsequence (that we still denote by w_k) such that $w_k \to u_1$ in $L^p(\partial\Omega)$. Let us define $\phi_k \in (W^{1,p}(\Omega))'$ as

$$\phi_k(u) = \lambda_k \int_{\partial\Omega} |w_k|^{p-2} w_k u \, \mathrm{d}\sigma$$

and $\phi \in (W^{1,p}(\Omega))'$ by

$$\phi(u) = \lambda_1 \int_{\partial\Omega} |u_1|^{p-2} u_1 u \, \mathrm{d}\sigma.$$

From the $L^p(\partial\Omega)$ convergence of w_k to u_1 we get that ϕ_k converges to ϕ in $(W^{1,p}(\Omega))'$. Using the continuity of A_p given by Lemma 3.1 we get that the sequence w_k converge strongly in $W^{1,p}(\Omega)$. Therefore, passing to the limit in the weak form of (6.6) we get that u_1 is an eigenfunction with eigenvalue λ_1. By Lemma 6.1 we can assume that $u_1 > 0$ on $\partial\Omega$. By Egorov's theorem we can find a subset A_ε of $\partial\Omega$ such that $|A_\varepsilon| < \varepsilon$ and $w_k \to u_1 > 0$ uniformly in $\partial\Omega \setminus A_\varepsilon$. This contradicts the fact that, by (6.9), we have, for every k, $|\partial\Omega_k^-| = \partial\Omega \cap \{w_k < 0\} \geqslant C\lambda_k^{-(N-1)/(p-1)}$. This result completes the proof. □

Now, we go back to our original problem, the asymptotic behavior of the best Sobolev trace constant when considered over the family Ω_μ.

Let us look now to the case $\mu \to +\infty$. In this case we find, as before, that the behavior strongly depends on p and q. We prove the following theorem.

THEOREM 6.6. *Let $\beta_{pq} = (qN - pN + p)/q$. It holds*:

(1) *If $1 < q < p$, $0 < c_1\mu^{\beta_{pq}-1} \leqslant S(\Omega_\mu, p, q) \leqslant c_2\mu^{\beta_{pq}-1}$.*

(2) *If $p \leqslant q < p_*$, $0 < c_1 \leqslant S(\Omega_\mu, p, q) \leqslant c_2 < \infty$.*

For the lower bound in (2) *in the case $p < q < p_*$, we have to assume that the corresponding extremals v_μ rescaled such that $\max_{\overline{\Omega}} v_\mu = 1$ verify $|\nabla v_\mu| \leqslant C\mu$. Moreover, for all cases, we have that the corresponding extremals u_μ rescaled as in Theorem* 6.3 *concentrates at the boundary, in the sense that*

$$\int_\Omega |v_\mu|^p \, \mathrm{d}x \leqslant C\mu^{-\beta_{pq}} \to 0 \quad \text{as } \mu \to +\infty \text{ if } q \geqslant p,$$

and

$$\int_{\Omega} |v_\mu|^p \, \mathrm{d}x \leqslant C\mu^{-1} \to 0 \qquad \textit{as } \mu \to +\infty \textit{ if } q < p,$$

with

$$\int_{\partial\Omega} |v_\mu|^q \, \mathrm{d}\sigma = 1.$$

As before, the behavior of the Sobolev trace constant depends on p and q. We have that, as $\mu \to +\infty$,

$$\begin{array}{ll} S \to 0 & \text{if } \beta_{pq} - 1 < 0, \ \text{i.e., } q < p, \\ 0 < c_1 \leqslant S \leqslant c_2 < \infty & \text{if } \beta_{pq} - 1 \geqslant 0, \ \text{i.e., } q \geqslant p. \end{array}$$

The hypothesis $|\nabla v_\mu| \leqslant C\mu$ is a regularity assumption, see [29] and [119] for regularity results. As a consequence we have that the extremals do not develop a peak if $1 < q < p$ as in this case we have that

$$c_1 \leqslant \int_{\partial\Omega} |v_\mu|^p \, \mathrm{d}\sigma \leqslant c_2$$

and

$$\int_{\partial\Omega} |v_\mu|^q \, \mathrm{d}\sigma = 1.$$

For $p = q$ it is proved in [100] that the first eigenvalue $\lambda_1(\Omega_\mu) = S(\Omega_\mu, p, p)$ is isolated and simple, see Theorem 6.5. As a consequence of this if Ω is a ball, the extremal v_μ is radial and hence it does not develop a peak. Finally, for $q > p$ the extremals develop peaking concentration phenomena in the sense that, for every $a > 0$, $a^p|\partial\Omega \cap \{v_\mu > a\}| \to 0$ as $\mu \to +\infty$, with $\max_{\overline{\Omega}} v_\mu = 1$. This is in concordance with the results of [46] where for $p = 2$, $q > 2$ they find that the extremals concentrate, with the formation of a peak near a point of the boundary where the curvature maximizes. We believe that for $q > p$, extremals develop a single peak as in the case $p = 2$. Nevertheless that kind of analysis needs some fine knowledge of the limit problem in $\mathbb{R}^N_+$ that is not yet available for the p-Laplacian.

Let us give an idea of the proof of the lower bounds. In the case $p = q$ we can obtain the lower bound by an approximation procedure. We replace $W^{1,p}(\Omega)$ by an increasing sequence of subspaces in the minimization problem. Then we prove a convergence result and find a uniform bound from below for the approximating problems. We believe that this idea can be used in other contexts. For the case $q > p$ we use our assumption $|\nabla v_\mu| \leqslant C\mu$ to prove a reverse Hölder inequality for the extremals on the boundary that allows us to reduce to the case $p = q$.

Finally, for large μ, in the case $p = q$ we can prove that every eigenvalue is bounded.

THEOREM 6.7. *Let* $\lambda_1(\Omega_\mu) \leqslant \lambda(\Omega_\mu) \leqslant \lambda_k(\Omega_\mu)$ *be an eigenvalue of* (6.4) *in* Ω_μ (*variational or not*). *Then there exists two constants,* $C_1, C_2 > 0$, *independent of* μ *such that* $0 < C_1 \leqslant \lambda(\Omega_\mu) \leqslant C_2 < +\infty$ *for every* μ *large.*

Now we continue our study of the dependence of the best constant $S(\Omega, p, q)$ and extremals on the domain by considering the best Sobolev constant in thin domains. Now we consider a different family of domains. Let $N = n + k$ and define the family

$$\Omega_\mu = \left\{(\mu x, y) \mid (x, y) \in \Omega,\ x \in \mathbb{R}^n, y \in \mathbb{R}^k\right\}.$$

Remark that for small values of μ, Ω_μ is a narrow domain in the x direction.

Our first result shows that, when the domain is very narrow, the problem of looking at the trace of a function is equivalent, in some sense, to the problem of the immersion of the function in the projection of the domain over the y variables. More precisely, we define the projection

$$P(\Omega) = \left\{y \in \mathbb{R}^k \;\middle|\; \exists x \in \mathbb{R}^n \text{ with } (x, y) \in \Omega\right\}$$

and consider the weighted Sobolev embedding $W^{1,p}(P(\Omega), \alpha) \hookrightarrow L^q(P(\Omega), \beta)$ with associated best constant given by

$$\overline{S}_{\alpha,\beta}\big(P(\Omega), p, q\big) = \inf_{v \in W^{1,p}(P(\Omega),\alpha)} \frac{\int_{P(\Omega)} (|\nabla v|^p + |v|^p)\alpha(y)\,\mathrm{d}y}{(\int_{P(\Omega)} |v|^q\, \beta(y)\,\mathrm{d}y)^{p/q}}.$$

We have the following theorem.

THEOREM 6.8. *Let* $1 \leqslant q < p_*$. *Then there exist two nonnegative weights* $\alpha, \beta \in L^\infty(P(\Omega))$ *such that*

$$\lim_{\mu \to 0+} \frac{S(\Omega_\mu, p, q)}{\mu^{(nq-np+p)/q}} = \overline{S}_{\alpha,\beta}\big(P(\Omega), p, q\big)$$

and if we scale the extremals u_μ *of* $S(\Omega_\mu, p, q)$ *to the original domain* Ω *as* $v_\mu(x, y) = u_\mu(\mu x, y)$, $(x, y) \in \Omega$, *normalized as* $\|u_\mu\|^q_{L^q(\partial\Omega_\mu)} = \mu^{n-1}$, *then* $v_\mu \to v = v(y)$ *strongly in* $W^{1,p}(\Omega)$, *where* $v \in W^{1,p}(P(\Omega), \alpha)$ *is an extremal for* $\overline{S}_{\alpha,\beta}(P(\Omega), p, q)$.

We want to remark that the weights α and β can be determined in terms of the geometry of Ω. In fact, $\alpha(y) = |\Omega_y|$ where Ω_y is the section at level y of Ω.

To clarify the content of the result, assume that Ω is a product, $\Omega = \Omega_1 \times \Omega_2$ where $\Omega_1 \subset \mathbb{R}^n$ and $\Omega_2 \subset \mathbb{R}^k$. Then $\Omega_\mu = \mu\Omega_1 \times \Omega_2 = \{(\mu x, y) \mid x \in \Omega_1,\ y \in \Omega_2\}$. As in Theorem 6.8, let us call u_μ an extremal corresponding to Ω_μ and define $v_\mu(x, y) = u_\mu(\mu x, y)$. We have that $v_\mu \in W^{1,p}(\Omega)$ and

$$\frac{S(\Omega_\mu, p, q)}{\mu^{(nq-np+p)/q}} = \frac{\int_\Omega |(\mu^{-1}\nabla_x v_\mu, \nabla_y v_\mu)|^p + |v_\mu|^p\,\mathrm{d}x\,\mathrm{d}y}{(\int_{\partial\Omega_1 \times \Omega_2} |v_\mu|^q\,\mathrm{d}\sigma_x\,\mathrm{d}y + \mu \int_{\Omega_1 \times \partial\Omega_2} |v_\mu|^q\,\mathrm{d}x\,\mathrm{d}\sigma_y)^{p/q}},$$

where $\nabla_x u = (u_{x_1}, \dots, u_{x_n})$ and $\nabla_y u = (u_{y_1}, \dots, u_{y_k})$. The normalization imposed in Theorem 6.8 in this case reduces to

$$\int_{\partial\Omega_1\times\Omega_2} |v_\mu|^q \, \mathrm{d}\sigma_x \, \mathrm{d}y + \mu \int_{\Omega_1\times\partial\Omega_2} |v_\mu|^q \, \mathrm{d}x \, \mathrm{d}\sigma_y = 1. \tag{6.10}$$

In this simpler case, the weight functions α, β are constants and can be computed explicitly, in fact, $\alpha(y) = |\Omega_1|$ and $\beta(y) = |\partial\Omega_1|$. Hence, Theorem 6.8 reads as follows.

THEOREM 6.9. *Let $1 \leqslant q < p_*$ and $\Omega = \Omega_1 \times \Omega_2$. Then*

$$\lim_{\mu\to 0+} \frac{S(\Omega_\mu, p, q)}{\mu^{(nq-np+p)/q}} = \frac{|\Omega_1|}{|\partial\Omega_1|^{p/q}} \overline{S}(\Omega_2),$$

where $\overline{S}(\Omega_2) = \overline{S}_{1,1}(\Omega_2, p, q)$ is the usual Sobolev constant. Moreover, if we scale the extremals u_μ to the original domain Ω as $v_\mu(x, y) = u_\mu(\mu x, y)$, $x \in \Omega_1$, $y \in \Omega_2$, normalized by (6.10)*, then $v_\mu \to v = v(y)$ strongly in $W^{1,p}(\Omega)$, where $v \in W^{1,p}(\Omega_2)$ is an extremal for $\overline{S}(\Omega_2)$.*

Observe, that the critical exponent for the Sobolev embedding, $W^{1,p}(\Omega_2) \hookrightarrow L^q(\Omega_2)$ valid for $1 \leqslant q < pk/(k-p)$, is larger than the one for the Sobolev trace embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$, which holds for $1 \leqslant q < p(k+n-1)/(k+n-p)$.

Again, in the special case $p = q$, the problem becomes a nonlinear eigenvalue problem. Following [71] (see also [49]), a sequence of variational eigenvalues λ_j can be characterized by

$$\lambda_j = \inf_{C\in\mathcal{C}_j} \max_{u\in C} \frac{\|u\|^p_{W^{1,p}(\Omega)}}{\|u\|^p_{L^p(\partial\Omega)}}, \tag{6.11}$$

where

$$\mathcal{C}_j = \bigl\{\Phi\bigl(S^{j-1}\bigr) \subset W^{1,p}(\Omega) \bigm| \Phi : S^{j-1} \to W^{1,p}(\Omega) \setminus \{0\} \text{ is continuous and odd}\bigr\}$$

and S^{j-1} is the unit sphere of $\mathbb{R}^j$. These eigenvalues differ slightly from the ones considered in [71]. However, the same arguments used there apply proving that in fact $\{\lambda_j\}$ is an unbounded sequence of eigenvalues.

When μ goes to zero, there is a limit problem which is a weighted eigenvalue problem on the projection $P(\Omega)$. Let α and β be the weights given by Theorem 6.8 and consider the following eigenvalue problem

$$\begin{cases} -\operatorname{div}\bigl(\alpha|\nabla v|^{p-2}\nabla v\bigr) + \alpha|v|^{p-2}v = \bar{\lambda}\beta|v|^{p-2}v & \text{in } P(\Omega), \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial P(\Omega). \end{cases} \tag{6.12}$$

For problem (6.12), one can define the sequence

$$\bar{\lambda}_j = \inf_{C\in\overline{\mathcal{C}}_j} \max_{u\in C} \frac{\int_{P(\Omega)}(|\nabla u|^p + |u|^p)\alpha \, dy}{\int_{P(\Omega)} |u|^p \beta \, dy}, \tag{6.13}$$

where

$$\overline{\mathcal{C}}_j = \big\{\Phi\big(S^{j-1}\big) \subset W^{1,p}\big(P(\Omega)\big)\colon \ \Phi : S^{j-1} \to W^{1,p}\big(P(\Omega)\big) \setminus \{0\} \text{ is continuous and odd}\big\}.$$

Once again, applying the Ljusternik–Schnirelmann critical point theory one could check that $\{\bar{\lambda}_j\}$ is an unbounded sequence of eigenvalues for (6.12). However, this fact is a direct consequence of our next result.

THEOREM 6.10. *Let $\lambda_{j,\mu}$ given by* (6.11) *in Ω_μ and let $u_{j,\mu}$ be an associated eigenfunction normalized as in Theorem* 6.8. *Then*

$$\lim_{\mu\to 0} \frac{\lambda_{j,\mu}}{\mu} = \bar{\lambda}_j,$$

where $\bar{\lambda}_j$ is defined by (6.13) *and is an eigenvalue of* (6.12). *Also, along a subsequence, $v_{j,\mu}(x,y) = u_{j,\mu}(\mu x, y)$ converges strongly in $W^{1,p}(\Omega)$ to a function $\bar{v}_j = \bar{v}_j(y)$ which is an eigenfunction of* (6.12) *with eigenvalue $\bar{\lambda}_j$.*

Observe that the first eigenvalue λ_1 coincides with the best Sobolev trace constant $S(\Omega, p, p)$. Hence, for $p = q$ and for the first eigenvalue, Theorem 6.8 and Theorem 6.10 coincide.

As before, in the case $\Omega = \Omega_1 \times \Omega_2$, the limit problem has a simpler form, that is,

$$\begin{cases} -\Delta_p v + |v|^{p-2}v = \frac{|\partial\Omega_1|}{|\Omega_1|}\bar{\lambda}|v|^{p-2}v & \text{in } \Omega_2, \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega_2. \end{cases}$$

However, Theorem 6.10 conserves the same statement.

Our last result is concerned with the following fact: once the domain has been contracted in the x direction, we can now try to contract it in the y direction and see if the limit coincides with the one obtained by contracting the domain in every direction at the same time. Surprisingly, this is not the case. In fact, we obtain the following theorem.

THEOREM 6.11. *Let $\Omega = \Omega_1 \times \Omega_2$ and consider $\Omega_{\mu,\nu} = \{(\mu x, \nu y)\colon (x,y) \in \Omega\}$, then*

$$\lim_{\nu\to 0}\left(\lim_{\mu\to 0} \frac{S(\Omega_{\mu,\nu}, p, q)}{\mu^{(nq-np+p)/q}\nu^{(kq-kp)/q}}\right) = \frac{|\Omega|}{(|\partial\Omega_1||\Omega_2|)^{p/q}}.$$

By the previous results, Theorem 6.3, we have

$$\lim_{\mu\to 0}\frac{S(\mu\Omega, p,q)}{\mu^{(Nq-Np+p)/q}}=\frac{|\Omega|}{|\partial\Omega|^{p/q}}\neq\frac{|\Omega|}{(|\partial\Omega_1||\Omega_2|)^{p/q}}.$$

This shows that the double limit $\lim_{(\mu,\nu)\to(0,0)} S(\Omega_{\mu,\nu}, p,q)$ does not exists.

For a general domain Ω we have

$$\lim_{\nu\to 0}\left(\lim_{\mu\to 0}\frac{S(\Omega_{\mu,\nu}, p,q)}{\mu^{(nq-np+p)/q}\nu^{(kq-kp)/q}}\right)=\frac{|\Omega|}{(\int_{P(\Omega)}\beta\,\mathrm{d}y)^{p/q}}.$$

To prove this fact we assume that the immersion $W^{1,p}(P(\Omega),\alpha)\hookrightarrow L^q(P(\Omega),\beta)$ is compact. To see in which cases this holds, see [66].

7. Symmetry of extremals

The aim of this section is to study of the following problem: Given a ball of radius ρ, $B(0,\rho)$, in $\mathbb{R}^N$, $N\geqslant 3$, decide whether or not there exists a radial extremal for the embedding

$$H^1\big(B(0,\rho)\big)\hookrightarrow L^q\big(\partial B(0,\rho)\big).$$

First, let us introduce our motivation. Recall that the best constant for the Sobolev trace embedding

$$S(\Omega,2,q)=\inf_{v\in H^1(\Omega)\setminus H^1_0(\Omega)}\frac{\int_\Omega|\nabla v|^2+|v|^2\,\mathrm{d}x}{(\int_{\partial\Omega}|v|^q\,\mathrm{d}\sigma)^{2/q}}. \tag{7.1}$$

As we noticed before, the best constant $S(\Omega,2,q)$ is not homogeneous under dilatations. In fact, we have

$$S(\mu\Omega,2,q)=\mu^\beta\inf_{v\in H^1(\Omega)\setminus H^1_0(\Omega)}\frac{\int_\Omega\mu^{-2}|\nabla v|^2+|v|^2\,\mathrm{d}x}{(\int_{\partial\Omega}|v|^q\,\mathrm{d}\sigma)^{2/q}},$$

where $\beta=(Nq-2N+2)/q$. For $1\leqslant q<2_*=2(N-1)/(N-2)$, the embedding is compact, so we have existence of extremals, that is, functions where the infimum is attained. These extremals are weak solutions of the following problem

$$\begin{cases}\Delta u=u & \text{in } \Omega,\\ \frac{\partial u}{\partial\eta}=\lambda|u|^{q-2}u & \text{on } \partial\Omega.\end{cases}$$

The asymptotic behavior of $S(\mu\Omega, p,q)$ in expanding ($\mu\to\infty$) and contracting domains ($\mu\to 0$) was studied in the previous section, see also [46] and [73]. In [46] it is proved that for expanding domains and $q > 2$, $S(\mu\Omega, 2, q) \to S(\mathbb{R}^N_+, 2, q)$. In [73], see Theorem 6.3, it is shown that

$$\lim_{\mu \to 0+} \frac{S(\mu\Omega, 2, q)}{\mu^\beta} = \frac{|\Omega|}{|\partial\Omega|^{2/q}}.$$

As we mentioned in the previous section, the behavior of the extremals for (7.1) in expanding and contracting domains is also studied in [46] and [73]. For expanding domains, it is proved in [46] that the extremals develop a peak near a point where the mean curvature of the boundary is a maximum. For contracting domains, we have that the extremals, when rescaled to the original domain as $v(x) = u(\mu x)$, $x \in \Omega$, and normalized with $\|v\|_{L^q(\partial\Omega)} = 1$, are nearly constant in the sense that $\lim_{\mu\to 0} v = |\partial\Omega|^{-1/q}$ in $H^1(\Omega)$.

A big difference between the Sobolev trace theorem and the Sobolev embedding theorem arises in the behavior of extremals. Namely, if Ω is a ball, $\Omega = B(0, \rho)$, as the extremals do not change sign, from results of [82] the extremals for the usual Sobolev embedding, $H^1_0(B(0, 1)) \hookrightarrow L^q(B(0, 1))$, are radial while, if q exceeds 2 and ρ is large, extremals for (7.1) are not, since they develop a peaking concentration phenomena as is described in [46].

The above discussion leads naturally to the purpose of this section: the study of the symmetry properties for the extremals of the Sobolev trace embedding in small balls. We find that the symmetry properties of the extremals depend on the size of the ball. Our first result describes when there exists a radial extremal.

THEOREM 7.1. *Let $2_* = 2(N-1)/(N-2)$ be the critical exponent for the Sobolev trace immersion. Concerning symmetry properties of the extremals for the embedding $H^1(B(0,\rho)) \hookrightarrow L^q(\partial B(0,\rho))$ there holds*:

(1) *Let $1 < q \leqslant 2$. For every $\rho > 0$ there exists a radial extremal.*

(2) *Let $2 < q < 2_*$. There exists $\rho_0 > 0$ such that, for every $\rho < \rho_0$, there is a unique positive extremal u, normalized such that $\|u(\rho x)\|_{L^q(\partial B(0,1))} = 1$; moreover, this extremal is a radial function. However, for large values of ρ, there is no radial extremal.*

(3) *Let $q = 2_*$. There exists $\rho_0 > 0$ such that, for every $\rho < \rho_0$, there is a positive radial extremal.*

The main ingredient of the proof of the symmetry result for small balls is the implicit function theorem. We remark that the moving planes technique cannot be applied to obtain symmetry results in this case, as the extremals for large ρ are not radial.

For small balls, using the symmetry result in balls, for the critical exponent $2_* = 2 \times (N-1)/(N-2)$, we can prove existence of extremals, which turns out to be radial functions. For general domains Ω, see [3], where it is proved that a extremal exists if the domain (bounded or not) verifies that it contains a point at the boundary with strictly positive curvature. We remark that the existence of extremals for the critical exponent is not trivial, this is due to the lack of compactness. This result has to be compared with the case of the immersion $H^1_0(B(0,\rho)) \to L^{2^*}(B(0,\rho))$ where it is well known that, by Pohozaev identity, there is no positive solution regardless the size of the ball for the critical exponent $2^* = 2N/(N-2)$. However, there exist solutions for topologically nontrivial domains.

In [92] Lami-Dozo and Torne studied the symmetry and symmetry breaking of the extremals in a ball for general $1 < p < \infty$. Consider

$$S\big(B(0,\rho), p, q\big) = \inf_{u \in W^{1,p}(\Omega) \setminus W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p + |u|^p \, dx}{(\int_{\partial\Omega} |u|^q \, d\sigma)^{p/q}}. \tag{7.2}$$

As we have already mentioned for subcritical q, $1 < q < p(N-1)/(N-p)$, the best constant is attained and we can assume that extremals are positive in $B(0,\rho)$. With the normalization (different from $\int_{\partial\Omega} u^q = 1$)

$$S\big(B(0,\rho), p, q\big)\left(\int_{\partial B(0,\rho)} u^q \, d\sigma\right)^{(p-q)/q} = 1,$$

the extremals are solutions of

$$\begin{cases} \Delta_p u = u^{p-1} & \text{in } B(0,\rho), \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \lambda u^{q-1} & \text{on } \partial B(0,\rho). \end{cases} \tag{7.3}$$

Let us call φ_1 the eigenfunction associated with the first eigenvalue λ_1 of the problem

$$\begin{cases} \Delta_p \varphi = |\varphi|^{p-2}\varphi & \text{in } B(0,\rho), \\ |\nabla \varphi|^{p-2} \frac{\partial \varphi}{\partial \nu} = \lambda |\varphi|^{p-2}\varphi & \text{on } \partial B(0,\rho). \end{cases} \tag{7.4}$$

Recall that the first eigenvalue is isolated and simple, see Section 6 and [100], therefore φ_1 is radial. We have the following theorem.

THEOREM 7.2. *Let $\rho > 0$ and $1 < p < \infty$ be fixed.*
(1) *If there exists a radial minimizer of* (7.2) *then it is a multiple of φ_1.*
(2) *Assume that there exists a radial minimizer for $S(B(0,\rho), p, q_0)$ then any minimizer for $S(B(0,\rho), p, q)$ with $q < q_0$ is radial and a multiple of φ_1.*
(3) *Let $1 < q < p$. Then the solution of* (7.3) *is unique and it is a multiple of φ_1. In particular, any extremal is a multiple of φ_1.*

Radial symmetry is lost when ρ or q is sufficiently large. Define the function

$$Q(\rho) = \frac{1}{\lambda_1(\rho)^{p/(p-1)}}\left(1 - (N-1)\frac{\lambda_1(\rho)}{\rho}\right) + 1.$$

We have the following theorem.

THEOREM 7.3. *Let $1 < p < \infty$ be fixed.*
(1) *Let $\rho > 0$. If $q > Q(\rho)$ then there is no radial minimizer for $S(B(0,\rho), p, q)$.*
(2) *Let $p < q < p(N-1)/(N-p)$. There exists $R(q)$ such that for any $\rho > R(q)$ there is no radial minimizer for $S(B(0,\rho), p, q)$.*

The main ideas of the proofs of these theorems are as follows. The proof of uniqueness of the solution of (7.3) for $1 < q < p$ follows from Picone's identity and the weak formulation. In fact, assume that there exist two positive solutions u and v, then

$$
\begin{aligned}
0 &\leqslant \int_{B(0,\rho)} |\nabla u|^p \,\mathrm{d}x - \int_{B(0,\rho)} |\nabla v|^{p-2}\nabla v \nabla\left(\frac{u^p}{v^{p-1}}\right)\mathrm{d}x \\
&= -\int_{B(0,\rho)} u^p \,\mathrm{d}x + \int_{\partial B(0,\rho)} u^q \,\mathrm{d}\sigma + \int_{B(0,\rho)} v^{p-1}\frac{u^p}{v^{p-1}}\,\mathrm{d}x \\
&\quad - \int_{\partial B(0,\rho)} v^{q-1}\frac{u^p}{v^{p-1}}\,\mathrm{d}\sigma \\
&= \int_{\partial B(0,\rho)} u^q \,\mathrm{d}\sigma - \int_{\partial B(0,\rho)} v^{q-p}u^p \,\mathrm{d}\sigma \\
&= \int_{\partial B(0,\rho)} u^p\left(u^{q-p} - v^{q-p}\right)\mathrm{d}\sigma.
\end{aligned}
\tag{7.5}
$$

We can interchange the roles of u and v in (7.5) and finally obtain

$$
0 \leqslant \int_{\partial B(0,\rho)} \left(u^p - v^p\right)\left(u^{q-p} - v^{q-p}\right)\mathrm{d}\sigma.
$$

Since $q < p$, the integrand is nonpositive and hence $u = v$ on $\partial B(0,\rho)$. The uniqueness follows from the uniqueness of the Dirichlet problem for $\Delta_p u = u^{p-1}$.

The function $Q(\rho)$ appears when one try to prove that the extremals are not radial for large q. In fact, let u_0 denote the positive radial solution of $\Delta_p u_0 = u_0^{p-1}$ in $\mathbb{R}^N$ normalized such that $u_0 = 1$ on $\partial B(0,\rho)$. For any $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$, denote by $x^t = (x_1 - t, x_2, \ldots, x_N)$ and consider the function

$$
\Phi(t) = \frac{\int_{B(0,\rho)} |\nabla u_0(x^t)|^p + u_0^p(x^t)\,\mathrm{d}x}{(\int_{\partial B(0,\rho)} u_0^q(x^t)\,\mathrm{d}\sigma)^{p/q}}.
$$

After some calculations, see [92] for the details, we get

$$
\Phi'(0) = 0
$$

and

$$
\Phi''(0) = C\left(1 - (N-1)\frac{\lambda_1(\rho)}{\rho} - (q-1)\lambda_1(\rho)^{p/(p-1)}\right).
$$

Hence, when $q > Q(\rho)$ we have $\Phi''(0) < 0$ and hence $t = 0$ is a local maximum for Φ. Therefore u_0 is not a minimizer.

One remarkable fact concerning the first eigenvalue $\lambda_1(\rho)$ of (7.4) is that it verifies the differential equation

$$\lambda_1'(\rho) = 1 - (p-1)\lambda_1(\rho)^{p/(p-1)} - (N-1)\frac{\lambda_1(\rho)}{\rho},$$

with the initial condition $\lambda_1(0) = 1$. The proof of this fact is as follows, see [92] for details. Let u_0 denote the positive radial solution of $\Delta_p u_0 = u_0^{p-1}$ in $\mathbb{R}^N$ normalized such that $u_0(0) = 1$. For any $\rho > 0$ the first eigenfunction of (7.4) is given by the restriction of u_0 to $B(0, \rho)$. From the boundary condition we get

$$\lambda_1(\rho) = \frac{u_0'(\rho)^{p-1}}{u_0(\rho)^{p-1}}.$$

Differentiating with respect to ρ and using the equation verified by u_0,

$$\left(\rho^{N-1}\left|u_0'(\rho)\right|^{p-2}u_0'(\rho)\right)' = \rho^{N-1}u_0^{p-1}(\rho),$$

we get the desired differential equation

$$\lambda_1'(\rho) = 1 - (p-1)\lambda_1(\rho)^{p/(p-1)} - (N-1)\frac{\lambda_1(\rho)}{\rho}.$$

To obtain the initial condition $\lambda_1(0) = 1$ we just observe that $\lambda_1(\rho) \to 1$ as $\rho \to 0$, see Theorem 6.3.

The technique of spherical symmetrization, also known as foliated Schwarz symmetrization, is well suited for the study of the symmetry properties of nonradial extremals for our problem. The definition of *spherical symmetrization* (see [90]) is as follows. Given a measurable set $A \subset \mathbb{R}^N$, the spherical symmetrization A^* of A is defined as follows: For each r, take $A \cap \partial B(0,r)$ and replace it by the spherical cap of the same area and center re_N. The union of these caps is A^*.

Let u be an extremal and let u^* denote the spherical symmetrization of u with respect to the north pole, e_N. It is well known that, for any ball $B(0, \rho)$, we have

$$\begin{aligned} &\left\|u^*\right\|_{W^{1,p}(B(0,\rho))} \leqslant \|u\|_{W^{1,p}(B(0,\rho))} \quad \text{and} \\ &\left\|u^*\right\|_{L^p(\partial B(0,\rho))} = \|u\|_{L^q(\partial B(0,\rho))}. \end{aligned} \tag{7.6}$$

Hence, u^* is also a minimizer. Remark that the restriction of u^* to any sphere centered at the origin and contained in $B(0, \rho)$ is an increasing function of the geodesic distance from the north pole. This fact together with the maximum principle imply that u^* concentrates at a single point on the boundary of $B(0, \rho)$.

For the special case when $p = 2$, it is shown in [48] that either the inequality in (7.6) is strict or u and u^* coincide on every sphere up to a rotation. This implies that any minimizer is spherically symmetric.

8. Behavior of the best Sobolev trace constant and extremals in domains with holes

In this section, using results from [76], we study the best Sobolev trace constant corresponding to the embedding of $W^{1,p}(\Omega)$ into $L^q(\partial\Omega)$ for functions that vanish on a fixed subset of Ω, that we will call A. We consider subcritical exponents $1 \leqslant q < p_* = p(N-1)/(N-p)$ so that the immersion $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$ is compact. To begin with, we consider a subset $A \subset \Omega$ with positive measure and define the best Sobolev trace constant associated with this set, that is,

$$S_A = \inf\left\{ \frac{\int_\Omega (|\nabla u|^p + |u|^p)\,\mathrm{d}x}{(\int_{\partial\Omega} |u|^q \,\mathrm{d}S)^{p/q}} \colon u \in W^{1,p}(\Omega), u|_{\partial\Omega} \not\equiv 0 \text{ and } u|_A = 0 \right\}. \tag{8.1}$$

Since along this section we are interested in the dependence of this best constant on A, we have dropped the explicit dependence of S on (Ω, p, q).

As a consequence of the compact immersion, there exist extremals for S_A. An extremal for S_A is a weak solution to

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = 0 & \text{in } \Omega \setminus A, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda |u|^{q-2}u & \text{on } \partial\Omega, \\ u = 0 & \text{in } A, \end{cases} \tag{8.2}$$

where λ depends on the normalization of u. For instance, if $\|u\|_{L^q(\partial\Omega)} = 1$ then $\lambda = S_A$. The existence of weak solutions to (8.2) when $A = \emptyset$ has been studied in Section 6, see [71]. Of special importance is the case $q = p$ in which the equation and boundary condition in (8.2) have the same homogeneity. In this case, (8.2) can be considered as a nonlinear eigenvalue problem. This nonlinear eigenvalue problem, in the case $A = \emptyset$, has been already studied in this work (Section 6), see also [71] and [100]. So we are studying a nonlinear generalization of an eigenvalue problem by adding the restriction that the functions involved vanish on a subset A.

Optimal design problems are usually formulated as problems of minimization of the energy, stored in the design under a prescribed loading. Solutions of these problems are unstable to perturbations of the loading. The stable optimal design problem is formulated as minimization of the stored energy of the project under the most unfavorable loading. This most dangerous loading is one that maximizes the stored energy over the class of admissible functions. The problem is reduced to minimization of Steklov eigenvalues. See [28].

In view of the above discussion, our first concern is to consider the following optimization problem: For a fixed $0 < a < |\Omega|$, find a set A_0 of measure a that minimizes S_A among all subsets $A \subset \Omega$ of measure a. We have that such a set exists and, in the case that Ω is a ball, that there exists an optimal set that is spherically symmetric in the sense of [90] (see Section 7 for the definition). Moreover, in the case $p = 2$, every optimal set is spherically symmetric. On the other hand, we also have that there does not exists a set A that maximizes S_A. Also, $\sup S_A = +\infty$ where the supremum is taken over all sets A of given measure a. We also get the continuity of S_A with respect to the "hole" A.

Optimization problems for eigenvalues of elliptic operators has been widely studied in the past, and is still an area of intensive research. In [48] the author studies an optimization problem for the second Neumann eigenvalue of the Laplacian with $A \subset \partial\Omega$. Our approach to the optimization problem follows closely the one in [48]. Optimal design problems have been widely studied not only for eigenvalue problems.

Now we state the main results of this section. Our first result is the sequential lower semicontinuity of S_A.

THEOREM 8.1. *Let $A_n \subset \Omega$ be sets of positive measure such that*

$$\chi_{A_n} \overset{*}{\rightharpoonup} \chi_{A_0} \quad \text{in } L^\infty(\Omega),$$

where χ_A is the characteristic function of the set A. Then

$$S_{A_0} \leqslant \liminf_{n\to\infty} S_{A_n},$$

where S_A is given by (8.1).

We remark that the continuity is not true in general. This semicontinuity result suggest that a minimizer for S_A among sets A of fixed positive Lebesgue measure exists. However, there is a major difficulty here because of the fact that sets of prescribed positive Lebesgue measure are not compact with respect to the topology of Theorem 8.1. The result concerning the existence of an optimal design for the constant S_A is as follows.

THEOREM 8.2. *Given $0 < a < |\Omega|$, let us define*

$$S(a) := \inf_{A\subset\Omega, |A|=a} S_A.$$

Then, there exists a set $A_0 \subset \Omega$ such that $|A_0| = a$ and $S_{A_0} = S(a)$. On the other hand, there is no upper bound for $S(a)$. Let $0 < a < |\Omega|$. Then

$$\sup_{A\subset\Omega, |A|=a} S_A = \infty.$$

Next we study symmetry properties of optimal sets A_0 in the special case where Ω is a ball. To this end, we use the definition of *spherical symmetrization* (see [90] and Section 7). We have the following result.

THEOREM 8.3. *Let $\Omega = B(0,1)$ and $0 < a < |B(0,1)|$. Then, there exists an optimal hole of measure a which is spherically symmetric, that is, $A^* = A$. Moreover, when $p = 2$ every optimal hole is spherically symmetric.*

Now we state the results that allow us to consider the case of measure zero, that is, $|A| = 0$. For simplicity we will consider closed sets A. When trying to give sense to a best Sobolev trace constant for functions that vanish in a set of zero Lebesgue measure a different approach has to be made. We consider the space $W_A^{1,p}(\Omega) = \overline{C_0^\infty(\overline{\Omega} \setminus A)}$, where the closure is taken in $W^{1,p}$ norm. That is, $W_A^{1,p}(\Omega)$ stands for the set of functions of the Sobolev space $W^{1,p}(\Omega)$ that can be approximated by smooth functions that vanish in a neighborhood of A.

In this context the best Sobolev trace constant is defined as

$$\mathbf{S}_A = \inf_{u\in W_A^{1,p}(\Omega)\setminus W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p + |u|^p \,\mathrm{d}x}{(\int_{\partial\Omega} |u|^q \,\mathrm{d}S)^{p/q}}.$$

In this case this problem only makes sense if A is a set with positive p-capacity (see (8.3)). More precisely, we have that $\mathbf{S}_A = \mathbf{S}_\emptyset$ if and only if the p-capacity of A is zero. Note that $\mathbf{S}_\emptyset$ is the usual Sobolev trace constant from $W^{1,p}(\Omega)$ onto $L^q(\partial\Omega)$. Observe that the constants S_A and $\mathbf{S}_A$ need not be the same.

First, we study when $\mathbf{S}_A$ is equal to the usual Sobolev trace constant, that is, when $\mathbf{S}_A = \mathbf{S}_\emptyset$. For this purpose we recall the definition of p-capacity. For $A\subset\Omega$, define

$$\begin{aligned}&\mathrm{Cap}_p(A)\\ &\quad = \inf\left\{\int_{\mathbb{R}^n} |\nabla\phi|^p \,\mathrm{d}x \;\middle|\; \phi\in W^{1,p}\big(\mathbb{R}^N\big)\cap C^\infty\big(\mathbb{R}^N\big) \text{ and } A\subset\{\phi\geqslant 1\}^\circ\right\}.\end{aligned} \tag{8.3}$$

We have the following theorem.

THEOREM 8.4. *Let $A\subset\Omega$. Then $W_A^{1,p}(\Omega) = W^{1,p}(\Omega)$ if and only if* $\mathrm{Cap}_p(A) = 0$.

As a corollary we obtain:

COROLLARY 8.1. $\mathrm{Cap}_p(A) = 0$ *if and only if* $\mathbf{S}_A = \mathbf{S}_\emptyset$.

Next, we look for the dependence of $\mathbf{S}_A$ under perturbations of A. We find that $\mathbf{S}_A$ is continuous with respect to A in the topology given by the Hausdorff distance.

THEOREM 8.5. *Let $A, A_n\subset\Omega$ be closed sets such that $d(A_n, A)\to 0$ as $n\to\infty$ where $d(A_n, A)$ is the Hausdorff distance between A_n and A. Then $|\mathbf{S}_{A_n} - \mathbf{S}_A|\to 0$ when $n\to\infty$ and if we denote by u_n an extremal for $\mathbf{S}_{A_n}$ normalized such that $\|u_n\|_{L^q(\partial\Omega)} = 1$, there exists a subsequence u_{n_k} such that* $\lim_{k\to\infty} u_{n_k} = u$, *strongly in $W^{1,p}(\Omega)$, and u is an extremal for $\mathbf{S}_A$.*

9. The Sobolev trace embedding with the critical exponent

In this section we look at problems with the critical Sobolev trace exponent in the boundary condition. Existence result for elliptic problems with critical Sobolev exponents have deserved a great deal of attention since the pioneering work [22] and is an intensive area of research nowadays.

Now we describe the results of [35]. The authors study the problem

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } B^+(0,1), \\ u = 0 & \text{on } \partial B^+(0,1) \cap \{x_N > 0\}, \\ -\frac{\partial u}{x_N} = u^{N/(N-2)} & \text{on } \partial B^+(0,1) \cap \{x_N = 0\}. \end{cases} \tag{9.1}$$

Here $B^+(0,1)$ stands for the half-ball $\{|x| < 1, x_N > 0\}$.

The following result parallels the one in [22] for Dirichlet boundary conditions.

THEOREM 9.1. *Let μ_1 be the first eigenvalue of the problem*

$$\begin{cases} \Delta \phi + \mu \phi = 0 & \text{in } B^+(0,1), \\ \phi = 0 & \text{on } \partial B^+(0,1) \cap \{x_N > 0\}, \\ -\frac{\partial \phi}{x_N} = 0 & \text{on } \partial B^+(0,1) \cap \{x_N = 0\}. \end{cases}$$

Then

(1) *if $N \geqslant 4$ then a positive solution to problem* (9.1) *exists if and only if* $0 < \lambda < \mu_1$;

(2) *for $N = 3$ there is no positive solution of* (9.1) *for $\lambda \geqslant \mu_1 = \pi^2$ while for $\pi^2/4 < \lambda < \pi^2$ positive solutions exist. There is a value $\lambda^* \in (0, \pi^2/4)$ such that no positive solution exists for $-\infty < \lambda < \lambda^*$.*

Remark that, as happens for the Dirichlet problem (see [22]), $N = 3$ is a *critical* dimension.

The main reason to deal with the half-ball $B^+(0,1)$ is the fact that solutions are $C^{2,\alpha}(\overline{B^+(0,1)})$ and have cylindrical symmetry. For the nonexistence proof the authors use a sharp Pohozaev identity. For the existence part they consider the minimization problem

$$A_\lambda = \inf_{u \in H^1(\Omega), u=0 \text{ on } \{x_N=0\}} \left\{ \frac{\|\nabla u\|^2_{L^2(\Omega)} - \lambda \|u\|^2_{L^2(\Omega)}}{\|u\|^2_{L^{2(N-1)/(N-2)}(\partial B^+(0,1) \cap \{x_N > 0\})}} \right\},$$

and they prove that when $0 < \lambda < \mu_1$ ($N \geqslant 4$) or $\lambda^* < \lambda < \mu_1$ ($N = 3$) there is the case when A_λ lies below a critical level. Hence the concentration–compactness results of [96,97] can be used to obtain the existence of a minimizer for A_λ that turns out to be a solution of (9.1).

✓ In [3] it is studied another linear perturbation of the Laplace equation with a critical nonlinearity at the boundary, namely,

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ -\frac{\partial u}{\partial \nu} = u^{N/(N-2)} + \lambda u & \text{on } \Gamma_N. \end{cases} \tag{9.2}$$

Here we assume that the boundary of Ω splits into two parts Γ_D and Γ_N. If we denote by $\tilde{\mu}_1$ the first eigenvalue of the problem $\Delta \phi = 0$ with $\phi = 0$ on Γ_D and $\frac{\partial \phi}{\partial \nu} = \tilde{\mu} \phi$ on Γ_N, then problem (9.2) has a positive solution if and only if $0 < \lambda < \tilde{\mu}_1$, no matter the dimension.

Hence, although (9.2) and (9.1) are linear perturbations of the same problem their behavior for $N = 3$ is very different.

For the existence result instead of A_λ they consider minimizing

$$B_\lambda = \inf\left\{ \frac{\|\nabla u\|^2_{L^2(\Omega)} - \lambda \|u\|^2_{L^2(\Gamma_N)}}{\|u\|^2_{L^{2(N-1)/(N-2)}(\Gamma_N)}} \colon u \in H^1(\Omega), u = 0 \text{ on } \Gamma_D \right\},$$

and prove that the minimum is attained if and only if $0 < \lambda < \tilde{\mu}_1$.

To explain the different behavior of problems (9.2) and (9.1) for $N = 3$, we observe that the linear perturbation taking place on an $(N-1)$-dimensional manifold instead of an N-dimensional domain typically reduces the critical dimension by 1, which leads to the fact that there is no critical dimension for problem (9.2).

✓ Concerning the symmetry properties of the extremals of the Sobolev trace constant, it is proved in [65] that if Ω is a ball of sufficiently small radius, then the extremals are radial functions, see Section 7. Also in [65] the authors use this result to prove that there exists a radial extremal for the immersion $H^1(B(0,\rho)) \to L^{2_*}(\partial B(0,\rho))$ if the radius ρ is small enough. See also [3] for other geometric conditions that leads to existence of extremals in the case $p = 2$. Concerning the existence of extremals for the Sobolev trace theorem with the critical exponent in a general smooth bounded domain Ω, the main result of this section is the following theorem.

THEOREM 9.2. *Let Ω be a bounded smooth domain in $\mathbb{R}^N$ such that*

$$\frac{|\Omega|}{|\partial\Omega|^{p/p_*}} < \frac{1}{K(N,p)}, \tag{9.3}$$

where $K(N,p)$ is the trace constant for $\mathbb{R}^N_+$. Then there exists an extremal for the immersion $W^{1,p}(\Omega) \to L^{p_*}(\partial\Omega)$.

The proof of Theorem 9.2 uses the same approach as in [15], properly adapted to our new context, see [75] for the details. The other key ingredient in the proof is the result of [20] where the author compute the optimal constant $K(N,p)$. See also [94] for a similar result in the case $p = 2$.

REMARK 9.1. Let Ω be any smooth bounded domain in $\mathbb{R}^N$ and let

$$\Omega_\mu = \mu\Omega = \{\mu x \mid x \in \Omega\},$$

where $\mu > 0$. We observe that when μ is small enough, precisely

$$\mu < \frac{1}{K(N,p)^{1/p}} \frac{|\partial\Omega|^{1/p_*}}{|\Omega|^{1/p}},$$

then Ω_μ verifies the hypotheses of Theorem 9.2 and hence there is an extremal for the immersion $W^{1,p}(\Omega_\mu) \to L^{p_*}(\partial\Omega_\mu)$.

REMARK 9.2. We observe that with the same ideas and computations, we can consider a problem of the form

$$\begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = a(x)u^{p_*-1} & \text{on } \partial\Omega, \end{cases}$$

with $a \in L^\infty(\partial\Omega)$ bounded away from zero. This corresponds to a Sobolev trace immersion with a weight a, on the boundary. In this case the condition on Ω and the weight function a is

$$|\Omega|\Big(\sup_{\partial\Omega} a\Big)^{p/p_*} < \frac{1}{K(N,p)}\Big(\int_{\partial\Omega} a\,\mathrm{d}\sigma\Big)^{p/p_*}.$$

REMARK 9.3. From the proof of Theorem 9.2, see [75], we obtain the existence of extremals for every domain Ω that satisfies

$$S(\Omega, p, p_*) < \frac{1}{K(N,p)}. \tag{9.4}$$

Condition (9.3) is the simplest geometric condition that ensures (9.4).

✓ Now, let us look at the special case of studying the best constant for the Sobolev trace embedding from $W^{1,1}(\Omega)$ into $L^1(\partial\Omega)$. We follow the presentation of [11], see also [104].

As $W^{1,1}(\Omega) \hookrightarrow L^1(\partial\Omega)$ we have that there exists a constant S such that the following inequality holds, $S\|u\|_{L^1(\partial\Omega)} \leqslant \|u\|_{W^{1,1}(\Omega)}$ for all $u \in W^{1,1}(\Omega)$. The best constant for this embedding is the largest λ such that the above inequality holds, that is,

$$\begin{aligned} S(\Omega,1,1) &= \lambda_1(\Omega) \\ &= \inf\Big\{\int_\Omega |u| + \int_\Omega |\nabla u|\colon u \in W^{1,1}(\Omega), \int_{\partial\Omega} |u| = 1\Big\}. \end{aligned} \tag{9.5}$$

Our main interest now is to study the dependence of the best constant $\lambda_1(\Omega)$ and extremals (functions where the constant is attained) on the domain. We remark again that the existence of extremals is not trivial, due to the lack of compactness of the embedding.

For $1 < p \leqslant N$, let us consider the variational problem

$$\begin{aligned} S(\Omega,p,p) &= \lambda_p(\Omega) \\ &= \inf\Big\{\int_\Omega |u|^p + \int_\Omega |\nabla u|^p\colon u \in W^{1,p}(\Omega), \int_{\partial\Omega} |u|^p = 1\Big\}. \end{aligned} \tag{9.6}$$

This is the best constant for the trace map from $W^{1,p}(\Omega)$ into $L^p(\partial\Omega)$. Due to the compactness of the embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$, it is well known (see, for instance, [71]) that problem (9.6) has a minimizer in $W^{1,p}(\Omega)$. These extremals are weak solutions of the following problem

$$\begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda |u|^{p-2}u & \text{on } \partial\Omega, \end{cases} \tag{9.7}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, $\frac{\partial}{\partial \nu}$ is the outer unit normal derivative and if we use the normalization $\|u\|_{L^p(\partial\Omega)} = 1$, one can check that $\lambda = \lambda_p(\Omega)$, see [100].

Our first result says that $\lambda_1(\Omega)$ is the limit as $p \searrow 1$ of $\lambda_p(\Omega)$ and provides a bound for $\lambda_1(\Omega)$.

THEOREM 9.3. *We have* $\lim_{p\searrow 1} \lambda_p(\Omega) = \lambda_1(\Omega)$, *that is,* $\lim_{p\searrow 1} S(\Omega, p, p) = S(\Omega, 1, 1)$ *and* $\lambda_1(\Omega) \leqslant \min\{|\Omega|/P(\Omega), 1\}$; *here* $P(\Omega)$ *stands for the perimeter of* Ω.

Therefore, it seems natural to search for an extremal for $\lambda_1(\Omega)$ as the limit of extremals for $\lambda_p(\Omega)$ when $p \searrow 1$. Formally, if we take limit as $p \searrow 1$ in (9.7), we get

$$\begin{cases} \Delta_1 u := \operatorname{div}\left(\frac{Du}{|Du|}\right) = \frac{u}{|u|} & \text{in } \Omega, \\ \frac{Du}{|Du|} \cdot \nu = \lambda_1(\Omega)\frac{u}{|u|} & \text{on } \partial\Omega. \end{cases} \tag{9.8}$$

Hence we will look at Neumann problems involving the 1-Laplacian, $\Delta_1(u) = \operatorname{div}(Du/|Du|)$ in the context of bounded variation functions (that is a natural context for this type of problems). Following [8] (see also [10]) we give the following definition of solution of problem (9.8).

DEFINITION 9.1. A function $u \in BV(\Omega)$ is said to be a *solution* of problem (9.8) if there exists $z \in X_1(\Omega)$ with $\|z\|_\infty \leqslant 1$, $\tau \in L^\infty(\Omega)$ with $\|\tau\|_\infty \leqslant 1$ and $\theta \in L^\infty(\partial\Omega)$ with $\|\theta\|_\infty \leqslant 1$ such that

$$\operatorname{div}(z) = \tau \quad \text{in } D'(\Omega), \tag{9.9}$$

$$\tau u = |u| \quad \text{a.e. in } \Omega \quad \text{and} \quad (z, Du) = |Du| \quad \text{as measures}, \tag{9.10}$$

$$[z, \nu] = \lambda_1(\Omega)\theta \quad \text{and} \quad \theta u = |u|, \quad H^{N-1}\text{-a.e. on } \partial\Omega. \tag{9.11}$$

We shall say that Ω has the *trace-property* if there exists a vector field $z_\Omega \in L^\infty(\Omega, \mathbb{R}^N)$ with $\|z_\Omega\|_\infty \leqslant 1$ such that $\operatorname{div}(z_\Omega) \in L^\infty(\Omega)$ and $[z_\Omega, \nu] = \lambda_1(\Omega)$, H^{N-1}-a.e. on $\partial\Omega$. Our main result states that for any domain that has the trace-property, the best Sobolev trace constant, $\lambda_1(\Omega)$, is attained by a function in $L^1(\Omega)$ whose derivatives in the sense of distributions are bounded measures on Ω, that is, a function with bounded variation.

THEOREM 9.4. *Let* Ω *be a bounded open set in* $\mathbb{R}^N$ *with the trace-property. Then, there exists a nonnegative function of bounded variation which is a minimizer of the variational problem* (9.5) *and a solution of problem* (9.8).

We have that every bounded domain with $\lambda_1(\Omega) < 1$, has the trace-property. Hence we have proved that, if $\lambda_1(\Omega) < 1$ then there exists an extremal. Moreover, using results from [104], we can find examples of domains (a ball or an annulus) such that $\lambda_1(\Omega) = 1$ and verify the trace-property (and therefore they have extremals). We also have that every planar domain with a point of curvature greater than $1/2$ has $\lambda_1(\Omega) < 1$.

Now let us present some examples just to see how these definitions work.

EXAMPLE. Let $\Omega = B(0, R) \subset \mathbb{R}^N$, the ball in $\mathbb{R}^N$ centered in 0 of radius R. Then, if $z(x) := x/N$, we have $\operatorname{div}(z) = 1$ in $D'(\Omega)$ and $[z, \nu] = R/N = |\Omega|/P(\Omega)$, H^{N-1}-a.e. on $\partial\Omega$. Moreover, $\|z\|_\infty = R/N \leqslant 1$ if and only if $|\Omega|/P(\Omega) \leqslant 1$.

EXAMPLE. Let $\Omega = B(0, R) \setminus \overline{B(0, r)} \subset \mathbb{R}^N$ the annulus in $\mathbb{R}^N$ centered in 0 of radii R and r. Then, it is easy to see that if

$$z(x) := \left[\left(R^{N-1} + r^{N-1}\right) - (R + r)\frac{r^{N-1}R^{N-1}}{\|x\|^N}\right]\frac{x}{N(R^{N-1} + r^{N-1})},$$

we have $\operatorname{div}(z) = 1$ in $D'(\Omega)$ and $[z, \nu] = |\Omega|/P(\Omega)$, H^{N-1}-a.e. on $\partial\Omega$. Moreover, $\|z\|_\infty \leqslant 1$ if and only if $|\Omega|/P(\Omega) \leqslant 1$.

REMARK 9.4. Motron in [104] proves that if $\Omega = B(0, R)$ is the ball in $\mathbb{R}^N$ centered in 0 of radius R or $\Omega = B_R(0) \setminus \overline{B_r(0)}$, the annulus in $\mathbb{R}^N$ centered in 0 of radii R and r, then

$$\int_{\partial\Omega} |u| \, d\sigma \leqslant \frac{P(\Omega)}{|\Omega|} \int_\Omega |u| \, dx + \int_\Omega |\nabla u| \, dx \quad \forall u \in W^{1,1}(\Omega), \tag{9.12}$$

and equality holds in (9.12) if and only if u is constant.

From (9.12), it follows that if $\Omega = B(0, R) \subset \mathbb{R}^N$ or $\Omega = B(0, R) \setminus \overline{B(0, r)} \subset \mathbb{R}^N$, then

$$\lambda_1(\Omega) = \begin{cases} \frac{|\Omega|}{P(\Omega)} & \text{if } \frac{|\Omega|}{P(\Omega)} \leqslant 1, \\ 1 & \text{if } \frac{|\Omega|}{P(\Omega)} \geqslant 1. \end{cases}$$

Moreover, if $|\Omega|/P(\Omega) \leqslant 1$, then $u = (1/P(\Omega))\chi_\Omega$ is a minimizer of the variational problem (9.5), being the only minimizer in the case $|\Omega|/P(\Omega) = 1$, and if $|\Omega|/P(\Omega) > 1$, the variational problem (9.5) does not have minimizer.

In the following example we show that there exists bounded connected open sets Ω, with $|\Omega|/P(\Omega) < 1$, for which $\lambda_1(\Omega) < |\Omega|/P(\Omega)$.

EXAMPLE. For $\delta > 0$ and $0 < \alpha \leqslant \frac{\pi}{2}$, let

$$\Omega_{\delta,\alpha} := B(0, 1) \cup \left(B(0, 2 + \delta) \setminus \overline{B(0, 2)}\right) \cup \left\{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+ : x^2 + y^2 \leqslant 2, \operatorname{arctg}\left(\frac{y}{x}\right) < \alpha\right\}.$$

We have

$$\frac{|\Omega_{\delta,\alpha}|}{P(\Omega_{\delta,\alpha})} = \frac{\pi + \pi(\delta^2 + 4\delta) + \frac{3}{2}\alpha}{10\pi + 2\pi\delta + 2 - 3\alpha}.$$

Thus, for $0 < \delta \leqslant 1$ and $0 < \alpha \leqslant \frac{\pi}{2}$, we get $|\Omega_{\delta,\alpha}|/P(\Omega_{\delta,\alpha}) < 1$. Now, if we take $u := \chi_{B(0,2+\delta)\setminus\overline{B(0,2)}}$,

$$\lambda_1(\Omega_{\delta,\alpha}) \leqslant \frac{\int_{\Omega_{\delta,\alpha}} |Du|\,\mathrm{d}x + \int_{\Omega_{\delta,\alpha}} |u|\,\mathrm{d}x}{\int_{\partial\Omega_{\delta,\alpha}} |u|\,\mathrm{d}\sigma} = \frac{2\alpha + \pi(\delta^2 + 4\delta)}{8\pi + 2\delta\pi - 2\alpha}.$$

Then, it is easy to see that for δ and α small enough, we get $\lambda_1(\Omega_{\delta,\alpha}) < |\Omega_{\delta,\alpha}|/P(\Omega_{\delta,\alpha}) < 1$.

In the next example we will see that even we can take Ω convex.

EXAMPLE. Let Ω be the set in $\mathbb{R}^2$ with boundary the isosceles triangle with height k, base of length $2a$ and the two equal sides of length l. Let t the angle between the height and one of the equal side. Then

$$\frac{|\Omega|}{P(\Omega)} = \frac{ak}{2(a+l)} = \frac{ak}{a2(a + a/\sin t)} = \frac{k\sin t}{2(1+\sin t)}.$$

Let $E \subset \Omega$ be the set with boundary the isosceles triangle with height $k - r$, base of length $2b$ and the two equal sides of length $\tilde{l}$. Then, if $u := \chi_E$, we have

$$\begin{aligned}\lambda_1(\Omega) &\leqslant \frac{\int_\Omega |Du|\,\mathrm{d}x + \int_\Omega |u|\,\mathrm{d}x}{\int_{\partial\Omega} |u|\,\mathrm{d}\sigma} = \frac{2b + b(k-r)}{2\tilde{l}} = \frac{b(k+2-r)}{2b/\sin t} \\ &= \frac{\sin t}{2}(k+2-r).\end{aligned}$$

Hence, $\lambda_1(\Omega) < |\Omega|/P(\Omega) < 1$ if

$$k < \min\left\{(r-2)\frac{1+\sin t}{\sin t}, 2\frac{1+\sin t}{\sin t}\right\}.$$

Now, obviously, we can find k, r and α satisfying the above inequality, and consequently, we can obtain a convex, bounded open set Ω satisfying $\lambda_1(\Omega) < |\Omega|/P(\Omega) < 1$.

EXAMPLE. For $0 < \rho < r$ and $\delta > 0$, let

$$\Omega_{\rho,r,\delta} := B(0,\rho) \cup \big(B(0, r+\delta) \setminus B(0,r)\big) \subset \mathbb{R}^2.$$

We have

$$\frac{|\Omega_{\rho,r,\delta}|}{P(\Omega_{\rho,r,\delta})} = \frac{\delta^2 + \rho^2 + 2r\delta}{2(2r + \delta + \rho)}.$$

If we take $u := \chi_{B(0,\rho)}$ and $v := \chi_{B(0,r+\delta)\setminus\overline{B(0,r)}}$, then

$$\Phi(u) := \frac{\int_{\Omega_{\rho,r,\delta}} |Du| + \int_{\Omega_{\rho,r,\delta}} |u|}{\int_{\partial\Omega_{\rho,r,\delta}} |u|} = \frac{\rho}{2}$$

and

$$\Phi(v) := \frac{\int_{\Omega_{\rho,r,\delta}} |Dv| + \int_{\Omega_{\rho,r,\delta}} |v|}{\int_{\partial\Omega_{\rho,r,\delta}} |v|} = \frac{\delta}{2}.$$

Suppose that $0 < \rho < \delta \leqslant 2$. If we consider the vector field z in $\Omega_{\rho,r,\delta}$ defined by

$$z(x,y) := \begin{cases} \frac{\delta(x,y)}{2\rho} & \text{if } (x,y) \in B_\rho(0), \\ \left[\delta - (\delta + 2r)\frac{r(r+\delta)}{\|(x,y)\|}\right]\frac{(x,y)}{2(2r+\delta)} & \text{if } (x,y) \in B_{r+\delta}(0) \setminus B_r(0), \end{cases}$$

we have $\|z\|_\infty \leqslant 1$, $\operatorname{div}(z) = \tau$ in $D'(\Omega_{\rho,r,\delta})$, with $\tau = \frac{\delta}{\rho}\chi_{\partial B(0,\rho)} + \chi_{B(0,r+\rho)\setminus B(0,r)}$ and $[z,\nu] = \delta/2$, H^1-a.e. on $\partial\Omega_{\rho,r,\rho}$. Now, $\lambda_1(\Omega) \leqslant \Phi(u) = \rho/2$. Hence, $\lambda_1(\Omega) < \delta/2$. If $\delta = \rho$, we have

$$\lambda_1(\Omega_{\rho,r,\rho}) \leqslant \Phi(u) = \Phi(v) = \Phi(\chi_{\Omega_{\rho,r,\rho}}) = \frac{|\Omega_{\rho,r,\rho}|}{P(\Omega_{\rho,r,\rho})} = \frac{\rho}{2}.$$

Suppose that $\rho \leqslant 1$. Then, if we consider the vector field z in $\Omega_{\rho,r,\rho}$ defined by

$$z(x,y) := \begin{cases} \frac{(x,y)}{2} & \text{if } (x,y) \in B(0,\rho), \\ 0 & \text{if } (x,y) \in B(0,r+\rho) \setminus B(0,r), \end{cases}$$

we have $\operatorname{div}(z) = \chi_{B_\rho(0)}$ in $D'(\Omega_{\rho,r,\rho})$, and $[z,\nu] = \frac{\rho}{2}\chi_{\partial B(0,\rho)}$, H^1-a.e. on $\partial\Omega_{\rho,r,\rho}$. Therefore, u is a solution of the problem

$$\begin{cases} \Delta_1 w := \operatorname{div}\left(\frac{Dw}{|Du|}\right) = \frac{w}{|w|} & \text{in } \Omega_{\rho,r,\rho}, \\ \frac{Dw}{|Dw|} \cdot \nu = \frac{\rho}{2}\frac{w}{|w|} & \text{on } \partial\Omega_{\rho,r,\rho}. \end{cases} \tag{9.13}$$

Now, if we consider the vector field z in $\Omega_{\rho,r,\rho}$ defined by

$$z(x,y) := \begin{cases} 0 & \text{if } (x,y) \in B(0,\rho), \\ \left[\rho - (\rho + 2r)\frac{r(r+\rho)}{\|(x,y)\|}\right]\frac{(x,y)}{2(2r+\rho)} & \text{if } (x,y) \in B(0,r+\rho) \setminus B(0,r), \end{cases}$$

we have $\operatorname{div}(z) = \chi_{B(0,r+\rho)\setminus B(0,r)}$ in $D'(\Omega_{\rho,r,\rho})$ and $[z,\nu] = \frac{\rho}{2}\chi_{\partial(B(0,r+\rho)\setminus B(0,r))}$, H^1-a.e. on $\partial\Omega_{\rho,r,\rho}$. Therefore, v is also a solution of the problem (9.13). Moreover, in this case also $\chi_{\Omega_{\rho,r,\rho}}$ is a solution of the problem (9.13).

PROBLEM. Is $\lambda_1(\Omega_{\rho,r,\rho}) = \rho/2$?

The next example shows that there are bounded connected open sets Ω for which $\lambda_1(\Omega) < 1$ and $|\Omega|/P(\Omega) > 1$.

EXAMPLE. Let $\Omega :=]-k,0[\times]0,k[\cup\{0\}\times]0,\delta[\cup]0,1[\times]0,\delta[\subset\mathbb{R}^2$ be. Then

$$\frac{|\Omega|}{P(\Omega)} = \frac{k^2+\delta}{4k+2} > 1 \quad \text{if } k > 2+\sqrt{6-\delta}.$$

Now, if we take $u := \chi_{]0,1[\times]0,\delta[}$, we have

$$\lambda_1(\Omega) \leqslant \frac{\int_\Omega |Du| + \int_\Omega |u|}{\int_{\partial\Omega} |u|} = \frac{2\delta}{2+\delta} < 1 \quad \Longleftrightarrow \quad 0 < \delta < 2.$$

Therefore, for instance, if $\delta = 1$ and $k = 5$, we have $|\Omega|/P(\Omega) > 1$ and $\lambda_1(\Omega) < 1$.

In [104] it is considered the A–B program for $W^{1,1}(\Omega)$ and $L^1(\partial\Omega)$, that is, to find the best possible constants A and B such that

$$\int_{\partial\Omega} |u|\,\mathrm{d}\sigma \leqslant A\int_\Omega |\nabla u|\,\mathrm{d}x + B\int_\Omega |u|\,\mathrm{d}x.$$

It is proved that the best possible A is 1, and the best possible B is $P(\Omega)/|\Omega|$.

✓ In [1,108,109] (see also [26]) it is studied a problem with critical nonlinearity both in the equation and in the boundary condition, namely, they look for positive solutions of

$$\begin{cases} -\Delta u = |u|^{(N+2)/(N-2)} + f(x,u) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = u^{N/(N-2)} + g(x,u) & \text{on } \partial\Omega. \end{cases}$$

Under adequate hypotheses on f and g the problem admits a positive solution.

10. Dependence of the best Sobolev trace constant on the exponents

In this section we focus our attention on the dependence of the best Sobolev trace constant on the involved exponents, p, q.

For any $1 \leqslant p \leqslant \infty$, we define the Sobolev trace conjugate as

$$p_* = \begin{cases} \frac{p(N-1)}{N-p} & \text{if } p < N, \\ \infty & \text{if } p \geqslant N. \end{cases}$$

If $1 \leqslant q \leqslant p_*$ (with strict second inequality if $p = N$), we have the immersion $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$ and hence the following inequality holds

$$S\|u\|_{L^q(\partial\Omega)} \leqslant \|u\|_{W^{1,p}(\Omega)}$$

for all $u \in W^{1,p}(\Omega)$. As before, the best constant for this embedding is the largest S such that the above inequality holds, that is,

$$S(\Omega, p, q) = \inf_{u \in W^{1,p}(\Omega) \setminus W_0^{1,p}(\Omega)} \frac{(\int_\Omega |\nabla u|^p + |u|^p \, \mathrm{d}x)^{1/p}}{(\int_{\partial\Omega} |u|^q \, \mathrm{d}\sigma)^{1/q}}. \tag{10.1}$$

Moreover, if $1 \leqslant q < p_*$ the embedding is compact and as a consequence we have the existence of extremals, that is, functions where the infimum is attained, see [71]. These extremals are weak solutions of the following problem

$$\begin{cases} \Delta_p u = |u|^{p-2} u & \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \lambda |u|^{q-2} u & \text{on } \partial\Omega. \end{cases} \tag{10.2}$$

Using [119] and [123] we can assume that the extremals are positive, $u > 0$, in Ω. In the special case $p = q$, problem (10.2) becomes a nonlinear eigenvalue problem that was studied in [71,100]. From now on, let us call $u_{p,q}$ an extremal corresponding to the exponents (p, q).

The main purposes of this section are to study the possibility of a uniform bound (independent of (p, q)) on $S(\Omega, p, q)$ and to study the limit behavior of the best Sobolev trace constants $S(\Omega, p, q)$ as $p \to +\infty$ and as $q \to +\infty$ and look at the limit cases $p = \infty$, $1 \leqslant q \leqslant \infty$ and $N < p < \infty$, $q = \infty$. The first result of [64] is the following.

THEOREM 10.1. *Given $A \subset \{(p, q)\colon 1 \leqslant p \leqslant \infty, 1 \leqslant q \leqslant p_*\}$, a set of admissible (p, q), there exist constants C_1 and C_2 independent of $(p, q) \in A$ such that*

$$C_1 \leqslant S(\Omega, p, q) \leqslant C_2$$

if and only if A verifies the following property: there is no sequence $(p_n, q_n) \in A$ with $p_n \to N$ and $q_n \to \infty$.

Notice that Theorem 10.1 says that we can obtain a uniform bound for $S(\Omega, p, q)$ on A as long as $(p, q) \in A$ stays away from the point (N, ∞). Observe that the upper bound, $S(\Omega, p, q) \leqslant C_2$, follows easily by taking $u \equiv 1$ in (10.1) and holds even if we are close to (N, ∞). The main difficulty arises in the proof of the lower bound. This is due to the fact that there exist functions in $W^{1,N}(\Omega)$ that do not belong to $L^\infty(\partial\Omega)$.

As we mentioned before, one of our concerns is to analyze the case $p = \infty$ with $1 \leqslant q \leqslant \infty$, that is, the immersion $W^{1,\infty}(\Omega) \hookrightarrow L^q(\partial\Omega)$. The best constant is given by

$$S(\Omega, \infty, q) = \inf_{u \in W^{1,\infty}(\Omega) \setminus W_0^{1,\infty}(\Omega)} \left\{ \frac{\|\nabla u\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\Omega)}}{\|u\|_{L^q(\partial\Omega)}} \right\}.$$

From this expression it is easy to see that $S(\Omega,\infty,q) = 1/|\partial\Omega|^{1/q}$ and $S(\Omega,\infty,\infty) = 1$, with extremal $u_{\infty,q} = u_{\infty,\infty} \equiv 1$ in both cases (we normalize the extremals according to $\|u_{\infty,q}\|_{L^\infty(\partial\Omega)} = \|u_{\infty,\infty}\|_{L^\infty(\partial\Omega)} = 1$). We prove that $S(\Omega,\infty,\infty) = 1$ is the limit of $S(\Omega,p,q)$ as $p,q \to \infty$ and also $S(\Omega,\infty,q)$ is the limit of $S(\Omega,p,q)$ when $p \to \infty$.

THEOREM 10.2. *Let $S(\Omega,p,q)$ be the best Sobolev trace constant and $u_{p,q}$ be any extremal normalized such that $\|u_{p,q}\|_{L^\infty(\partial\Omega)} = 1$. Then*

$$\lim_{p,q\to\infty} S(\Omega,p,q) = S(\Omega,\infty,\infty) = 1$$

and, for any $1 < r < \infty$, as $p,q \to \infty$,

$$u_{p,q} \rightharpoonup u_{\infty,\infty} \equiv 1 \quad \textit{weakly in } W^{1,r}(\Omega),$$
$$u_{p,q} \to u_{\infty,\infty} \equiv 1 \quad \textit{strongly in } C^\alpha\big(\overline{\Omega}\big).$$

Moreover, for fixed $1 \leqslant q < \infty$,

$$\lim_{p\to\infty} S(\Omega,p,q) = S(\Omega,\infty,q) = \frac{1}{|\partial\Omega|^{1/q}}$$

and, for any $1 < r < \infty$, as $p \to \infty$,

$$u_{p,q} \rightharpoonup u_{\infty,q} \equiv 1 \quad \textit{weakly in } W^{1,r}(\Omega),$$
$$u_{p,q} \to u_{\infty,q} \equiv 1 \quad \textit{strongly in } C^\alpha\big(\overline{\Omega}\big).$$

The limit $q \to \infty$ with $p > N$ fixed is more subtle since we do not know a priori which is the extremal for the limit case. However we find an equation for the limit extremal.

THEOREM 10.3. *Let $p > N$. Then*

$$\lim_{q\to\infty} S(\Omega,p,q) = S(\Omega,p,\infty)$$

and, up to subsequences, as $q \to \infty$,

$$u_{p,q} \rightharpoonup u_{p,\infty} \quad \textit{weakly in } W^{1,p}(\Omega),$$
$$u_{p,q} \to u_{p,\infty} \quad \textit{strongly in } C^\alpha\big(\overline{\Omega}\big).$$

Moreover, there exists a measure $\mu \in C(\partial\Omega)^*$ with $\mu(\{u_{p,\infty} = 1\}) = 1$ such that $u_{p,\infty}$ is a weak solution of

$$\begin{cases} \Delta_p u = |u|^{p-2}u & \textit{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = S^p(\Omega,p,\infty)\mu\chi_{\{u\equiv 1\}} & \textit{on } \partial\Omega. \end{cases}$$

We observe that $W^{1,N}(\Omega) \not\hookrightarrow L^\infty(\partial\Omega)$. Hence we expect that the best constant $S(\Omega, p, q)$ goes to zero as $(p, q) \to (N, \infty)$. This is the content of the next result.

THEOREM 10.4. *The best constant $S(\Omega, p, q)$ goes to zero as $(p, q) \to (N, \infty)$, and moreover, for any $\alpha < (N-1)/N$, there exists a constant C such that*

$$S(\Omega, p, q) \leqslant C \max\left\{(p-N)_+, \frac{1}{q}\right\}^\alpha.$$

11. Elliptic systems with nonlinear boundary conditions

In this section we study existence and multiplicity results for elliptic systems coupled through nonlinear boundary conditions. We divide our exposition according to dealing with systems with or without variational structure.

Existence results for nonlinear elliptic systems have deserved a great deal of interest in recent years, in particular, when the nonlinear term appears as a source in the equation, complemented with Dirichlet boundary conditions. Problems without variational structure can be treated via fixed point arguments. There are two main classes of systems that can be treated variationally, Hamiltonian and gradient systems. The system is called Hamiltonian if there exists a function H such that the coupling nonlinearities verify $H_v = f$ and $H_u = g$, and is called gradient if there exists F with $\nabla F = (f, g)$. For these type of results see the survey [44].

Nonvariational systems. First, we address the existence problem without any variational assumption on f and g. We study the existence via topological methods of nonnegative solutions of the following elliptic system,

$$\begin{cases} \Delta u = u & \text{in } \Omega, \\ \Delta v = v, \end{cases} \tag{11.1}$$

with nonlinear coupling at the boundary given by

$$\begin{cases} \frac{\partial u}{\partial \nu} = f(x, u, v) & \text{on } \partial\Omega, \\ \frac{\partial v}{\partial \nu} = g(x, u, v). \end{cases} \tag{11.2}$$

Here $f, g : \partial\Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ are smooth positive functions with $f(x, 0, 0) = g(x, 0, 0) = 0$. Moreover, we deal with the “superlinear” case (see detailed assumptions (H1)–(H3) on f and g).

The topological method (a fixed point argument) we apply here has been used by several authors to deal with problems without variational structure and, as in our case, they were forced to impose some growth restrictions on f and g.

The proofs need some knowledge on the following eigenvalue problem that was our main subject of study in previous sections,

$$\begin{cases} \Delta\varphi = \varphi & \text{in } \Omega, \\ \frac{\partial\varphi}{\partial\nu} = \lambda\varphi & \text{on } \partial\Omega. \end{cases} \tag{11.3}$$

The main difficulty in carrying out the fixed point argument is to obtain L^∞ a priori bounds for (11.1)–(11.2). This difficulty is overcome by means of the blow-up technique introduced by Gidas and Spruck [83]. The key ingredient to make this technique work is a Liouville-type theorem for the system

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^N_+, \\ \Delta v = 0, \end{cases} \tag{11.4}$$

with boundary conditions

$$\begin{cases} \frac{\partial u}{\partial\nu} = v^p & \text{on } \partial\mathbb{R}^N_+, \\ \frac{\partial v}{\partial\nu} = u^q. \end{cases} \tag{11.5}$$

In [87], see also Section 4, the author studies the single equation

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^N_+, \\ \frac{\partial u}{\partial\nu} = u^p & \text{on } \partial\mathbb{R}^N_+. \end{cases} \tag{11.6}$$

There he proves that if $1 < p < N/(N-2)$ there is no nontrivial nonnegative classical solution of (11.6). In [88] such nonexistence-type result is applied to compute the blow-up rate of a parabolic problem.

One can adapt the moving plane technique to deal with the system (11.4)–(11.5) and obtain the same type of result with similar restrictions on the exponents (see Theorems 3.5 and 3.6).

We remark that we can also deal with the semilinear case,

$$\begin{cases} -\Delta u + u = r(x,u,v) & \text{in } \Omega, \\ -\Delta v + v = s(x,u,v), \end{cases}$$
$$\begin{cases} \frac{\partial u}{\partial\nu} = f(x,u,v) & \text{on } \partial\Omega, \\ \frac{\partial v}{\partial\nu} = g(x,u,v), \end{cases}$$

using the same ideas. Since the main novelty here comes from the boundary terms, we present our results for (11.1)–(11.2).

Let us give an idea of the arguments, see [70] for details. We want to apply the following fixed point theorem that can be found, for instance, in [43].

THEOREM 11.1. *Let C be a cone in a Banach space X and $S : C \to C$ a compact mapping such that $S(0) = 0$. Assume that there are real numbers $0 < r < R$ and $t > 0$ such that*

(1) *$x \neq tSx$ for $0 \leqslant t \leqslant 1$ and $x \in C$, $\|x\| = r$, and*

(2) *there exists a compact mapping $H : \overline{B}_R \times [0,\infty) \to C$ (where $B_\rho = \{x \in C; \|x\| < \rho\}$) such that*

(a) *$H(x,0) = S(x)$ for $\|x\| = R$,*

(b) *$H(x,t) \neq x$ for $\|x\| = R$ and $t > 0$,*

(c) *$H(x,t) = x$ has no solution $x \in \overline{B}_R$ for $t \geqslant t_0$.*

Then S has a fixed point in $U = \{x \in C; r < \|x\| < R\}$.

To apply this theorem, we proceed as follows. Consider the space

$$X = \{(u,v)\colon\ u, v \in C(\overline{\Omega})\}$$

with the norm $\|(u,v)\| = \|u\|_\infty + \|v\|_\infty$, which makes it a Banach space. Let $S : X \to X$ be the solution operator defined by $S(\phi,\psi) = (u,v)$, where (u,v) is the solution of

$$\begin{cases} \Delta u = u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = f(x,\phi,\psi) & \text{on } \partial\Omega, \end{cases}$$
$$\begin{cases} \Delta v = v & \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} = g(x,\phi,\psi) & \text{on } \partial\Omega. \end{cases}$$

We observe that a fixed point of S is a solution of (11.1)–(11.2).

Now, let us see that S verifies the hypothesis of the Theorem 11.1. By standard regularity theory [84], it follows that u and v are C^α, hence S is a compact operator. As $f(x,0,0) = g(x,0,0) = 0$ we have, by Hopf lemma, that $S(0) = 0$. Let C be the cone $C = \{(u,v) \in X; u \geqslant 0, v \geqslant 0\}$. It follows from the maximum principle that $S(C) \subset C$.

To verify (1) in Theorem 11.1 we argue by contradiction. Let us assume that, for every $r > 0$, there exists a $0 \leqslant t \leqslant 1$ and a pair (U,V) such that

$$\begin{cases} \Delta U = U & \text{in } \Omega, \\ \Delta V = V, \end{cases} \tag{11.7}$$

with

$$\begin{cases} \frac{\partial U}{\partial \nu} = tf(x,U,V) & \text{on } \partial\Omega, \\ \frac{\partial V}{\partial \nu} = tg(x,U,V). \end{cases}$$

We multiply the first equation of (11.7) by φ_1, the first eigenfunction of (11.3), and we obtain

$$\begin{aligned} 0 &= \int_\Omega (\Delta U - U)\varphi_1 \,\mathrm{d}x \\ &= \int_\Omega U(\Delta\varphi_1 - \varphi_1)\,\mathrm{d}x + t\int_{\partial\Omega} f(x,U,V)\varphi_1 \,\mathrm{d}\sigma - \int_{\partial\Omega} U \frac{\partial\varphi_1}{\partial\nu}\,\mathrm{d}\sigma. \end{aligned}$$

Hence

$$0 = t \int_{\partial\Omega} f(x, U, V)\varphi_1 \, dx - \lambda_1 \int_{\partial\Omega} U\varphi_1 \, d\sigma.$$

We assume that f and g are "superlinear", in fact, we make the following hypothesis that we call (H1),

$$\begin{aligned} &f(x, U, V) \leqslant \varepsilon(U + V) \quad \text{and} \\ &g(x, U, V) \leqslant \varepsilon(U + V) \quad \text{for small } \|(U, V)\|. \end{aligned} \tag{H1}$$

Using (H1) we obtain

$$\lambda_1 \int_{\partial\Omega} U\varphi_1 \, d\sigma \leqslant \varepsilon t \int_{\partial\Omega} (U + V)\varphi_1 \, d\sigma.$$

Analogously, for V we get

$$\lambda_1 \int_{\partial\Omega} V\varphi_1 \, d\sigma \leqslant \varepsilon t \int_{\partial\Omega} (U + V)\varphi_1 \, d\sigma.$$

Adding both inequalities we conclude $\lambda_1 \leqslant 2\varepsilon$, a contradiction if ε verifies $\varepsilon < \lambda_1/2$.

To see (2) we define H by $H((\phi, \psi), t) = S(\phi + t, \psi + t)$. Clearly (a) holds. To see (c) we have to impose any of the following conditions, we call this (H2):

(H2.i) There exists real numbers $\mu > \lambda_1$ and $C > 0$ such that

$$f(x, u, v) \geqslant \mu u - C$$

uniformly in $x \in \overline{\Omega}$ and $v \in \mathbb{R}_+$.

(H2.ii) There exists real numbers $\mu > \lambda_1$ and $C > 0$ such that

$$g(x, u, v) \geqslant \mu v - C$$

uniformly in $x \in \overline{\Omega}$ and $u \in \mathbb{R}_+$.

(H2.iii) There exists real numbers $\mu > \lambda_1$ and $C > 0$ such that

$$f(x, u, v) + g(x, u, v) \geqslant \mu(u + v) - C$$

uniformly in $x \in \overline{\Omega}$.

For instance, assume that (H2.i) holds. Then, for t large enough, we have

$$f(x, u + t, v + t) \geqslant \mu(u + t) - C \geqslant \mu u$$

with $\mu > \lambda_1$ and hence, for t large, u is a nonnegative solution of

$$\begin{cases} \Delta u = u & \text{in } \Omega, \\ \frac{\partial u}{\partial n} \geqslant \mu u & \text{on } \partial\Omega, \end{cases}$$

which contradicts the fact that λ_1 is the first eigenvalue.

The other cases can be handled in a similar way.

Finally, condition (b) is an immediate consequence of an a priori bound for the system

$$\begin{cases} \Delta u = u & \text{in } \Omega, \\ \Delta v = v, \end{cases} \tag{11.8}$$

with

$$\begin{cases} \frac{\partial u}{\partial \nu} = f(x, u+t, v+t) & \text{on } \partial\Omega, \\ \frac{\partial v}{\partial \nu} = g(x, u+t, v+t). \end{cases} \tag{11.9}$$

Hence we have proved our existence result, provided we have an a priori L^∞ bound for (11.8)–(11.9).

THEOREM 11.2. *Let f and g satisfy* (H1)–(H2). *If there exists a constant C such that for every solution (u, v) of* (11.1)–(11.2) *it holds $\|u\|_\infty, \|v\|_\infty \leqslant C$ then the system* (11.1)–(11.2) *has a nontrivial positive solution*.

Now our aim is to prove that, under further conditions on f and g, the nonnegative solutions of (11.1)–(11.2) are bounded in L^∞, so Theorem 11.2 applies. To do so, we apply the blow-up technique introduced by Gidas and Spruck [83]. We argue by contradiction. Assume that there is no such a priori bound. Then, there exists a sequence of positive solutions (u_n, v_n) with $\max\{\|u_n\|_\infty, \|v_n\|_\infty\} \to \infty$. Let β_1, β_2 be two positive numbers to be fixed. We can assume that $\|u_n\|_\infty \to \infty$ and $\|u_n\|_\infty^{\beta_2} \geqslant \|v_n\|_\infty^{\beta_1}$. As $\overline{\Omega}$ is compact and u_n is continuous, we can choose $x_n \in \overline{\Omega}$ such that $u_n(x_n) = \max_{\overline{\Omega}} u_n$. Moreover, it follows from the maximum principle that $x_n \in \partial\Omega$. Again, by the compactness of $\overline{\Omega}$, we can assume that $x_n \to x_0 \in \partial\Omega$. We define γ_n such that $\gamma_n^{\beta_1}\|u_n\|_\infty = 1$. This sequence, γ_n, goes to 0 as $n \to \infty$. Let

$$w_n(y) = \gamma_n^{\beta_1} u_n(\gamma_n y + x_n),$$
$$z_n(y) = \gamma_n^{\beta_2} v_n(\gamma_n y + x_n).$$

These functions are defined in $\Omega_n = \{y \in \mathbb{R}^N; \gamma_n y + x_n \in \Omega\}$. We observe that $0 \leqslant w_n$, $z_n \leqslant 1$ and $w_n(0) = 1$.

On f and g we impose the following condition (hypothesis (H3))

$$\begin{aligned} f(x, u, v) &= a(x)u^{p_{11}} + b(x)v^{p_{12}} + h_1(x, u, v), \\ g(x, u, v) &= c(x)u^{p_{21}} + d(x)v^{p_{22}} + h_2(x, u, v), \end{aligned} \tag{H3}$$

where $0 < k \leqslant a, b, c, d \leqslant K < \infty$ and h_i are lower-order terms

$$\left|h_i(x, u, v)\right| \leqslant c_i\left(1 + |u|^{\alpha_{i1}} + |v|^{\alpha_{i2}}\right).$$

Here the exponents α_{ij} satisfy $0 \leqslant \alpha_{ij} < p_{ij}$.

Hence, w_n and z_n satisfy

$$\begin{cases} \Delta w_n = \gamma_n^2 w_n & \text{in } \Omega_n, \\ \Delta z_n = \gamma_n^2 z_n, \end{cases} \tag{11.10}$$

with

$$\begin{cases} \frac{\partial w_n}{\partial \nu} = \gamma_n^{\beta_1(1-p_{11})+1} a w_n^{p_{11}} + \gamma_n^{\beta_1+1-\beta_2 p_{12}} b z_n^{p_{12}} \\ \qquad + \gamma_n^{\beta_1+1} h_1\big(\cdot, \gamma_n^{-\beta_1} w_n, \gamma_n^{-\beta_2} z_n\big) \quad \text{on } \partial\Omega_n, \\ \frac{\partial z_n}{\partial \nu} = \gamma_n^{\beta_2+1-\beta_1 p_{21}} c w_n^{p_{21}} + \gamma_n^{\beta_2(1-p_{22})+1} d z_n^{p_{22}} \\ \qquad + \gamma_n^{\beta_2+1} h_2\big(\cdot, \gamma_n^{-\beta_1} w_n, \gamma_n^{-\beta_2} z_n\big). \end{cases} \tag{11.11}$$

Now we want to pass to the limit in (11.10)–(11.11), so we need to know what happens with the coefficients of the leading terms.

We distinguish two cases in terms of p_{ij}, the weakly coupled and the strongly coupled case.

1. *Weakly coupled case.* We say that the system is weakly coupled if there exists β_1, β_2 such that

$$\beta_1(1-p_{11})+1=0, \qquad \beta_1+1-\beta_2 p_{12} > 0,$$
$$\beta_2(1-p_{22})+1=0, \qquad \beta_2+1-\beta_1 p_{21} > 0.$$

Thus, in this case, we choose

$$\beta_1 = \frac{1}{p_{11}-1}, \qquad \beta_2 = \frac{1}{p_{22}-1}.$$

These conditions impose

$$1 < p_{11}, p_{22}, \qquad p_{12} < \frac{p_{11}(p_{22}-1)}{p_{11}-1} \quad \text{and} \quad p_{21} < \frac{p_{22}(p_{11}-1)}{p_{22}-1}.$$

2. *Strongly coupled case.* We say that the system is strongly coupled if there exist β_1, β_2 such that

$$\beta_1(1-p_{11})+1>0, \qquad \beta_1+1-\beta_2 p_{12} = 0,$$
$$\beta_2(1-p_{22})+1>0, \qquad \beta_2+1-\beta_1 p_{21} = 0.$$

Thus, in this case, we choose

$$\beta_1 = \frac{p_{12}+1}{p_{12}p_{21}-1}, \qquad \beta_2 = \frac{p_{21}+1}{p_{12}p_{21}-1}.$$

These conditions impose

$$1 < p_{21}, p_{12}, \qquad p_{11} < 1 + \frac{p_{21}p_{12} - 1}{p_{12} + 1} \quad \text{and} \quad p_{22} < 1 + \frac{p_{21}p_{12} - 1}{p_{21} + 1}.$$

First we deal with the weakly coupled case. As w_n, z_n are C^α (see [84]) and f, g are smooth, we have that w_n, z_n are uniformly bounded in $C^{1+\alpha}$. Hence, by standard Schauder theory [84], we obtain that w_n, z_n are uniformly bounded in $C^{2+\alpha}$. Using a compactness argument we can assume that $(w_n, z_n) \to (w, z)$ in $C^{2+\beta} \times C^{2+\beta}$ with $\beta < \alpha$. We observe that the domains Ω_n approaches to $\mathbb{R}^N_+$. Therefore, passing to the limit, we obtain a nontrivial nonnegative bounded solution w of

$$\begin{cases} \Delta w = 0 & \text{in } \mathbb{R}^N_+, \\ \frac{\partial w}{\partial \nu} = a(x_0) w^{p_{11}} & \text{on } \partial\mathbb{R}^N_+. \end{cases} \tag{11.12}$$

As we mentioned in Section 4, Hu in [87] proved the following nonexistence theorem.

THEOREM 11.3. *The only nonnegative classical solution of* (11.12) *is* $w \equiv 0$ *when* $1 < p_{11} < N/(N-2)$ (p_{11} *is subcritical*) *if* $N \geqslant 3$ *or* $0 < p_{11}$ *if* $N = 2$.

Using Theorem 11.3 we get a contradiction and this proves the a priori bound in the weakly coupled case. In summary, we have proved the following result.

THEOREM 11.4. *Assume that the system* (11.1)–(11.2) *satisfy* (H3) *and is weakly coupled. If* $1 < p_{11}, p_{22} < N/(N-2)$ $(N \geqslant 3)$ *or* $0 < p_{11}, p_{22}$ $(N = 2)$ *then there exists a constant* C *such that every nonnegative solution verifies* $\|u\|_\infty, \|v\|_\infty \leqslant C$.

Next, we deal with the strongly coupled case. Passing to the limit as in the previous case, we obtain a nontrivial, nonnegative solution of

$$\begin{cases} \Delta w = 0 & \text{in } \mathbb{R}^N_+, \\ \Delta z = 0, \end{cases} \tag{11.13}$$

with boundary conditions

$$\begin{cases} \frac{\partial w}{\partial \nu} = b(x_0) z^{p_{12}} & \text{on } \partial\mathbb{R}^N_+, \\ \frac{\partial z}{\partial \nu} = c(x_0) w^{p_{21}}. \end{cases} \tag{11.14}$$

For this problem, using the moving planes technique, we have the following Liouville-type theorems, see [71],

THEOREM 11.5. *Suppose* $N \geqslant 3$ *and* $p_{12}, p_{21} \leqslant N/(N-2)$ *but not both equal to* $N/(N-2)$, *with* $p_{12}p_{21} > 1$. *Let* (w, z) *be a classical nonnegative solution of* (11.13)–(11.14), *then* $w \equiv z \equiv 0$.

For the case $N = 2$ we have to suppose that w or z is bounded, and we obtain the same conclusion with no restriction on the exponents p_{12}, p_{21}. More precisely, we have the following theorem.

THEOREM 11.6. *Let $N = 2$ and $p_{12}, p_{21} > 0$. Let (w, z) be a classical nonnegative solution of* (11.13)–(11.14) *with w bounded, then $w \equiv z \equiv 0$.*

Again, applying Theorems 11.5 and 11.6 we get a contradiction in the strongly coupled case. In summary, we have proved the following theorem.

THEOREM 11.7. *Assume that the system* (11.1)–(11.2) *satisfies* (H3) *and is strongly coupled. If $1 < p_{12}p_{21}$ and $p_{12}, p_{21} \leqslant N/(N-2)$ but not both equal $(N \geqslant 3)$ or $0 < p_{12}, p_{21}$ $(N = 2)$, then there exists a constant C such that every nonnegative solution (u, v) verifies $\|u\|_\infty, \|v\|_\infty \leqslant C$.*

✓ Now we assume that the nonlinearities in the boundary conditions have some variational structure. As we have mentioned, there are two main classes of systems that can be treated variationally, Hamiltonian and gradient systems. First, we deal with a Hamiltonian problem. As we have already seen, problems with no variational structure can be treated via fixed point arguments.

Hamiltonian elliptic systems. We study the existence of nontrivial solutions of the elliptic system

$$\begin{cases} \Delta u = u & \text{in } \Omega, \\ \Delta v = v, \end{cases} \tag{11.15}$$

with nonlinear coupling at the boundary given by

$$\begin{cases} \frac{\partial u}{\partial \nu} = H_v(x, u, v) & \text{on } \partial\Omega, \\ \frac{\partial v}{\partial \nu} = H_u(x, u, v). \end{cases} \tag{11.16}$$

Here $H : \partial\Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a smooth positive function (say C^1) with growth control on H and its first derivatives.

The results obtained for this Hamiltonian problem are strongly inspired by [45]. There the authors study

$$\begin{cases} -\Delta u = H_v(x, u, v) & \text{in } \Omega, \\ -\Delta v = H_u(x, u, v), \end{cases}$$

complemented with Dirichlet boundary conditions. They prove existence of strong solutions using the same variational arguments we use here and, as in our case, they were forced to impose similar growth restrictions on H.

The crucial part here is to find the proper functional setting for (11.15)–(11.16) that allows us to treat our problem variationally. We accomplish this by defining a self-adjoint operator that takes into account the boundary conditions together with the equations and considering its fractional powers that verify a suitable "integration by parts" formula. Once we have done this, the proof follows the steps used in [45]. For the proof we use a linking theorem in a version of [62]. Linking theorems have been proved to be a useful tool in order to obtain existence result of elliptic problems.

The precise assumptions on the Hamiltonian H are

$$\big|H(x,u,v)\big| \leqslant C\big(|u|^{p+1}+|v|^{q+1}+1\big), \tag{11.17}$$

and for $r>0$ small, if $|(u,v)|\leqslant r$,

$$\big|H(x,u,v)\big| \leqslant C\big(|u|^{\alpha}+|v|^{\beta}\big), \tag{11.18}$$

where the exponents $p+1\geqslant\alpha>p>0$ and $q+1\geqslant\beta>q>0$ satisfy

$$1>\frac{1}{\alpha}+\frac{1}{\beta}, \tag{11.19}$$

$$\max\left\{\frac{p}{\alpha}+\frac{q}{\beta};\ \frac{q}{q+1}\frac{p+1}{\alpha}+\frac{p}{p+1}\frac{q+1}{\beta}\right\}<1+\frac{1}{N-1} \tag{11.20}$$

$$\frac{p}{p+1}\frac{q+1}{\beta}<1 \quad\text{and}\quad \frac{q}{q+1}\frac{p+1}{\alpha}<1. \tag{11.21}$$

If $N\geqslant 4$ we have to impose the additional hypothesis

$$\max\left\{\frac{p}{\alpha};\ \frac{q}{\beta};\ \frac{q}{q+1}\frac{p+1}{\alpha};\ \frac{p}{p+1}\frac{q+1}{\beta}\right\}<\frac{N+1}{2(N-1)}. \tag{11.22}$$

When $\alpha=p+1$ and $\beta=q+1$, conditions (11.19), (11.20) and (11.22) become

$$1>\frac{1}{p+1}+\frac{1}{q+1}>1-\frac{1}{N-1}, \qquad p,q\leqslant\frac{N+1}{N-3}\quad\text{if } N\geqslant 4.$$

REMARK 11.1. These hypothesis, (11.19)–(11.22), imply that there exists s and t with $s+t=1$, $s,t>1/4$ such that

$$\frac{\alpha-p}{\alpha}>\frac{1}{2}-\frac{2s-1/2}{N-1}, \qquad \frac{\beta-q}{\beta}>\frac{1}{2}-\frac{2t-1/2}{N-1},$$

$$1-\frac{p(q+1)}{\beta(p+1)}>\frac{1}{2}-\frac{2s-1/2}{N-1}, \qquad 1-\frac{q(p+1)}{\alpha(q+1)}>\frac{1}{2}-\frac{2t-1/2}{N-1}.$$

On the derivatives of H we impose the following:

$$\begin{aligned}\left|\frac{\partial H}{\partial u}(x,u,v)\right| &\leqslant C\big(|u|^{p}+|v|^{p(q+1)/(p+1)}+1\big),\\ \left|\frac{\partial H}{\partial v}(x,u,v)\right| &\leqslant C\big(|u|^{q(p+1)/(q+1)}+|v|^{q}+1\big).\end{aligned} \tag{11.23}$$

And for R large, if $|(u,v)| \geqslant R$,

$$\frac{1}{\alpha}\frac{\partial H}{\partial u}(x,u,v)u + \frac{1}{\beta}\frac{\partial H}{\partial v}(x,u,v)v \geqslant H(x,u,v) > 0, \tag{11.24}$$

we observe that from (11.24), it follows that (see [62])

$$\left|H(x,u,v)\right| \geqslant c\big(|u|^{\alpha} + |v|^{\beta}\big) - C. \tag{11.25}$$

We have the following theorem.

THEOREM 11.8. *Let us assume that* $H : \partial\Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *verifies* (11.17)–(11.24). *Then there exists a nontrivial solution to* (11.15)–(11.16).

Next, we look for positive solutions. If we also impose

$$\frac{\partial H}{\partial u}(x,u,v), \frac{\partial H}{\partial v}(x,u,v) \geqslant 0 \quad \text{for all } u,v \geqslant 0, \tag{11.26}$$

$$\begin{aligned} &\frac{\partial H}{\partial u}(x,u,v) = 0 \quad \text{when } u = 0,\\ &\frac{\partial H}{\partial v}(x,u,v) = 0 \quad \text{when } v = 0, \end{aligned} \tag{11.27}$$

we can prove the following theorem.

THEOREM 11.9. *If* $H : \partial\Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *verifies* (11.17)–(11.24) *and* (11.26)–(11.27), *then there exists at least one positive solution to* (11.15)–(11.16).

Multiplicity. Now we study the existence of infinitely many nontrivial solutions of the elliptic system (11.15)–(11.16). To study multiplicity of solutions we are inspired by the articles [17] and [63] where the authors study a Hamiltonian system with Dirichlet boundary conditions in a bounded domain.

As before the crucial part in the nonlinear boundary conditions case, is to find the proper functional setting for (11.15)–(11.16) that allows us to treat our problem variationally. We assume the same hypotheses on the Hamiltonian H as before, but adding the following symmetry condition

$$H(x,u,v) = H(x,-u,-v). \tag{11.28}$$

We have the following theorem.

THEOREM 11.10. *Assume that* $H : \partial\Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *satisfies* (11.17)–(11.28). *Then there exists a sequence of nontrivial strong solutions* $\{u_n, v_n\}$ *to* (11.1)–(11.2) *such that* $\|u_n\|_{W^{2,(q+1)/(q)}(\Omega)} + \|v_n\|_{W^{2,(p+1)/(p)}(\Omega)} \to \infty$.

✓ In [125] and [112] the previous results were generalized by using a new version of the linking theorem with the same functional setting described below. This new version of the linking argument may have applications to other elliptic problems.

Now let us describe the functional setting that allows us to treat (11.15)–(11.16) variationally. Let us consider the space $L^2(\Omega) \times L^2(\partial\Omega)$ which is a Hilbert space with inner product, that we will denote by $\langle\cdot,\cdot\rangle$, given by

$$\big\langle(u,v),(\phi,\psi)\big\rangle = \int_\Omega u\phi\,\mathrm{d}x + \int_{\partial\Omega} v\psi\,\mathrm{d}\sigma.$$

Now, let $A: D(A) \subset L^2(\Omega)\times L^2(\partial\Omega) \to L^2(\Omega)\times L^2(\partial\Omega)$ be the operator given by

$$A(u, u|_{\partial\Omega}) = \left(-\Delta u + u, \frac{\partial u}{\partial \eta}\right),$$

where $D(A) = \{(u, u|_{\partial\Omega}) \mid u \in H^2(\Omega)\}$. We claim that $D(A)$ is dense in $L^2(\Omega) \times L^2(\partial\Omega)$. In fact, let $(f,g) \in C(\overline{\Omega}) \times C(\partial\Omega)$, take $\varepsilon > 0$ and consider $\Omega_\varepsilon = \{x \in \Omega \mid \operatorname{dist}(x,\partial\Omega) > \varepsilon\}$. Now we choose $u \in C^2(\overline{\Omega_\varepsilon})$ such that $\|u - f\|_{L^2(\Omega_\varepsilon)}$ is small. As $\partial\Omega$ is smooth, we can extend u to the whole $\overline{\Omega}$ in such a way that $u \in C^2(\overline{\Omega})$ and $\|u-g\|_{L^2(\partial\Omega)}$ is also small. As ε is arbitrary and $C(\overline{\Omega}) \times C(\partial\Omega)$ is dense in $L^2(\Omega)\times L^2(\partial\Omega)$ the claim follows.

We observe that A is invertible with inverse given by $A^{-1}(f,g) = (u, u|_{\partial\Omega})$, where u is the solution of

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega,\\ \frac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega. \end{cases}$$

By standard regularity theory, see [84], page 214, it follows that A^{-1} is bounded and compact. Therefore, $R(A) = L^2(\Omega)\times L^2(\partial\Omega)$ thus in order to see that A (and hence A^{-1}) is self-adjoint it remains to check that A is symmetric. To see this, let $u, v \in D(A)$, and by Green's formula we have

$$\begin{aligned}\langle Au, v\rangle &= \int_\Omega (-\Delta u + u)v\,\mathrm{d}x + \int_{\partial\Omega} \frac{\partial u}{\partial\nu} v\,\mathrm{d}\sigma\\ &= \int_\Omega u(-\Delta v + v)\,\mathrm{d}x + \int_{\partial\Omega} u\frac{\partial v}{\partial \nu}\,\mathrm{d}\sigma = \langle u, Av\rangle,\end{aligned}$$

therefore, A is symmetric. Moreover, A (and hence A^{-1}) is positive. In fact, let $u \in D(A)$ and using again Green's formula,

$$\langle Au, u\rangle = \int_\Omega(-\Delta u + u)u\,\mathrm{d}x + \int_{\partial\Omega}\frac{\partial u}{\partial\nu}u\,\mathrm{d}\sigma = \int_\Omega |\nabla u|^2 + u^2\,\mathrm{d}x \geqslant 0.$$

Therefore, there exists a sequence of eigenvalues $(\lambda_n) \subset \mathbb{R}$ with eigenfunctions $(\phi_n, \psi_n) \in L^2(\Omega) \times L^2(\partial\Omega)$ such that $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n \leqslant \cdots \nearrow +\infty$ and $\phi_n \in H^2(\Omega)$, $\phi_n|_{\partial\Omega} = \psi_n$,

$$\begin{cases} -\Delta\phi_n + \phi_n = \lambda_n\phi_n & \text{in } \Omega, \\ \frac{\partial\phi_n}{\partial\nu} = \lambda_n\phi_n & \text{on } \partial\Omega. \end{cases}$$

Let us consider the fractional powers of A, for $0 < s < 1$, $A^s : D(A^s) \to L^2(\Omega) \times L^2(\partial\Omega)$, A^s is given by

$$A^s u = \sum_{n=1}^{\infty} \lambda_n^s a_n(\phi_n, \psi_n),$$

where $u = \sum a_n(\phi_n, \psi_n)$. We call $E^s = D(A^s)$. E^s is a Hilbert space with inner product, that we denote by $(\cdot,\cdot)_{E^s}$, given by $(u, \phi)_{E^s} = \langle A^s u, A^s\phi\rangle$. Let us see that $E^s \subset H^{2s}(\Omega)$. In fact, if we define $A_1 : H^2(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$, $A_1 u = -\Delta u + u$, and $A_2 : H^2(\Omega) \subset D(A_2) \subset L^2(\partial\Omega) \to L^2(\partial\Omega)$, $A_2 u = \frac{\partial u}{\partial\nu}$. Then $\widetilde{A} = (A_1, A_2)$ verify $A = \widetilde{A}|_{(u,u)}$, $u \in D(A_1) \cap D(A_2)$, and hence $A^s = \widetilde{A}^s|_{(u,u)}$, $u \in D(A_1^s) \cap D(A_2^s)$. As $D(A_1) = H^2(\Omega) \subset D(A_2)$, we have $D(A_1^s) \subset D(A_2^s)$, therefore $E^s = D(A^s) = D(A_1^s)$. Now, by the results of [118], as Ω is smooth, it follows that $E^s = D(A_1^s) \subset H^{2s}(\Omega)$. So we have the following inclusions

$$E^s \hookrightarrow H^{2s}(\Omega) \hookrightarrow H^{2s-1/2}(\partial\Omega) \hookrightarrow L^p(\partial\Omega).$$

More precisely, we have the following immersion theorem.

THEOREM 11.11. *Given $s > 1/4$ and $p \geqslant 1$ so that $\frac{1}{p} \geqslant \frac{1}{2} - \frac{2s-1/2}{N-1}$ the inclusion map $i : E^s \to L^p(\partial\Omega)$ is well defined and bounded. Moreover, if above we have strict inequality, then the inclusion is compact.*

Let us now set $E = E^s \times E^t$ where $s + t = 1$, s, t given by Remark 11.1 and define $B : E \times E \to \mathbb{R}$ by $B((u, v), (\phi, \psi)) = \langle A^s u, A^t\psi\rangle + \langle A^s\phi, A^t v\rangle$. E is a Hilbert space with the usual product structure, and hence B is a bounded, bilinear, symmetric form. Therefore, there exists a unique bounded, self-adjoint, linear operator $L : E \to E$, such that $B(z, \gamma) = (Lz, \gamma)_E$. Now we define

$$Q(z) = \frac{1}{2}B(z, z) = \frac{1}{2}(Lz, z)_E = \langle A^s u, A^t v\rangle.$$

By (11.17), Remark 11.1 and Theorem 11.11, we can define $H : E \to \mathbb{R}$ as

$$H(u, v) = \int_{\partial\Omega} H(x, u, v)\, \mathrm{d}\sigma.$$

PROPOSITION 11.1. *The functional H defined above is of class C^1 and its derivative is given by*

$$H'(u,v)(\phi,\psi) = \int_{\partial\Omega} H_u(x,u,v)\phi \,\mathrm{d}\sigma + \int_{\partial\Omega} H_v(x,u,v)\psi \,\mathrm{d}\sigma. \tag{11.29}$$

Moreover, H' is compact.

PROOF. From (11.23) we have

$$\int_{\partial\Omega} \left| \frac{\partial H}{\partial u}(x,u,v)\phi \right| \mathrm{d}\sigma \leqslant C \int_{\partial\Omega} \bigl(|u|^p + |v|^{p(q+1)/(p+1)} + 1\bigr)|\phi| \,\mathrm{d}\sigma.$$

By Hölder's inequality and Theorem 11.11, we have

$$\int_{\partial\Omega} \left| \frac{\partial H}{\partial u}(x,u,v)\phi \right| \mathrm{d}\sigma \leqslant C\bigl(\|u\|_{E^s}^p + \|v\|_{E^t}^{p(q+1)/(p+1)} + 1\bigr)\|\phi\|_{E^s}.$$

In a similar way we obtain the analogous inequality for H_v.

Thus H' is well defined and bounded in E. Next, a standard argument gives that H is Fréchet differentiable with H' continuous. The fact that H' is compact comes from Theorem 11.11 (see [110] for the details). □

Now we can define the functional $F : E \to \mathbb{R}$ as

$$F(u,v) = Q(u,v) - H(u,v).$$

The functional F is of class C^1 and it has the structure needed in order to apply minimax techniques.

To end the description of the functional setting let us now give the definition of weak solution of (11.15)–(11.16).

DEFINITION 11.1. We say that $z = (u,v) \in E = E^s \times E^t$ is an (s,t)-weak solution of (11.15)–(11.16) if z is a critical point of F. In other words, for every $(\phi,\psi) \in E$, we have

$$\langle A^s u, A^t \psi\rangle + \langle A^s \phi, A^t v\rangle - \int_{\partial\Omega} H_u(x,u,v)\phi \,\mathrm{d}\sigma - \int_{\partial\Omega} H_v(x,u,v)\psi \,\mathrm{d}\sigma = 0.$$

With this definition it only remains to show that F has critical points using a linking argument [68,69,112,125].

Gradient elliptic systems. Now we assume that we are dealing with a gradient elliptic system

$$\begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } \Omega, \\ \Delta_q v = |v|^{q-2}v, \end{cases} \tag{11.30}$$

with nonlinear coupling at the boundary given by

$$\begin{cases} |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = F_u(x,u,v) & \text{on } \partial\Omega, \\ |\nabla v|^{q-2}\frac{\partial v}{\partial \nu} = F_v(x,u,v). \end{cases} \tag{11.31}$$

Here (F_u, F_v) is the gradient of some positive potential $F : \partial\Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with precise hypotheses that we state below.

As before, for (weak) solutions of (11.30)–(11.31) we understand critical points of the functional

$$\mathcal{F}(u,v) = \int_\Omega \frac{|\nabla u|^p}{p} + \frac{|u|^p}{p}\,\mathrm{d}x + \int_\Omega \frac{|\nabla v|^q}{q} + \frac{|v|^q}{q}\,\mathrm{d}x - \int_{\partial\Omega} F(x,u,v)\,\mathrm{d}\sigma. \tag{11.32}$$

The geometry of $\mathcal{F}$ is similar to the one of the functional

$$\mathcal{F}_1(u) = \frac{1}{p}\int_\Omega |\nabla u|^p + |u|^p\,\mathrm{d}x - \int_{\partial\Omega} F(x,u)\,\mathrm{d}\sigma$$

which corresponds to a single quasilinear equation with nonlinear boundary conditions. The functional $\mathcal{F}_1$ was studied in [71] where essentially the case $F(x,u) = |u|^r$ was consider. However, some interesting phenomena appear in (11.32) due to the coupling in the system (11.30)–(11.31). Our results for (11.30)–(11.31) generalize the ones in [71] both to systems and to more general potentials. In [17] the functional

$$\overline{\mathcal{F}}(u,v) = \int_\Omega |\nabla u|^p + |\nabla v|^q\,\mathrm{d}x - \int_\Omega \overline{F}(x,u,v)\,\mathrm{d}x$$

was analyzed. In this paper we extend their results to the nonlinear boundary condition case and moreover some new results are obtained. For instance, multiplicity results in the subcritical case with an oddness condition on F and mainly, existence results with critical growth.

Let us introduce the precise assumptions of F. From now on, we fix $1 < p, q < N$, and so the functional $\mathcal{F}$ will be defined in the Banach space $W^{1,p}(\Omega) \times W^{1,q}(\Omega)$. Of course, the growth of F has to be controlled in order for $\mathcal{F}$ to make sense for $(u,v) \in W^{1,p}(\Omega) \times W^{1,q}(\Omega)$. According to the Sobolev trace embedding, we impose

$$\left|F(x,u,v)\right| \leqslant C\left(1 + |u|^{p_*} + |v|^{q_*}\right), \tag{F$_1$}$$

where $p_* = p(N-1)/(N-p)$ and $q_* = q(N-1)/(N-q)$ are the critical Sobolev trace exponents and C is some positive constant. With (F$_1$), as $W^{1,p}(\Omega) \to L^{p_*}(\partial\Omega)$ and $W^{1,q}(\Omega) \to L^{q_*}(\partial\Omega)$ by the Sobolev trace theorem, we have that $\mathcal{F}$ is well defined.

In order to apply variational techniques, we need the functional $\mathcal{F}$ to be C^1. To this end, (F_1) is not enough. One has to consider the stronger assumption

$$\begin{aligned} \left|F_u(x,u,v)\right| &\leqslant C\left(1+|u|^{p_*-1}+|v|^{q_*(p_*-1)/p_*}\right), \\ \left|F_v(x,u,v)\right| &\leqslant C\left(1+|v|^{q_*-1}+|u|^{p_*(q_*-1)/q_*}\right). \end{aligned} \tag{F$_2$}$$

One can easily check that (F_2) implies (F_1) and under (F_2), it follows that critical points of $\mathcal{F}$ are weak solutions of (11.30)–(11.31).

Now, the geometry of $\mathcal{F}$ depends strongly on the precise growth of the potential F. That is, on the exponents r and s in the inequality

$$\left|F(x,u,v)\right| \leqslant C\left(1+|u|^r+|v|^s\right), \tag{F$_3$}$$

where $r \leqslant p_*$ and $s \leqslant q_*$.

Of course, the case of interest is

$$F(x,0,0)=F_u(x,0,0)=F_v(x,0,0)=0 \quad \text{for } x \in \partial\Omega, \tag{F$_4$}$$

then $u \equiv v \equiv 0$ is a trivial solution of the system (11.30)–(11.31).

We will distinguish mainly four different cases:

(1) $r < p$ and $s < q$. (Sublinear-like.)
(2) $r = p$ and $s = q$. (Resonant.)
(3) $p < r < p_*$ and $q < s < q_*$. (Superlinear-like, subcritical.)
(4) $r = p_*$ and $s = q_*$. (Critical.)

First, we turn our attention to the superlinear and subcritical case (3). In order to verify the Palais–Smale condition, we need to impose the following assumption: There exist $R > 0$, θ_p and θ_q with $\theta_p < 1/p$, $\theta_q < 1/q$, such that

$$0 < F(x,u,v) \leqslant \theta_p u F_u(x,u,v) + \theta_q v F_v(x,u,v) \tag{F$_5$}$$

for all $x \in \partial\Omega$ and $|u|, |v| \geqslant R$. We have the following theorem.

THEOREM 11.12. *Assume that the potential F satisfies* (F_2), (F_3) *with r, s as in* (3), (F_4) *and* (F_5). *Assume, moreover, that there exists constants $c > 0$ and $\varepsilon > 0$ and $p_* > \bar{r} > p$, $q_* > \bar{s} > q$ such that*

$$\left|F(x,u,v)\right| \leqslant c\left(|u|^{\bar{r}}+|v|^{\bar{s}}\right) \quad \text{for } x \in \partial\Omega,\ |u|,|v| \leqslant \varepsilon. \tag{F$_6$}$$

Then $\mathcal{F}$ has a critical point. If, moreover, F is even then $\mathcal{F}$ has infinitely many critical points which are unbounded in $W^{1,p}(\Omega) \times W^{1,q}(\Omega)$.

Case (1) is similar in nature to a sublinear problem for the usual Laplacian. So, direct minimization yields a nontrivial solution. However, under a hypothesis similar to (F_5) we can show the existence of infinitely many solutions (of course, with an oddness assumption on F). The condition is: There exists $R > 0$, θ_p and θ_q with $\theta_p < 1/p$, $\theta_q < 1/q$, such that

$$\theta_p u F_u(x,u,v) + \theta_q v F_v(x,u,v) - F(x,u,v) \geqslant -c\big(|u|^r + |v|^s\big) \tag{F$_7$}$$

for all $x \in \partial\Omega$ and $|u|, |v| \geqslant R$. We have the following theorem.

THEOREM 11.13. *Assume that the potential F satisfies* (F$_2$), (F$_3$) *with r, s as in* (1), *and* (F$_4$). *Then $\mathcal{F}$ has a nontrivial critical point providing there exists a constant $R > 0$, $\theta < 1$ and a continuous function $K : \partial\Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that*

$$\begin{aligned} &F\big(x, t^{1/p}u, t^{1/q}v\big) \geqslant t^\theta K(x,u,v) \\ &\quad \text{for } x \in \partial\Omega,\ |u|, |v| \leqslant R \text{ and small } t > 0. \end{aligned} \tag{F$_8$}$$

If, moreover, F is even and (F$_7$) *holds, then $\mathcal{F}$ has infinitely many critical points which form a compact set in $W^{1,p}(\Omega) \times W^{1,q}(\Omega)$.*

The case (2) is a resonant problem. So there is an underlying (nonlinear) eigenvalue problem. In this case, it is natural to assume a condition on F that implies that the functional $\mathcal{F}$ satisfies the so-called *Cerami condition*. This assumption is as follows: There are positive constants c, R, a, b with $0 < a < p$, $0 < b < q$ such that

$$\frac{1}{p} u F_u(x,u,v) + \frac{1}{q} v F_v(x,u,v) - F(x,u,v) \geqslant c\big(|u|^a + |v|^b\big) \tag{F$_9$}$$

for $x \in \partial\Omega,\ |u|, |v| > R$.

In order to avoid resonance, we need to understand the underlying eigenvalue problem. A similar eigenvalue problem for the Dirichlet boundary condition case, was introduced in [17]. Let $G : \mathbb{R}^2 \to [0, \infty)$ be a C^1 positive even function such that

$$G\big(t^{1/p}u, t^{1/q}v\big) = tG(u,v), \tag{G$_1$}$$

$$G(u,v) \leqslant k\big(|u|^p + |v|^q\big). \tag{G$_2$}$$

The eigenvalue problem is

$$\begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } \Omega, \\ \Delta_q v = |v|^{q-2}v, \end{cases}$$
$$\begin{cases} |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} - aG_u = \lambda |u|^{p-2}u & \text{on } \partial\Omega, \\ |\nabla v|^{q-2}\frac{\partial v}{\partial \nu} - aG_v = \lambda |v|^{q-2}v, \end{cases} \tag{11.33}$$

where $a \in L^\infty(\partial\Omega)$.

We will see that problem (11.33) has a first eigenvalue $\lambda_1(a)$. So in order to avoid resonance, we assume that there exists positive numbers R and ε, and $a, b \in L^\infty(\partial\Omega)$ such that

$$\begin{aligned} &\lambda_1(a) < 0, \qquad F(x,u,v) \geqslant a(x)G(u,v), \quad |u|,|v| \geqslant R, \\ &\lambda_1(b) > 0, \qquad F(x,u,v) \leqslant b(x)\overline{G}(u,v), \quad |u|,|v| \leqslant \varepsilon, \end{aligned} \tag{F$_{10}$}$$

where G and $\overline{G}$ satisfy (G_1). We have the following theorem.

THEOREM 11.14. *Assume that the potential F satisfies* (F_2), (F_3) *with r and s as in* (2), (F_4), (F_9) *and* (F_{10}). *Then the functional $\mathcal{F}$ has a nontrivial critical point.*

Now we turn our attention to the critical case (4). As it is well known, the compactness in the immersion $W^{1,p}(\Omega) \times W^{1,q}(\Omega) \to L^{p_*}(\partial\Omega) \times L^{q_*}(\partial\Omega)$ fails, so the functional $\mathcal{F}$ does not verify the Palais–Smale condition. However, by applying the concentration–compactness method (see [96,97]), we can prove that $\mathcal{F}$ satisfy a *local Palais–Smale* condition that will suffices to apply the usual variational techniques.

The hypotheses on the potential F are

$$F(x,u,v) = F_\lambda(x,u,v) = F^{\mathrm{c}}(x,u,v) + \lambda F^{\mathrm{s}}(x,u,v), \tag{F$_{11}$}$$

where F^{c} is the *critical* part of F_λ and F^{s} is a subcritical perturbation, that verifies (F_3) with r and s as in (1) or (3).

The hypotheses on the critical part F^{c} are: There exist two constants $c, C > 0$ such that

$$c\big(|u|^{p_*} + |v|^{q_*}\big) \leqslant F^{\mathrm{c}}_u(x,u,v)u + F^{\mathrm{c}}_v(x,u,v)v \leqslant C\big(|u|^{p_*} + |v|^{q_*}\big), \tag{F$^{\mathrm{c}}_1$}$$

$$c\big(|u|^{p_*} + |v|^{q_*}\big) \leqslant \theta_p F^{\mathrm{c}}_u(x,u,v)u + \theta_q F^{\mathrm{c}}_v(x,u,v)v - F^{\mathrm{c}}(x,u,v), \tag{F$^{\mathrm{c}}_2$}$$

where θ_p and θ_q are defined for the two cases in (F_5) and (F_7).

For the subcritical perturbation F^{s}, we need also to impose the following condition,

$$F^{\mathrm{s}}_u(x,u,v)u + F^{\mathrm{s}}_v(x,u,v)v \leqslant C\big(1 + |u|^r + |v|^s\big). \tag{F$^{\mathrm{s}}_1$}$$

We have the following theorem.

THEOREM 11.15. *Assume that F_λ satisfies* (F_{11}) *with F^{s} satisfying* (F_2), (F_3) *with r and s as in* (3), (F_4), (F_5) *and* (F^{s}_1) *and F^{c} satisfying* (F^{c}_1) *and* (F^{c}_2). *Then there exists a constant $\overline{\Lambda} > 0$ such that, if $\lambda > \overline{\Lambda}$, $\mathcal{F}$ has a critical point in $W^{1,p}(\Omega) \times W^{1,q}(\Omega)$.*

Finally, for sublinear perturbations we have the theorem.

THEOREM 11.16. *Assume that F_λ satisfies* (F_{11}) *with F^{s} satisfying* (F_2), (F_3) *with r and s as in* (1), (F_4), (F_7) *and* (F^{s}_1) *and F^{c} satisfying* (F^{c}_1) *and* (F^{c}_2). *Then there exists a constant $\Lambda > 0$ such that, if $0 < \lambda < \Lambda$, $\mathcal{F}$ has a nontrivial critical point. Moreover, if F is even then $\mathcal{F}$ has infinitely many critical points in $W^{1,p}(\Omega) \times W^{1,q}(\Omega)$.*

12. Other results

In this section we collect more results for elliptic problems with nonlinear boundary conditions.

✓ Brock in [23] prove the following isoperimetric result concerning eigenvalues of the Steklov problem,

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda \rho u & \text{on } \partial\Omega, \end{cases} \tag{12.1}$$

where $\rho \in L^\infty(\partial\Omega)$ is positive.

THEOREM 12.1. *Let $|\Omega| = |B(0, R)|$ and λ_i the sequence of eigenvalues of* (12.1), *then*

$$\sum_{i=2}^{N+1} \frac{1}{\lambda_i} \geqslant \frac{N\omega_N R^{N-1}}{\int_{\partial\Omega} 1/\rho \, \mathrm{d}\sigma}.$$

✓ In [41] Davila and Montenegro study nonnegative solutions of the problem

$$\begin{cases} \Delta u = u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = -u^{-\beta} + f(x, u) & \text{on } \partial\Omega \cap \{u > 0\}. \end{cases}$$

Remark that for this problem a *free boundary* appears, the solution is allowed to vanish on some part of the boundary and hence it develops a free boundary, the boundary of the set $\{u > 0\} \cap \partial\Omega$. The authors show that, under adequate hypotheses on f, there exists a nontrivial maximal solution and also find conditions under which it develops a free boundary. Also, some regularity results are provided.

✓ Let us recall that in [38] it is proved the well-known maximum/antimaximum principle for the linear Laplace operator with zero Dirichlet boundary condition. Denoting by μ_1 the first eigenvalue of the Laplace operator (with zero Dirichlet boundary condition), it states that "*given any positive $h \in L^r(\Omega)$, $r > N$, there exists $\varepsilon = \varepsilon(h) > 0$ such that every solution u of $-\Delta u = \mu u + h$ in Ω with $u|_{\partial\Omega} = 0$ satisfies either $u < 0$ in Ω provided that $\mu_1 < \mu < \mu_1 + \varepsilon$* (antimaximum principle), *or $u > 0$ in Ω if $\mu < \mu_1$*" (maximum principle).

Our main goal now is to extend the maximum/antimaximum principle to the case of elliptic equations with nonlinear boundary conditions,

$$\begin{cases} -\Delta u + u = h & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda u + g & \text{on } \partial\Omega. \end{cases} \tag{12.2}$$

We have the following result, see [14] for a proof.

THEOREM 12.2. *Let $h \in L^r(\Omega)$ and $g \in L^s(\partial\Omega)$ with $r > N/2$ and $s > N - 1$. Let*

$$A = \int_\Omega h\varphi_1 \, \mathrm{d}x + \int_{\partial\Omega} g\varphi_1 \, \mathrm{d}\sigma.$$

(1) *If* $A > 0$ *then there exists* $\varepsilon > 0$ *such that every solution of* (12.2) *with* $\lambda_1 - \varepsilon < \lambda < \lambda_1$ *verifies* $u > 0$ *in* Ω *and every solution of* (12.2) *with* $\lambda_1 < \lambda < \lambda_1 + \varepsilon$ *verifies* $u < 0$ *in* $\overline{\Omega}$.
(2) *If* $A < 0$ *then there exists* $\varepsilon > 0$ *such that every solution of* (12.2) *with* $\lambda_1 - \varepsilon < \lambda < \lambda_1$ *verifies* $u < 0$ *in* Ω *and every solution of* (12.2) *with* $\lambda_1 < \lambda < \lambda_1 + \varepsilon$ *verifies* $u > 0$ *in* $\overline{\Omega}$.
(3) *If* $A = 0$ *then every solution of* (12.2) *with* $\lambda \neq \lambda_1$ *changes sign in* $\overline{\Omega}$.

With respect to the proof of this theorem, we emphasize that the main idea is to look at the problem from a bifurcation (from infinity) point of view like in [13].

Note that for the Neumann boundary condition we are just assuming that h is in $L^r(\Omega)$ with $r > N/2$. This is in contrast with the case of Dirichlet boundary condition in [38] where it is assumed $r > N$. This strengthened assumption is necessary for the Dirichlet case. Indeed, it is proved in [116] that there exists a positive function $h \in L^N(\Omega)$ for which the antimaximum principle does not hold; that is, the space $L^r(\Omega)$ with $r > N$ is sharp for this principle.

We also have that $L^{N/2}(\Omega)$ and $L^{N-1}(\partial\Omega)$ are sharp for the antimaximum principle in the case of Neumann boundary conditions. First, let us see that antimaximum principle does not hold, in general, for $h \in L^{N/2}(\Omega)$.

THEOREM 12.3. *There exists a positive function* $h \in L^{N/2}(\Omega)$ *such that the solution of* (12.2) *with* $g \equiv 0$ *and* $\lambda > \lambda_1$ *is positive somewhere in* Ω.

Also, the hypothesis $g \in L^{N-1}(\partial\Omega)$ is not sufficient to obtain the antimaximum principle.

THEOREM 12.4. *There exist a bounded domain* Ω *and a positive function* $g \in L^{N-1}(\partial\Omega)$ *such that the solution of* (12.2) *with* $h \equiv 0$ *and* $\lambda > \lambda_1$ *is positive somewhere in* Ω.

✓ The authors of [19] study questions of existence, uniqueness and continuous dependence for semilinear elliptic equations with nonlinear boundary conditions, $\beta(u) - \Delta u \ni f$ on Ω, $\frac{\partial u}{\partial \nu}\gamma(u) \in 0$ on $\partial\Omega$, where $f \in L^1(\Omega)$. The nonlinearities β and γ are maximal monotone graphs in $\mathbb{R}$. In particular, the authors obtain results concerning the continuous dependence of the solutions on the nonlinearities in the problem, which in turn implies analogous results for a related parabolic problem.

See also [9] and [12] where it is proved existence and uniqueness of solutions to equations of the form $u - \operatorname{div} a(x, Du(x)) = f$ in Ω, supplied with the boundary condition $-\frac{\partial u}{\partial \nu_a} \in \beta(u)$ on $\partial\Omega$, where $f \in L^1(\Omega)$; and $\partial/(\partial\nu_a)$ is the Neumann boundary operator associated to a, that is, $\partial u/\partial\nu_a \equiv \langle a(x, Du(x)), \eta\rangle$ and β is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $0 \in \beta(0)$. As a consequence of the existence–uniqueness result, the authors associate an m-complete accretive operator in $L^1(\Omega)$ to the corresponding parabolic equation, which permits them to study that equation from the point of view of nonlinear semigroup theory.

✓ Now, following [42], we deal with self-similar solutions of the porous medium equation in a half-space with a nonlinear boundary condition. We study existence and symmetry of nonnegative solutions of the following problem

$$\begin{cases} \Delta u = u^\alpha & \text{in } \mathbb{R}^N_+, \\ \frac{\partial u}{\partial \nu} = u & \text{on } \partial\mathbb{R}^N_+, \end{cases} \tag{12.3}$$

where $0 < \alpha < 1$. This elliptic problem appears naturally when one considers self-similar blowing up solutions of the porous medium equation ($m > 1$)

$$\begin{cases} v_t = \Delta v^m & \text{in } \mathbb{R}^N_+ \times (0, T), \\ \frac{\partial v^m}{\partial \nu} = v^m & \text{on } \partial\mathbb{R}^N_+ \times (0, T). \end{cases} \tag{12.4}$$

The blow-up problem for the porous medium equation has deserved a great deal of attention, see for example the survey [78]. In the study of blow-up problems, self-similar profiles are used to study the fine asymptotic behavior of a solution of the parabolic equation near its blow-up time. It often happens that the spatial shape of the solution near blow-up is close to a self-similar profile, see [78].

In our case, assume that $v(x,t)$ is a solution of (12.4) with blow-up time T. Then the rescaled function $z(x,t) = (T-t)^{1/(m-1)} v(x,t)$ should converge as $t \nearrow T$ to a stationary profile $z(x)$ satisfying

$$\begin{cases} \Delta z^m = \frac{1}{m-1} z & \text{in } \mathbb{R}^N_+, \\ \frac{\partial z^m}{\partial \nu} = z^m & \text{on } \partial\mathbb{R}^N_+, \end{cases}$$

as is often the case when dealing with parabolic problems. Then $u(x) = cz(x)^m$ is a solution of (12.3) with $\alpha = 1/m$ for a suitable choice of the constant c. On the other hand, given a nonnegative solution $u(x)$ of (12.3), $z(x) = (u(x)/c)^{1/m}$ gives rise to a special solution to (12.4) (in self-similar form) blowing up at time T, of the form $v(x,t) = (T-t)^{-1/(m-1)} z(x)$. Remark that in our case the self-similar scaling does not change the spatial variable, and hence the blow-up set is given by the support of $z(x)$. Therefore there is an interest in studying self-similar profiles, in our case solutions of (12.3). In order to motivate our study, let us recall what is known for the problem

$$v_t = \Delta v^m + v^m \quad \text{in } \mathbb{R}^N \times (0, T). \tag{12.5}$$

Problem (12.5) admits self-similar solutions. In this case the profile $z(x)$ is a solution of

$$0 = \Delta z^m + z^m - \frac{1}{m-1} z \quad \text{in } \mathbb{R}^N. \tag{12.6}$$

One way to look for solutions of (12.6) is to search for radial ones. The existence of a radial compactly supported nontrivial solution reduces to the study of an ODE and was done in [40]. Moreover, a symmetry analysis using moving planes implies that every solution with finite energy has compact support and is composed by a finite number of radial "bumps" located such that their supports do not intersect, see [40].

Concerning the existence of solutions of (12.3), let us observe that in one space dimension we are facing an ODE that can be solved explicitly. It turns out that there exists only one compactly supported solution in $\mathbb{R}_+$, $u(x) = c_1((c_2 - x)_+)^{2/(1-\alpha)}$. Unfortunately, for $N \geqslant 2$, an easy inspection of problem (12.3) shows that there is no hope to look for radial solutions since they cannot verify the boundary condition. Therefore, in the case under study, the elliptic problem remains a PDE that cannot be solved by ODE methods. However, the problem has still some natural symmetry in the tangential variables. In fact, if we call a point $x \in \mathbb{R}^N_+$, $x = (x', x_N)$ $(x' \in \mathbb{R}^{N-1})$, we can search for solutions that are radial in the tangential variables, that is,

$$u(x) = u\big(\big|x'\big|, x_N\big). \tag{12.7}$$

It has to be noted that this symmetry assumption does not reduce the problem to an ODE. The first result of [42] reads as follows.

THEOREM 12.5. *There exists a nontrivial, nonnegative, compactly supported solution of* (12.3) *of the form* (12.7).

Next, we use the moving planes device (with a moving plane parallel to the x_N direction) to prove the following result that justifies our symmetry assumption in Theorem 12.5.

THEOREM 12.6. *Let $u \in H^1(\mathbb{R}^N_+)$ be a nonnegative solution of* (12.3) *with connected support. Then u is compactly supported and radial in the tangential variables, that is, it has the form* (12.7).

The problem of uniqueness of solutions to (12.3) with compact support remains open. In the case of equation (12.6), it is known that solutions with compact support are unique except for translations, but the argument relies strongly on ODE techniques.

When this analysis is performed we can obtain some easy corollaries concerning problem (12.4).

COROLLARY 12.1. *Every nonnegative nontrivial solution of* (12.4) *blows up in finite time.*

The proof of this fact follows by contradiction. Assume that v is a global nontrivial solution. As v is a supersolution of the porous medium equation its support expands, [123], and eventually covers the support of a self-similar profile z. The proof ends just with the use of a comparison argument using a self-similar solution with T large enough as subsolution.

COROLLARY 12.2. *There exists a solution of* (12.4) *with a blow-up set composed by an arbitrary number of connected components.*

In fact, we may consider a self-similar solution with a profile $z(x)$ composed by n disjoint copies of the compactly supported solution provided by Theorem 12.5.

✓ Now we study weak solutions for the p-Laplacian with a nonlinear boundary condition at resonance, see [101]. We look for conditions that provide existence of weak solutions for the problem

$$\begin{cases} \Delta_p u = |u|^{p-2}u + f(x,u) & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda |u|^{p-2}u - h(x,u) & \text{on } \partial\Omega. \end{cases} \tag{12.8}$$

We assume that the perturbations $f:\Omega \times \mathbb{R} \to \mathbb{R}$ and $h:\partial\Omega \times \mathbb{R} \to \mathbb{R}$ are bounded Carathéodory functions. In a variational approach, the functional associated to the problem is

$$\begin{aligned} J_\lambda(u) = {}& \frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x + \frac{1}{p}\int_\Omega |u|^p \,\mathrm{d}x - \frac{\lambda}{p}\int_{\partial\Omega} |u|^p \,\mathrm{d}\sigma \\ & + \int_\Omega F(x,u)\,\mathrm{d}x + \int_{\partial\Omega} H(x,u)\,\mathrm{d}\sigma, \end{aligned}$$

where F and H are primitives of f and h with respect to u, respectively. Weak solutions of (12.8) are critical points of J_λ in $W^{1,p}(\Omega)$.

Let us introduce some motivation to deal with (12.8). As we mentioned before, we will say that λ is an eigenvalue for the p-Laplacian with a nonlinear boundary condition if the problem,

$$\begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda |u|^{p-2}u & \text{on } \partial\Omega, \end{cases} \tag{12.9}$$

has nontrivial solutions. The set of solutions (called eigenfunctions) for a given λ will be denoted by A_λ. Resonance problems are well known in the literature. For example, for the resonance problem for the p-Laplacian with Dirichlet boundary conditions see [13,49] and references therein. In problem (12.8) we have a perturbation of the eigenvalue problem (12.9) given by the two nonlinear terms $f(x,u)$ and $h(x,u)$. Following ideas from [49], we prove the following result, that establishes Landesman–Lazer-type conditions on the nonlinear perturbation terms in order to have existence of weak solutions for (12.8).

THEOREM 12.7. *Let* $f^\pm := \lim_{t\to\pm\infty} f(x,t)$, $h^\pm := \lim_{t\to\pm\infty} h(x,t)$. *Assume that there exist* $\bar f \in L^q(\Omega)$ *and* $\bar h \in L^q(\partial\Omega)$ *such that* $|f(x,t)| \leqslant \bar f$ $\forall (x,t) \in \Omega \times \mathbb{R}$ *and* $|h(x,t)| \leqslant \bar h$ $\forall (x,t) \in \partial\Omega \times \mathbb{R}$ (*where* $q = p/p-1$). *Also assume that either*

$$\begin{aligned} (\mathrm{LL})^+_\lambda\colon \; & \int_{\{v>0\cap\Omega\}} f^+ v + \int_{\{v>0\cap\partial\Omega\}} h^+ v \\ & + \int_{\{v<0\cap\Omega\}} f^- v + \int_{\{v<0\cap\partial\Omega\}} h^- v > 0 \quad \textit{for all } v \in A_\lambda \setminus \{0\} \end{aligned}$$

or

$$(\mathrm{LL})^-_\lambda\colon \int_{\{v>0\cap\Omega\}} f^+ v + \int_{\{v>0\cap\partial\Omega\}} h^+ v + \int_{\{v<0\cap\Omega\}} f^- v + \int_{\{v<0\cap\partial\Omega\}} h^- v < 0 \quad \textit{for all } v \in A_\lambda\setminus\{0\},$$

then (12.8) *has a weak solution.*

Observe that in the case where λ is not an eigenvalue the hypotheses trivially hold. The integral conditions (of Landesman–Lazer type) that we impose for f and h are used to prove a Palais–Smale condition for the functional J_λ associated to the problem (12.8). Observe that these conditions involve an integral balance (with the eingenfunctions v as weights) between f and h. Hence we allow perturbations both in the equation and in the boundary condition.

Let us have a close look at the conditions for the first eigenvalue. As the first eigenvalue is isolated and simple with an eigenfunction that does not change sign in Ω (we call it ϕ_1 and assume $\phi_1 > 0$ in $\overline{\Omega}$) [100] and Section 6, the conditions involved in Theorem 12.7 for λ_1 read as

$$(\mathrm{LL})^+_{\lambda_1}\colon \int_\Omega f^+\phi_1\,\mathrm{d}x + \int_{\partial\Omega} h^+\phi_1\,\mathrm{d}\sigma > 0 \quad \text{and}$$
$$\int_\Omega f^-\phi_1\,\mathrm{d}x + \int_{\partial\Omega} h^-\phi_1\,\mathrm{d}\sigma < 0$$

or

$$(\mathrm{LL})^-_{\lambda_1}\colon \int_\Omega f^+\phi_1\,\mathrm{d}x + \int_{\partial\Omega} h^+\phi_1\,\mathrm{d}\sigma < 0 \quad \text{and}$$
$$\int_\Omega f^-\phi_1\,\mathrm{d}x + \int_{\partial\Omega} h^-\phi_1\,\mathrm{d}\sigma > 0.$$

For this case, $\lambda = \lambda_1$, we will prove a more general result which improve the conditions on f and h. In [13] the resonance problem for the Dirichlet problem was analyzed using bifurcation theory. If we adapt the arguments of [13] to our situation, using bifurcation techniques to deal with (12.8), we can improve the previous result by measuring the speed and the form at which f and h approaches the limits $f^\pm$ and $h^\pm$, see [101] for the details.

✓ We end this section with the study of the Fučik spectrum and a resonance problem for the p-Laplacian with an asymmetric nonlinearity at the nonlinear boundary condition. We want to find existence results for elliptic problems involving the p-Laplacian with a nonlinear flux boundary condition. More precisely, we look for solutions of

$$\begin{cases} \Delta_p u = A(x,u) & \text{in } \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = B(x,u) & \text{on } \partial\Omega. \end{cases} \tag{12.10}$$

We assume that $A(x,u)$ and $B(x,u)$ behave like

$$A(x,u) = |u|^{p-2}u - f(x,u),$$
$$B(x,u) = \alpha\big(u^+\big)^{p-1} - \beta(u^-)^{p-1} + h(x,u),$$

where f and h are lower-order terms. Hence we are assuming that we deal with a resonance problem with an asymmetric nonlinearity at the flux boundary condition.

To study this problem, as happens for the usual Dirichlet boundary conditions, we first have to make an analysis of the associated spectrum. The Fučik spectrum for the p-Laplacian with a nonlinear boundary condition is defined as the set $\widetilde{\Sigma}_p$ of $(\alpha,\beta) \in \mathbb{R}^2$ such that

$$\begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \alpha\big(u^+\big)^{p-1} - \beta(u^-)^{p-1} & \text{on } \partial\Omega \end{cases} \tag{12.11}$$

has a nontrivial solution. If $\alpha = \beta = \lambda$ then we arrive to our old friend, the eigenvalue problem,

$$\begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda|u|^{p-2}u & \text{on } \partial\Omega. \end{cases} \tag{12.12}$$

Let us go back to (12.11). We observe that if λ is an eigenvalue for (12.12) then the point (λ,λ) belongs to $\widetilde{\Sigma}_p$. Moreover, the lines $\mathbb{R} \times \{\lambda_1\}$ and $\{\lambda_1\} \times \mathbb{R}$ belong to $\widetilde{\Sigma}_p$, this follows from the fact that the first eigenfunction is positive.

The main results for this problem can be summarized as follows: first we have a first nontrivial curve in $\widetilde{\Sigma}_p$. We also have that the lines $\mathbb{R} \times \{\lambda_1\}$ and $\{\lambda_1\} \times \mathbb{R}$ are isolated in $\widetilde{\Sigma}_p$ and that the curve is the first nontrivial curve. This implies in particular that this curve passes through (λ_2,λ_2) which yields a variational characterization of the second eigenvalue λ_2 by a mountain-pass procedure. The first curve is monotone and its asymptotic behavior is similar to the Dirichlet problem if $p \leqslant N$ and in the case $p > N$ it is similar to the homogeneous Neumann problem. We also find a sequence of curves locally around a suitable sequence of variational eigenvalues (λ_k,λ_k).

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CHAPTER 6

Schrödinger Operators with Singular Potentials

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Contents

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Abstract

We describe classical and recent results on the spectral theory of Schrödinger and Pauli operators with singular electric and magnetic potentials.

1. Introduction

Analogous to Newton's equations, describing the motion of macroscopic bodies, from falling apples to space crafts orbiting the Earth, the Schrödinger equation describes microscopic phenomena, from the simplest atoms to processes inside stars. The differential operator appearing in the Schrödinger equation, the so-called *Schrödinger operator*, characterizes the physical system and it is therefore one of the most interesting objects in Mathematical Physics; its spectral theory has deep roots in nonrelativistic quantum mechanics. A number of mathematical notions and theories were inspired by the needs of the theory of the Schrödinger operator and its generalizations. Formally, the Schrödinger operator is obtained from the Hamilton function describing the classical system by some vaguely defined "quantization" procedure. So, if the classical Hamiltonian is $H(p,q) = p^2 + V(q)$, $p, q \in \mathbb{R}^d$ being the momenta and coordinates, the corresponding Schrödinger operator is $H = -\Delta + V(x)$ in $L^2(\mathbb{R}^d)$.[1] A number of questions arise immediately, in particular, about defining H as a self-adjoint operator in a proper Hilbert space and on determining its spectral properties on qualitative and quantitative level. As a result of the activity of many mathematicians starting in the 1950s, a rather complete set of answers to these questions was found, under regularity conditions on the potential V, more and more relaxed with time. Later, the researchers directed their interest also to the operators involving not only an electric field with potential $V(x)$ but also magnetic fields: the magnetic Schrödinger and Pauli operators. Here passage from regular to more singular potentials presented even harder mathematical problems. During the past several years, fundamental results concerning Schrödinger and Pauli operators with very singular potentials were obtained. So far these results are restricted to the qualitative questions of spectral theory, self-adjoint realizations, semiboundedness, discreteness of the whole or the negative spectrum. The quantitative results on the spectrum lag behind at the moment.

There are a number of sources dealing with the spectral theory of the Schrödinger operator. We name, first of all, the books [13,17,27,29,32,54,63,79,112–114,120,123,131] and references therein.

An extensive presentation of the spectral theory, even restricted to the most important operators in Mathematical Physics, would be far beyond the frames of a 100-pages review paper. We decided to cover "extreme parts" of the theory. We describe the setting of the main problems, some classical ideas and methods, and then pass to the most recent results in the field, not pretending to cover the whole multitude. We concentrate in our exposition on the most singular cases. This means that we are interested in such results where the conditions on the magnetic and electric potentials are relaxed, as far as possible. This requires often a new understanding of the problem, new ideas, and always overcoming considerable technical difficulties. In many cases the need for such extensions of mathematical knowledge is inspired by physical and technical needs. This choice, however, restricts the selection of the material to be presented in our review. We do not provide a survey, for example, over the vast landscape of microlocal methods in spectral theory [58,123] (where spectacular developments follow one another during the last decennia) since these methods

[1] Traditionally, both $+V$ and $-V$ occur in the notation. Not breaking the traditions, we apply both notations in the text, specifying explicitly which one is used in a particular place, so this must not cause a confusion.

require considerable regularity of the problem. We do not touch upon the exciting topic of scattering theory describing the large time evolution of quantum systems; it surely deserves separate expositions. We also leave aside such important fields as the theory of operators with periodic, almost periodic and random coefficients, as well as the study of the asymptotic properties of eigenvalues and other spectral characteristics of Schrödinger operators, see, e.g., [120] for some basic facts and references about these topics.

Obviously, we cannot go into all technical details, which the interested reader has to look for in the literature, but we try to explain the main ideas and the driving forces of the proofs.

The structure of the chapter is the following. Part 1 (Sections 2–6) is devoted to the qualitative spectral analysis. We consider two central questions: how to define the Schrödinger-type operator as a self-adjoint operator in a suitable Hilbert space, and what information of the structure and location of the spectrum one can obtain.

We begin with a description of the three main objects considered in the chapter, the Schrödinger operator, the magnetic Schrödinger operator and the Pauli operator. Of course, the first one is just a particular case of the second operator. However, not only the tradition advises to consider the nonmagnetic operator as a separate object of study, but the results for this operator are most complete and, at the same time, methods of the spectral analysis are usually invented for the Schrödinger operator, and further adapted and extended to more complicated cases. The Pauli operator is the first stage in taking into account relativistic features: it involves the interaction of the spin with the magnetic field.

Having described our operators formally, we have to define them as self-adjoint operators in a Hilbert space. The operators are self-adjoint only formally, so the task is to describe the proper domain of definition and the action of the operators on this domain. Only then one can develop the spectral theory. We discuss the general self-adjointness problem in Section 3, where, in particular, we explain the perturbation approach. The comfortable case here is when the operator, defined initially on some "nice" set, is essentially self-adjoint, which means that there is just one "natural" self-adjoint realization of the operator. This usually happens in "regular" situations. Recently, however, a substantial progress took place even in this classical field, and we describe these results in Section 4.

In more singular cases, a less explicit method of quadratic forms is needed to define the operator. It requires by far less regularity of the potentials. One can apply this method, not only for defining the operator but also for studying its spectrum, as soon as the necessary facts from the theory of functional spaces are obtained. These facts concern conditions for certain functional inequalities to hold, relating the quadratic form of the operator (the Sobolev space norm of the function, in the simplest case) and the norm of the function in some weighted space. Physically, it amounts to controlling the potential energy of the system by the kinetic one. The mathematical problem lies in the study of boundedness, infinitesimal boundedness and compactness of embeddings of Sobolev-type spaces in spaces of functions square-integrable with respect to some specified measure. We devote a major part of Section 5 to the explanation of basic facts and a description of the most recent results in the field of such functional inequalities. Having abstract perturbation theorems (presented in Section 3) at hand, one arrives at self-adjointness results.

Beginning from Section 6 we proceed to spectral theory. The questions addressed here concern the qualitative structure of the spectrum: is the negative spectrum empty, finite, discrete, or is the whole spectrum discrete. In other words, it is about *eigenvalues*, or *bound states* in the physical terminology.

Generally, if E is an eigenvalue of the operator H describing the quantum system and u_E is the corresponding eigenfunction, then the evolution of the system, governed by the Schrödinger equation $-\mathrm{i}\frac{\partial\psi}{\partial t} = H\psi$ with the initial state $\psi(0) = u_E$, has the form $\psi(t) = u_E \exp(-\mathrm{i}tE)$. Thus the (physically sensible) absolute value of the solution does not depend on t and the system is in a stationary (bound) state. Such states correspond to closed orbits in Bohr's classical theory of quantum systems. The study of eigenvalues of Schrödinger-type operators is important for Physics and Chemistry, since they determine the main properties of the physical systems. In Mathematics, the properties of eigenvalues of various operators have deep relations in real and complex analysis, partial differential equations, geometry, stochastic analysis and other topics.

One of the most powerful instruments in the qualitative, as well as in quantitative spectral analysis, is the *Birman–Schwinger principle* which relates the properties of eigenvalues of Schrödinger-type operators to the properties of embedding operators. In Section 6 we discuss this principle, and using results on embedding operators, we give answers to the questions above, starting from the most simple, classical ones, requiring a considerable regularity of the potentials, up to the most recent developments related to much more singular situations. In some cases, the results we discuss here are final in the sense that they give necessary and sufficient conditions for the spectral properties in question to hold.

The second part of the chapter is devoted to the quantitative spectral analysis of our operators. Here, again, a selection of material was necessary, and our choice, of course, was influenced by our own research interests. We present several topics concerning estimates of eigenvalues of Schrödinger-type operators. Such results are important for physical applications, especially in quantum mechanics, and the corresponding formulas have deep physical roots.

We consider the *negative* eigenvalues $-E_j$ of the Schrödinger-type operators $H_V = H_0 - V$ $(= -\hbar^2\Delta - V)$ in $L^2(\mathbb{R}^d)$; here $\hbar$ is the Planck constant divided by 2π. Information on their number $N(0; H_V)$ and their sum are important for physical applications.

From the "old quantum theory" one has the somewhat mystical *Bohr–Sommerfeld quantization condition* that "each nice set in the classical *phase-space* with volume $(2\pi\hbar)^d$ can accommodate one eigenstate of H". Hence, heuristically, one expects that

$$N(0; H_V) \sim (2\pi\hbar)^{-d} \iint_{\{(x,\xi)\,|\,H(x,\xi)<0\}} \mathrm{d}x\,\mathrm{d}\xi,$$

where $H(x,\xi) = |\xi|^2 - V(x)$ is the corresponding (classical) *Hamilton function*. For the standard Schrödinger operator $-\Delta - V$ one obtains that

$$N(0; -\Delta - V) \sim L^c_{0,d} \int_{\mathbb{R}^d} V_+(x)^{d/2}\,\mathrm{d}x; \tag{1.1}$$

here $V_+(x) = \max\{V(x), 0\}$, and $L^c_{0,d} = (2\pi\hbar)^{-d}\omega_d$ is the *classical constant*, $\omega_d = \pi^{d/2}/\Gamma(1+\frac{d}{2})$ denotes the volume of the unit ball in $\mathbb{R}^d$, $\Gamma(\alpha)$, $\alpha > 0$, denotes the Gamma function $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}\mathrm{e}^{-t}\,\mathrm{d}t$, and the meaning of the symbol $\sim$ has to be further specified. The right-hand side of (1.1) is usually referred to as the *phase-space volume*.

It was only in the 1970s that this formula was mathematically justified. The result, the *CLR estimate* states that in dimensions $d \geqslant 3$ the left-hand side in (1.1) is estimated from above by the right-hand side times some constant, depending on the dimension d only, viz.

$$N(0; H_V) \leqslant C_d \int_{\mathbb{R}^d} V_+(x)^{d/2}\,\mathrm{d}x. \tag{1.2}$$

Estimate (1.2) belongs to the "singular" theory from the very beginning: the only condition is finiteness of the integral on the right-hand side.

This estimate, as well as its generalizations to other operators, proved to be very helpful in many problems in Spectral Theory and Mathematical Physics. To get a better understanding of it, improve the constant, and generalize it to other classes of operators, several alternative proofs were proposed, using quite different mathematical machinery. We describe here, in Section 7, several approaches to proving (1.2), including the most recent, and probably the most abstract one, based on the analysis of positive semigroups. Abstractness of the method enables one to carry over the estimate to a wider class of operators, including many operators with magnetic fields, discrete operators, operators on manifolds, some relativistic operators, etc. We present also the recent important generalization of the CLR estimate to the case of the Schrödinger operator acting in a space of vector-valued functions.

One of the important applications of the CLR estimate lies in finding asymptotic formulas for $N(0; H_V)$ as some of the parameters tend to zero (or ∞). For instance, when $\hbar \to 0$, one talks about the *semiclassical* asymptotics, and the uniformness of the estimate (1.2) was crucial in justifying the heuristic formula (1.1) asymptotically. It holds, again, as long as the integral on the right-hand side is finite. On the other hand, one can replace V by gV, g being a *coupling constant*, and get the asymptotics of $N(0; H_{gV})$ as $g \to \infty$ [15,88],

$$\lim_{g\to\infty} \frac{N(0; H_{gV})}{g^{d/2}} = (2\pi\hbar)^d \omega_d \int_{\mathbb{R}^d} V_+(x)^{d/2}\,\mathrm{d}x; \tag{1.3}$$

this is the strong-coupling limit.

In proving this and many other asymptotic results, one usually establishes the asymptotics first for some nice, say, smooth and compactly supported potentials and then applies the CLR estimate together with methods from perturbation theory of linear operators (see Section A.11) to extend the asymptotics to all potentials. We do not discuss this kind of asymptotic formulas in this chapter, referring the reader to [32,120] and references therein.

However, in relation to the asymptotics (1.3), the following question arises. Consider the sum of the negative eigenvalues taken to some power $\gamma > 0$,

$$S_{\gamma,d}(V) = \sum_{-E_j<0} E_j^\gamma.$$

By means of formal integration in (1.3), one arrives at the asymptotics

$$\lim_{g\to\infty} \frac{S_{\gamma,d}(V)}{g^{d/2+\gamma}} = L^c_{\gamma,d} \int_{\mathbb{R}^d} V_+(x)^{d/2+\gamma}\,\mathrm{d}x \tag{1.4}$$

with a certain constant $L^c_{\gamma,d}$. Is this, formally obtained, asymptotics correct?

Information about $S_{\gamma,d}(V)$ and especially estimates of the form

$$S_{\gamma,d}(V) \leqslant L_{\gamma,d} \int_{\mathbb{R}^d} V_+(x)^{d/2+\gamma}\,\mathrm{d}x, \tag{1.5}$$

which are called *Lieb–Thirring* inequalities, turned out to be crucial in the study of stability of matter in nonrelativistic and relativistic quantum mechanics. Again, as soon as the estimate (1.5) is established, the asymptotics of the form (1.4) follows by perturbational reasoning. The physical applications require knowledge of the optimal constant, and it is only recently that essential progress was made here. In Section 8 we discuss these, as well as classical results in detail.

In Section 9 we present an abstract approach to obtaining eigenvalue estimates for operators with magnetic fields. The basic point here is an important property, the *diamagnetic inequality*, saying, in application to the magnetic Schrödinger operator, that the kernel of the Green function of the heat semigroup generated by this operator is pointwise majorized by the Green function of the analogous semigroup generated by the nonmagnetic operator. We discuss both classical and recently found methods for establishing diamagnetic inequalities and present their use in spectral theory: any, sufficiently regular eigenvalue estimate for the nonmagnetic operator is inherited by the magnetic one. This general observation fails, however, in the case of the Pauli operator, since it does not possess the diamagnetic property, on the contrary, a *paramagnetic* property is excepted, with the opposite monotonicity.

One of the complicating features for Pauli operators is the presence of so-called *zero modes*. The Pauli operator, even without electric field, may have zero as an eigenvalue, and its multiplicity, especially if it is infinite, influences significantly the behavior of the spectrum under perturbations. We describe in Section 10 how these zero modes can arise and present the latest results dealing with their study.

In Section 11 we consider the perturbed Pauli operator. Its spectral properties are considerably more complicated and less known than the ones for the magnetic Schrödinger operator. In particular, one must not expect that an analogy of the CLR estimate is valid here. Lieb–Thirring-type inequalities are looked for, with stability of matter problems already waiting. Unlike the Schrödinger case, there is no natural candidate for the form of such an inequality. We present the proof of the Lieb–Thirring inequality for the case of a constant magnetic field, the hardest proof we have taken a risk to include, and then describe the latest developments concerning Lieb–Thirring inequalities for nonconstant magnetic fields.

The spectral theory of differential operators is based heavily on the abstract spectral theory of operators in Hilbert spaces. We think, moreover, that the more abstract formulation (or proof) of a result in the theory of partial differential equations is found, the deeper is the understanding of particular ideas and engines involved, and the more possibilities exist to generalize the results. Therefore we try to emphasize, where possible, the abstract operator theory content of the concrete considerations for differential operators.

We assume that the reader knows the basic facts about linear operator theory in Hilbert spaces; we include in the Appendix some more specialized information, concerning quadratic forms, variational description of the spectrum (e.g., Glazman's lemma) and the Birman–Schwinger principle, thus relating the spectral properties of the operator to properties of embeddings for Sobolev-type Hilbert spaces. The Appendix contains also some useful facts from the theory of compact operators in Hilbert spaces. To the interested reader we, nevertheless, strongly advise to preclude the study of the paper by a more fundamental study of the abstract theory found in, say, [17,114] or [155].

2. Main operators

In this review we consider three important operators of Mathematical Physics: the Schrödinger operator, the magnetic Schrödinger operator and the Pauli operator. We begin by defining them formally, as *differential expressions* or *differential operations*, without discussing particular regularity conditions for coefficients, reserving the word *operator* to operators in a Hilbert space, with a specified domain.

The Laplacian. The (negative) Laplace operator in $\mathbb{R}^d$ is denoted by $-\Delta$. The Hilbert space usually will be $L^2(\mathbb{R}^d)$ with Lebesgue measure $\mathrm{d}x$. We consider also the *Laplace–Beltrami operator* on Riemannian manifolds. Given a manifold M with metric tensor g_{jk} in local coordinates, we define the Laplace–Beltrami operator $-\Delta_g$ as

$$-\Delta_g = -g^{-1/2} \sum \partial_j g^{jk} g^{1/2} \partial_k, \tag{2.1}$$

where $\{g^{jk}\}$ is the matrix, inverse to $\{g_{jk}\}$, and $g = \det g_{jk}$. The Hilbert space here is $L^2(M, \rho(\mathrm{d}x))$, where the Riemannian measure ρ has local density $g^{1/2}$.

Magnetic Laplacian. Let $\mathcal{A}(x) = (A_j(x))_{j=1,\dots,d}$ be a real vector-valued function in $\mathbb{R}^d$, called the *magnetic potential*. The components of the *magnetic gradient* $\nabla_{\mathcal{A}} = (P_1, \dots, P_d)$ are $P_j = P_{j,\mathcal{A}} = \partial_j + \mathrm{i}A_j(x)$ and the magnetic Laplacian is defined by the differential operation

$$-\Delta_{\mathcal{A}} = -\sum_j P_j^2 = -\sum_j \bigl(\partial_j + \mathrm{i}A_j(x)\bigr)^2. \tag{2.2}$$

The matrix-valued function (in fact, distribution) $\mathbf{B} = \operatorname{curl} \mathcal{A} = \{B_{jk}\}_{j,k=1}^d$, $B_{jk} = (\partial_j A_k - \partial_k A_j)$, is called the *magnetic field*. Under certain, rather weak, regularity conditions, operators corresponding to different magnetic potentials but the same magnetic field $\mathbf{B}$ are related by a unitary equivalence relation, the *gauge transformation*,

$$-\Delta_{\widetilde{\mathcal{A}}} = \mathrm{e}^{-\mathrm{i}\phi}(-\Delta_{\mathcal{A}})\mathrm{e}^{\mathrm{i}\phi}, \tag{2.3}$$

provided

$$\nabla\phi = \widetilde{\mathcal{A}} - \mathcal{A}, \qquad \nabla\times\widetilde{\mathcal{A}} = \nabla\times\mathcal{A}. \tag{2.4}$$

This equivalence is discussed in [71,72]. The gauge ϕ must be a function in L^1_{loc}. If such an equivalence takes place, the operators with magnetic potentials $\mathcal{A}$, $\widetilde{\mathcal{A}}$ describe "the same physics" since it is only the absolute value of the wave function that has physical meaning. If, however, the gauge ϕ needed to formally transform $-\Delta_{\mathcal{A}}$ to $-\Delta_{\widetilde{\mathcal{A}}}$ is more singular, then the corresponding operators are not necessarily unitary equivalent; the most interesting example here is the Aharonov–Bohm effect, to be discussed later on.

For dimension $d=3$, since there are only three independent nonzero components in $\mathbf{B}$, one can compose a vector $\mathcal{B}$ by setting $B_1 = B_{23}$, $B_2 = B_{31}$, $B_3 = B_{12}$ which is also called the magnetic field.

In dimension $d=2$ the magnetic field $\mathbf{B}$ is reduced to just one element, $B = B_{12} = \partial_2 A_1 - \partial_1 A_2$. The same happens in the three-dimensional case, provided the field has constant direction, say, along the x_3-axis, and here one can find a gauge wherein $\mathcal{A} = (A_1(x_1,x_2), A_2(x_1,x_2), 0)$.

To define the magnetic Laplacian on a Riemannian manifold, it is convenient to consider the magnetic potential as a real 1-form $\mathcal{A} = \sum A_j\,\mathrm{d}x_j$ in local coordinates. The magnetic differential

$$\mathrm{d}_{\mathcal{A}} : u \mapsto \mathrm{d}u + \mathrm{i}u\mathcal{A}$$

maps functions to 1-forms, in other words, it defines a connection on the trivial linear bundle over M. Denoting the adjoint (with respect to the scalar product in the space of forms, $(\alpha,\beta) = \int g^{jk}\alpha_j\beta_k g^{1/2}\,\mathrm{d}x$) differential as $\mathrm{d}^*_{\mathcal{A}}$ (it maps 1-forms to functions), the magnetic Laplacian is defined as

$$-\Delta_{\mathcal{A}} = \mathrm{d}^*_{\mathcal{A}}\mathrm{d}_{\mathcal{A}}. \tag{2.5}$$

The magnetic field in this representation is a 2-form, $\mathbf{B} = \mathrm{d}\mathcal{A}$, the curvature of the connection. On manifolds, even for regular magnetic potentials $\widetilde{\mathcal{A}}, \mathcal{A}$, the equality of the corresponding magnetic fields does not necessarily imply gauge equivalence of the corresponding operators: the topology of the manifold M may create obstacles for this. More exactly, if the first co-homology group of M is nontrivial, the equality $\mathrm{d}\mathcal{A} = \mathrm{d}\widetilde{\mathcal{A}}$ does not necessarily imply existence of a smooth function ϕ such that $\mathrm{d}\phi = \widetilde{\mathcal{A}} - \mathcal{A}$. In the cases when it is possible, in particular, in the Euclidean space for sufficiently regular magnetic potential, one chooses the gauge in some convenient way, say to preserve a certain symmetry or to require $\mathrm{d}^*\mathcal{A} = 0$ – the so-called *Coulomb gauge*.

Pauli operator. We consider the Pauli operator in the most important cases, i.e., $d=2$ and $d=3$, and only for the Euclidean space.

Let σ_j, $j = 1,2,3$, be the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

satisfying the anticommutation relations $\sigma_j\sigma_k + \sigma_k\sigma_j = 2\delta_{jk}I$, and let $\sigma = (\sigma_1, \sigma_2, \sigma_3)$. For the magnetic potential $\mathcal{A} = (A_1, A_2, A_3)$, the Pauli operator is defined as

$$\mathcal{P}_{\mathcal{A}} = -\big(\sigma \cdot (\nabla + \mathrm{i}\mathcal{A})\big)^2 = -(\nabla + \mathrm{i}\mathcal{A})^2 + \sigma \cdot \mathcal{B}, \tag{2.6}$$

in dimension 3, with obvious modification for dimension 2.

Thus, the Pauli operator acts on two-component vector-valued functions and differs just by a matrix term $\sigma \cdot \mathcal{B}$ from the magnetic Laplacian. In dimension 2 as well as in dimension 3 with magnetic field having constant direction, in the proper gauge the Pauli operator splits into the direct sum of two magnetic Schödinger operators

$$\mathcal{P}_{\mathcal{A}} = \mathrm{Diag}\big(-(\nabla + \mathrm{i}\mathcal{A})^2 \pm B\big). \tag{2.7}$$

In dimension 2 the Pauli operator admits another representation, a convenient factorization (temporarily, just formally),

$$\mathcal{P}_{\mathcal{A}} = \begin{pmatrix} Q_+Q_- & 0 \\ 0 & Q_+Q_- \end{pmatrix}, \tag{2.8}$$

where $Q_\pm = (\partial_1 + \mathrm{i}A_1) \pm \mathrm{i}(\partial_2 + \mathrm{i}A_2)$, provided the Coulomb gauge $\operatorname{div}\mathcal{A} = 0$ is chosen.

Operators with electric potential. The operators defined above, with or without magnetic field, will be perturbed by an electric field. The electric *potential*, henceforth denoted by V, will be a real-valued measurable function or, possibly, a signed measure (in the latter case we also denote it by ρ). If V is a function, the differential operation will be denoted by $H_{\mathcal{A},V} = H_{\mathcal{A}} + V$ (or $H_{\mathcal{A},V} = H_{\mathcal{A}} - V$), where $H_{\mathcal{A}}$ is the magnetic Laplacian $-\Delta_{\mathcal{A}}$ (the usual Laplacian, if $\mathcal{A} = 0$), or the Pauli operator $\mathcal{P}_{\mathcal{A}}$. We keep the same formal notation even in the case of V being a measure.

3. Self-adjointness

We discuss in this section the problem of defining operators, whose spectral properties are to be analyzed later. The most important observation here, as well as in other related problems is that the key instrument in spectral analysis of differential operators is given by *functional inequalities*.

3.1. *Essential self-adjointness*

The first task one encounters when studying the differential expressions described above is to associate to them some self-adjoint operators in proper Hilbert spaces, which includes defining directly or indirectly the domain of the operator and the action of the operator on the functions in this domain. The requirement of operators being self-adjoint follows from the condition of unitarity of the time evolution for the linear Schrödinger equation

$-\mathrm{i}\frac{\partial\psi}{\partial t} = H\psi$, where H is the operator in question. See the discussion in [112], Section X.1, on the physical meaning of self-adjointness.

In very few cases one can approach this task directly. Thus, for the (negative) Laplacian in $L^2(\mathbb{R}^d)$, the Fourier transform establishes self-adjointness of the operator on the domain $\mathfrak{D}(-\Delta) = \mathbf{H}^2(\mathbb{R}^d)$, the Sobolev space.[2] However, as soon as variable coefficients are present, such a direct approach does not work.

In more or less regular situations, one considers, for a given differential expression H, a symmetric operator H^0 defined on some dense subspace $\mathfrak{D}^0$ in the Hilbert space L^2. Most often, the space of compactly supported smooth functions, $\mathscr{D}$, serves as this initial domain. Then one applies the *closure* procedure. This means that one considers the closure of $\mathfrak{D}^0$ with respect to the norm $\|u\|_H^2 = \|Hu\|^2 + \|u\|^2$. The operator H^0 extends by continuity to this new domain, and may turn out to be self-adjoint there. In the affirmative case we say that the operator H^0 is *essentially self-adjoint* (on $\mathfrak{D}^0$) and this means that there is just one "natural" self-adjoint operator corresponding to the differential operation under scrutiny.

To implement this approach, one, first of all, needs that the coefficients of the operator are sufficiently regular, so that the differential expression, when acting on compactly supported smooth functions, produces functions in L^2. Thus, since the magnetic Laplacian can be represented as

$$-\Delta_{\mathcal{A}} u = -\Delta u - 2\mathrm{i}\mathcal{A}\cdot\nabla u + \big(\mathrm{i}\,\mathrm{div}\,\mathcal{A} + |\mathcal{A}|^2\big)u, \tag{3.1}$$

the components A_j of the magnetic potential must satisfy (see [72])

$$\mathrm{div}\,\mathcal{A} \in L^2_{\mathrm{loc}}, \quad \mathcal{A} \in L^4_{\mathrm{loc}}; \tag{3.2}$$

and, additionally, $\mathcal{B} = \mathrm{curl}\,\mathcal{A} \in L^2_{\mathrm{loc}}$ for the Pauli operator. The condition for the electric potential here is $V \in L^2_{\mathrm{loc}}$. For potentials having stronger but localized singularities, one can try considering a smaller initial domain $\mathfrak{D}^0$ of the operator, defining it on compactly supported smooth functions whose support does not touch the points of singularity of the coefficients.

If the operator H^0 can be defined in this way, it is symmetric, and the next task is to prove its essential self-adjointness. In Section 4 we will discuss some methods one uses to show this as well as some classical and recent results.

If the operator H^0 is not (proved to be) essentially self-adjoint, there may be an infinite set of self-adjoint extensions, the most interesting of which usually are realized by means of boundary conditions at singular points or at infinity. An extensive discussion of this topic can be found in [112], Section X.1. The important example we will consider in this chapter is the Aharonov–Bohm operator; see Sections 3.2 and 9.3.

For more singular potentials one even cannot define the action of the initial operator H^0 on compactly supported smooth functions, since formally applying the differential operation to such functions results in functions outside L^2 or even not functions. This means that the self-adjoint operator corresponding to the differential expression in question must

[2] Breaking the tradition, we denote Sobolev spaces by the boldface letter $\mathbf{H}$. The regular H is too overloaded in this field.

be somehow defined on nonsmooth functions. Being applied to the functions in the proper domain in the sense of distributions, the operation produces a sum of several distributions, such that being added together, strong singularities somehow cancel, resulting in a function that belongs to L^2; say, for the Schrödinger operator $-\Delta + V$ with V not in L^2_{loc} (or even a measure) one can consider the *maximal* operator defined on such functions $u \in L^2$ that both $-\Delta u$ and Vu make sense as distributions and their sum belongs to L^2. The maximal operator $H^{\max}$ thus defined, is, generally, not symmetric, and to obtain a self-adjoint operator one has to *restrict* $H^{\max}$ to some smaller domain. Since the description of the domain of the maximal operator is rather implicit, this approach is not very productive.

As in many other cases, a perturbation approach is rather efficient here. If we are given an operator H_0 which is already known to be essentially self-adjoint and we can show that the perturbation, V in our case, is in a certain sense weaker than H_0, then the proper perturbation theorem assures that the perturbed operator, $H_0 + V$, is essentially self-adjoint on a certain domain, sometimes even describing explicitly the domain of the self-adjoint extension.

The first basic result here is the Kato–Rellich theorem (see, e.g., [112], Theorem X.12).

DEFINITION 3.1. Let S and T be densely defined operators in a Hilbert space. Suppose that $\mathfrak{D}(S) \supset \mathfrak{D}(T)$ and, for certain positive numbers a, b and all elements in $\mathfrak{D}(T)$, the inequality

$$\|Su\|^2 \leqslant a\|Tu\|^2 + b\|u\|^2 \tag{3.3}$$

holds or, what is equivalent,

$$\|Su\| \leqslant a\|Tu\| + b\|u\| \tag{3.4}$$

(with some other b). Then the operator S is called *bounded with respect to T* (or, shorter, T-bounded) with relative bound a. If for any positive a, a number b exists such that (3.4) holds, the operator S is called *infinitesimally T-bounded*.

THEOREM 3.2 (Kato–Rellich (with addition by Wüst)). *Let T be a self-adjoint operator. Suppose that the symmetric operator S is T-bounded with relative bound smaller than* 1. *Then the operator $T + S$ is self-adjoint with $\mathfrak{D}(T + S) = \mathfrak{D}(T)$ and essentially self-adjoint on any subspace where T is essentially self-adjoint. If the relative bound equals* 1, *the operator $T + S$ is essentially self-adjoint on $\mathfrak{D}(T)$.*

Thus, if the relative bound equals one, the theorem does not give a direct description of the domain of the self-adjoint extension of the operator $T + S$.

As a classical example demonstrating how the Kato–Rellich theorem works, we consider the Schrödinger operator in $\mathbb{R}^3$ [61].

EXAMPLE 3.3 (Kato's theorem). Suppose that V can be represented as a sum $V_1 + V_2$, where V_1, V_2 are real-valued functions, such that $V_1 \in L^2$, $V_2 \in L^\infty$ (henceforth this will be denoted by $V \in L^2 + L^\infty$). Then the operator $-\Delta + V$ is self-adjoint on $\mathbf{H}^2(\mathbb{R}^3)$ and essentially self-adjoint on $\mathscr{D}(\mathbb{R}^3)$.

The proof is based on an *embedding theorem*: For any $a > 0$ there exists a b such that

$$\|u\|_{L^\infty} \leqslant a\|\Delta u\|_{L^2} + b\|u\|_{L^2}$$

for any $u \in \mathscr{D}(\mathbb{R}^3)$ (see, e.g., [112], Theorem IX.28). This inequality, together with

$$\|Vu\|_{L^2} \leqslant \|V_1\|_{L^2}\|u\|_{L^\infty} + \|V_2\|_{L^\infty}\|u\|_{L^2},$$

gives (3.4). So, the operator of multiplication by V is infinitesimally $-\Delta$-bounded, and the Kato–Rellich theorem applies. This example hints that in more general situations, one needs more advanced functional inequalities (embedding theorems) to establish self-adjointness.

The condition of the perturbation being relatively bounded is rather restrictive. In particular, it is not fulfilled if the potential V tends to $+\infty$ or $-\infty$ at infinity. Important examples here are the harmonic oscillator $V = c|x|^2$ and the Stark potential (a constant electric field, $V(x) = \mathbf{k} \cdot x$). We will discuss the advanced methods for handling such situations in Section 4.

3.2. *Quadratic forms*

As it was mentioned above, the Schrödinger operator H_V in $\mathbb{R}^d$ can be defined on $\mathscr{D}(\mathbb{R}^d)$ only provided $V \in L^2_{\mathrm{loc}}$. If the potential V has stronger local singularities, or especially, if it is a measure, not absolutely continuous with respect to the Lebesgue (or Riemannian) measure, one cannot apply the differential operation to arbitrary functions in $\mathscr{D}(\mathbb{R}^d)$ and therefore the direct definition of the minimal operator (which we plan to extend further) encounters obstacles. The powerful method to handle such singular situations, as well as operators with magnetic fields, is the method of quadratic forms.

First, some abstract theory. Having a self-adjoint operator H with domain $\mathfrak{D}(H)$, under the condition that H is lower semibounded, $\langle Hu, u\rangle \geqslant -C\|u\|^2$ for some $C > 0$, one can consider the *quadratic form*, $\mathfrak{h}[u] = \langle Hu, u\rangle, u \in \mathfrak{D}(H)$. The quadratic form $\mathfrak{h}$ can be extended by continuity to the closure $\mathfrak{D}(\mathfrak{h})$ of $\mathfrak{D}(H)$ with respect to the metric $\mathfrak{h}[u] + (C + 1)\|u\|^2$. The resulting quadratic form will be denoted by $QF(H)$, and its domain $\mathfrak{D}(\mathfrak{h})$ also by $\mathfrak{Q}(H)$. Thus the quadratic form can be constructed if the self-adjoint semibounded operator H is given. An important fact in operator theory is that the inverse construction also works. Having a quadratic form defined on some dense subspace, satisfying certain conditions specified below, one can, in a unique way, associate to it a self-adjoint operator. The conditions are the following. To justify the name "quadratic form", $\mathfrak{h}$ must be positively homogeneous, $\mathfrak{h}[tu] = |t|^2\mathfrak{h}[u]$. The parallelogram identity must hold, $\mathfrak{h}[u + v] + \mathfrak{h}[u - v] = 2\mathfrak{h}[u] + 2\mathfrak{h}[v]$; the necessity of these conditions follows automatically from the relation $\mathfrak{h}[u] = \langle Hu, u\rangle$. Furthermore, the form must be lower semibounded and, the last but the hardest to check, it must be *closed*; the latter means that for a sequence of elements $u_j \in \mathfrak{D}(\mathfrak{h})$, the relations $\|u_j - u\| \to 0$ and $\mathfrak{h}[u_j - u_k] \to 0$ imply $u \in \mathfrak{D}(\mathfrak{h})$ and $\mathfrak{h}[u_j - u] \to 0$. If these conditions hold, a canonical procedure enables one to construct the unique self-adjoint operator H such that $\mathfrak{h} = QF(H)$. First, one has to construct the sesquilinear form $\mathfrak{h}[u,v]$ corresponding to the quadratic form, by means of the polarization identity, see, e.g., [155], Theorem 1.6, and secondly, one defines the domain of the operator H as consisting of such $u \in \mathfrak{D}(\mathfrak{h})$ that the functional $\phi_u[v] = \mathfrak{h}[v,u]$ defined for $v \in \mathfrak{D}(\mathfrak{h})$, is bounded in the basic Hilbert space. For such u, the action of the operator H on the element u is defined by means of identity $\langle v, Hu\rangle = \mathfrak{h}[v,u]$; the element Hu exists and is unique, due to the Riesz representation theorem.

Thus, in concrete situations, in order to define the operator having a quadratic form, one has to prove that the quadratic form is semibounded and closed (homogeneity and the parallelogram identity usually can be verified immediately). However, the closedness condition can be relaxed. We say that the form $\mathfrak{h}^0$, defined on the dense linear set $\mathfrak{D}(\mathfrak{h}^0)$ is *closable* if for any sequence $u_j \in \mathfrak{D}(\mathfrak{h}^0)$, $\|u_j\| \to 0$, the property $\mathfrak{h}^0[u_j - u_k] \to 0$ implies $\mathfrak{h}^0[u_j] \to 0$. If the form is closable, one can consider, as we have done it with the form $\langle Hu, u\rangle$ above, the closure of $\mathfrak{D}(\mathfrak{h}^0)$ in the metric $\mathfrak{h}[u] + C\|u\|^2$ with proper C. The form $\mathfrak{h}^0$ extends there by continuity to a closed form $\mathfrak{h}$. Closability of a quadratic form is much easier to establish than closedness, however, in singular situations even this can be a hard technical task. We present here two examples; the first one shows how the proof of closability usually goes in simple cases, the second produces a form which is not closable.

EXAMPLE 3.4. Let $V \geqslant 1$ be a real-valued function in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Consider the form $\mathfrak{h}^0[u] = \int(|\nabla u|^2 + V|u|^2)\,\mathrm{d}x$, with domain $\mathfrak{D}(\mathfrak{h}^0) = \mathscr{D}(\mathbb{R}^d)$. Supposing that $\mathfrak{h}^0[u_j - u_k] \to 0$, we can deduce, from known completeness of the Sobolev space $\mathbf{H}^1$ and the weighted space L^2_V, that the sequence u_j converges to some U_1 in the Sobolev norm, as well as it converges to some U_2 in the norm of L^2_V. At the same time, since u_j converges to zero in L^2, there is a subsequence which converges to zero almost everywhere. Taking yet another subsequence, we find that it converges to U_2 almost everywhere, whence $U_2 = 0$. Finally, convergence in $\mathbf{H}^1$ implies convergence in L^2, so $u_j \to U_1$ in L^2 and therefore $U_1 = 0$. In conclusion, $\mathfrak{h}^0[u_j] \to 0$ and therefore closability is proved.

The crucial point in the reasoning above was the possibility of separating the terms in the form $\mathfrak{h}^0$ and therefore enabling one to use already known completeness of Sobolev and weighted L^2 spaces. This trick does not go through if we drop the condition of positivity of the potential V. We will see later how one handles this case.

Now, the second example.

EXAMPLE 3.5. Let $\mathfrak{h}^0[u] = |u(x_0)|^2$, $\mathfrak{D}(\mathfrak{h}^0) = \mathscr{D}(\mathbb{R}^d)$, where x_0 is some fixed point in $\mathbb{R}^d$. Fix some function $\phi \in \mathscr{D}(\mathbb{R}^d)$, such that $\phi(0) = 1$. Consider the sequence $u_j(x) = \phi(j(x - x_0))$. This sequence converges to zero in L^2, $\mathfrak{h}^0[u_j - u_k] = 0$, however $\mathfrak{h}^0[u_j] \nrightarrow 0$, and therefore the form is not closable.

The method of forms enables one to construct certain special self-adjoint extensions of *symmetric* semibounded operators. Having such an operator H^0 with domain $\mathfrak{D}(H^0)$, one considers the form $\mathfrak{h}^0[u] = \langle Hu, u\rangle$ with domain $\mathfrak{D}(\mathfrak{h}^0) = \mathfrak{D}(H^0)$. This form is automatically closable (see, for example, [112], Theorem X.23, and [13]) and by closing the form and then finding the corresponding self-adjoint operator, we obtain the self-adjoint operator which is called the *Friedrichs extension* of the symmetric operator H^0. Quite often (but not always) this extension turns out to be the most physically reasonable of all extensions. It is the absence of the Friedrichs extension that is one of the serious obstacles in the study of nonsemibounded operators, for example, Dirac and Maxwell operators.

When considering the magnetic Laplacian and Pauli operators without electric potential, the quadratic forms used to define the operators are respectively

$$\mathfrak{h}^0_{\mathcal{A}}[u] = \int_{\mathbb{R}^d} |\nabla u + \mathrm{i}\mathcal{A}u|^2 \, \mathrm{d}x \tag{3.5}$$

and

$$\mathfrak{p}^0_{\mathcal{A}}[u] = \int_{\mathbb{R}^d} \big|\sigma \cdot (\nabla u + \mathrm{i}\mathcal{A}u)\big|^2 \, \mathrm{d}x \tag{3.6}$$

defined on the initial domain $\mathscr{D}(\mathbb{R}^d)$. These expressions are obtained by formal integration by parts in quadratic forms of the magnetic Laplacian and Pauli operators described in Section 2. If the magnetic field is sufficiently regular, so that the minimal symmetric operators can be defined on $\mathscr{D}(\mathbb{R}^d)$, these forms, after closure, produce the same self-adjoint operators as the ones obtained above by means of the procedure of closing symmetric operators.

If, however, these regularity conditions are broken, it is the quadratic forms (3.5) and (3.6) that must be used directly to define the operator. The conditions on the forms can be considerably relaxed. It suffices that the components A_j of the magnetic field belong locally to L^2; this is the natural requirement for the forms to be defined on $\mathscr{D}(\mathbb{R}^d)$. It is rather easy (however, involving some technical tricks) to prove that for such magnetic potentials the forms (3.5) and (3.6) are closable (see, for example, [136]) and therefore define self-adjoint operators. The same proof applies to the case of operators on Riemannian manifolds, with the form (3.5) modified accordingly [20]. In the case of more singular magnetic potentials, one chooses the same expression for the form, but with different domain. For example, for the Aharonov–Bohm magnetic potential in $\mathbb{R}^2$ [5],

$$\mathcal{A}(x_1, x_2) = \alpha\big(x_2 r^{-2}, -x_1 r^{-2}\big), \quad r = \big(x_1^2 + x_2^2\big)^{-1/2}, \tag{3.7}$$

which does not belong to L^2_{loc}, one may consider, as the domain of the quadratic form, the subspace $\mathscr{D}(\mathbb{R}^2 \setminus \{0\})$, which produces a self-adjoint operator, denoted $-\Delta_{\mathrm{AB}}$, identical to the Friedrichs extension of the symmetric operator defined on the same domain. Alternatively, as a maximal domain for the quadratic form, one may take such functions $u \in L^2$ for which $(\nabla u + \mathrm{i}\mathcal{A}u) \in L^2$. This procedure generates another self-adjoint operator. In fact, there is a four-parameter family of self-adjoint operators corresponding to the same differential expression, and many of them have physical sense; see, e.g., the discussion in [3,122,148].

As soon as the electric potential V is present, some additional perturbational reasoning must be used. Example 3.4 shows how to handle the case of a positive potential, but it is more convenient to give an abstract formulation, including, among others, the following case.

PROPOSITION 3.6. *Let the semibounded forms* $\mathfrak{h}_j[u]$ *with domains* $\mathfrak{D}(\mathfrak{h}_j)$, $j = 1, 2$, *be closable, and let* $\mathfrak{D}(\mathfrak{h}_1) \cap \mathfrak{D}(\mathfrak{h}_2)$ *be dense. Then the form* $\mathfrak{h} = \mathfrak{h}_1 + \mathfrak{h}_2$ *with domain* $\mathfrak{D}(\mathfrak{h}_1) \cap \mathfrak{D}(\mathfrak{h}_2)$ *is closable.*

PROOF. Since both forms $\mathfrak{h}_j$ are semibounded, for a sequence $u_k \in \mathfrak{D}(\mathfrak{h}_1) \cap \mathfrak{D}(\mathfrak{h}_2)$ the relation $\mathfrak{h}[u_k - u_l] \to 0$ implies $\mathfrak{h}_j[u_k - u_l] \to 0$, and therefore, closability of $\mathfrak{h}_j$ implies that $\mathfrak{h}_j[u_k] \to 0$ and, finally, $\mathfrak{h}[u_k] \to 0$. □

For the Schrödinger or Pauli operators with lower semibounded electric potential V this fact enables one to define the self-adjoint operator provided $V \in L^1_{\mathrm{loc}}$. If V is even more singular, i.e., it is a measure ρ including a singular component, the form $\int |u|^2 \rho(\mathrm{d}x)$ is not closable on $\mathscr{D}$ (it can be shown similarly to Example 3.5), and this limits the applicability of the proposition.

In order to handle such singular situations, one has to modify Proposition 3.6 in the following way. Having a closable form $\mathfrak{t}$, defined on $\mathfrak{D}(\mathfrak{t})$, we say that the form $\mathfrak{s}$, *not necessarily semibounded*, defined on an dense set $\mathfrak{D}(\mathfrak{s}) \subset \mathfrak{D}(\mathfrak{t})$ is *closable with respect to* $\mathfrak{t}$ if for any sequence $u_k \in \mathfrak{D}(\mathfrak{s})$, the relations $\|u_k\| \to 0$, $\mathfrak{t}[u_k] \to 0$ and $\mathfrak{s}[u_k - u_l] \to 0$, $k, l \to \infty$, imply $\mathfrak{s}[u_k] \to 0$. In particular, if the form $\mathfrak{s}$ is *bounded with respect to* $\mathfrak{t}$, which means that $|\mathfrak{s}[u]| \leqslant a\mathfrak{t}[u] + b\|u\|^2$, $u \in \mathfrak{D}(\mathfrak{s})$, it is automatically closable with respect to $\mathfrak{t}$. This notion enables one to establish a perturbation criterion for closedness of quadratic forms, which generalizes the KLMN theorem (see [112], Theorem X.17).

THEOREM 3.7 (KLMN). *Let* $\mathfrak{t}$ *with domain* $\mathfrak{D}(\mathfrak{t})$ *be a semibounded closable quadratic form*; *let the form* $\mathfrak{s}$ *be closable with respect to* $\mathfrak{t}$ *and, moreover, assume that*

$$\mathfrak{s}[u] \geqslant -a\mathfrak{t}[u] - b\|u\|^2 \tag{3.8}$$

holds for some $a < 1$ *and some* b, *for all* $u \in \mathfrak{D}(\mathfrak{t})$. *Then the form* $\mathfrak{t} + \mathfrak{s}$ *is semibounded and closable on* $\mathfrak{D}(\mathfrak{t})$, *and thus defines a unique self-adjoint operator.*

The KLMN theorem in its usual formulation requires, in addition to (3.8), a similar estimate from above, which can be a too restrictive condition.

PROOF OF THEOREM 3.7. One can just repeat the reasoning in the proof of KLMN theorem in [112]. In fact, the estimate (3.8) implies that the form $\mathfrak{t} + \mathfrak{s}$ is semibounded from below and that

$$(\mathfrak{t} + \mathfrak{s})[u] \geqslant (1 - a)\mathfrak{t}[u] - b\|u\|^2. \tag{3.9}$$

So, if we have a sequence u_k such that $\|u_k\| \to 0$ and $(\mathfrak{t} + \mathfrak{s})[u_k - u_l] \to 0$, (3.9) gives us that $\mathfrak{t}[u_k - u_l] \to 0$. Now, since $\mathfrak{t}$ is closable, $\mathfrak{t}[u_k] \to 0$ and, finally, relative closability of $\mathfrak{s}$ implies that $\mathfrak{s}[u_k] \to 0$. □

In Section 5 we will discuss some recent results concerning finding concrete analytical conditions for relative closedness and for estimates of the form (3.8). Here we just give a simple example.

EXAMPLE 3.8. Let $d \geqslant 3$, $V \in L^1_{\text{loc}}(\mathbb{R}^d)$, and let its negative part V_- belong to $L^{d/2}(\mathbb{R}^d)$. Then the form

$$\mathfrak{h}_V[u] = \int_{\mathbb{R}^d} \big(|\nabla u|^2 + V|u|^2\big)\,\mathrm{d}x \tag{3.10}$$

is semibounded and closable on $\mathscr{D}$ and thus defines a unique self-adjoint operator.

The main analytical fact in the study of this example is the *Sobolev inequality* (see, e.g., [91,112]), viz.

$$\int_{\mathbb{R}^d} |\nabla u|^2\,\mathrm{d}x \geqslant C_d\left(\int_{\mathbb{R}^d} |u|^q\,\mathrm{d}x\right)^{2/q} \quad \text{for } u \in \mathscr{D}\big(\mathbb{R}^d\big);\ q = \frac{d}{d-2}. \tag{3.11}$$

Fix some K such that $\int ((V-K)_-)^{d/2}\,\mathrm{d}x < (2C_d)^{-d/2}$. Then, using that $V \geqslant -K - (V-K)_-$, the Hölder inequality and the Sobolev inequality, in this order, yield

$$\begin{aligned}\int_{\mathbb{R}^d} V|u|^2\,\mathrm{d}x &\geqslant -K\|u\|^2_{L^2} - \int_{\mathbb{R}^d} (V - K_-)|u|^2\,\mathrm{d}x \\ &\geqslant -K\|u\|^2_{L^2} - \left(\int_{\mathbb{R}^d}\big((V-K)_-\big)^{d/2}\,\mathrm{d}x\right)^{2/d}\left(\int_{\mathbb{R}^d}|u|^q\,\mathrm{d}x\right)^{2/q} \\ &\geqslant -K\|u\|^2_{L^2} - \frac{1}{2}\int_{\mathbb{R}^d}|\nabla u|^2\,\mathrm{d}x.\end{aligned}$$

Thus one of the hypotheses of Theorem 3.7 is satisfied. To check relative closedness is now easy.

Finally, we compare the conditions imposed on V by the Kato–Rellich theorem and by the quadratic form method in the case of dimension $d = 3$. The Kato–Rellich approach, see Example 3.3, requires that the potential belongs to $L^2 + L^\infty$. On the other hand, the quadratic form definition can be applied provided $V \in L^1_{\text{loc}}$, with $V_- \in L^{3/2}$. Thus the restrictions on the growth of the potential and on the strength of singularities are both considerably relaxed. On the other hand, the quadratic form approach gives a rather implicit description of the domain of the self-adjoint operator and of the action of the operator on the functions in this domain.

In the next example we present a perturbation which is not closable and which is not bounded with respect to the leading form but, nevertheless, is closable with respect to the leading term, and thus the Schrödinger operator can be defined.

EXAMPLE 3.9. Let $d = 1$ and $\mathfrak{t}[u] = \int_{\mathbb{R}^1} |u'|^2\,\mathrm{d}x$, $\mathfrak{D}(\mathfrak{t}) = \mathscr{D}(\mathbb{R}^1)$. Define the measure ρ on the line, $\rho = \sum_n n^2\delta(x-n)$. Then the form $\mathfrak{s}[u] = \int |u|^2\,\mathrm{d}\rho$ is closable with respect to $\mathfrak{t}$. Since it is semibounded, the sum $\mathfrak{t}+\mathfrak{s}$ defines a self-adjoint operator in $L^2(\mathbb{R}^1)$, which is natural to associate to the differential operation $-\mathrm{d}^2/\mathrm{d}x^2 + \rho$.

It is easy to show that the form $\mathfrak{s}$ is not closable in L^2; just repeat the construction in Example 3.5. To prove that it is closable with respect to $\mathfrak{t}$, we use the inequality $|u(x_0)|^2 \leqslant \int |u'|^2\,\mathrm{d}x + \int |u|^2\,\mathrm{d}x$, $u \in \mathscr{D}(\mathbb{R}^1)$ (the easiest way to prove it is to apply to the right-hand side of

$$\left|u(x_0)\right|^2 = 2\int_{-\infty}^{x_0} \Re\left(u'\bar{u}\right)\mathrm{d}x,$$

the Cauchy–Schwartz inequality). Therefore, having a sequence u_j, such that $\|u_j\| \to 0$, $\mathfrak{t}[u_j] \to 0$ and $\mathfrak{s}[u_j - u_k] \to 0$, we can verify that $u_j(n) \to 0$ for each n. Denote by U_j the sequence $nu_j(n), n \in \mathbb{Z}$. Since $\mathfrak{s}[u_j - u_k] \to 0$, U_j is a Cauchy sequence in l^2, and therefore it converges in l^2 to some U_0, $\mathfrak{s}[u_j - u_0] \to 0$. It follows that $u_j(n) \to u_0(n)$ for all n and therefore $u_0(n) = 0$, whence $\mathfrak{s}[u_j] \to 0$.

In Section 5.2 we discuss general closability conditions.

When we pass to operators with magnetic fields, it is again certain inequalities that play a crucial part. The main principle is that the presence of the magnetic field for the Schrödinger operator improves the crucial estimates. The most simple realization of this *diamagneticity* principle is the "weak diamagnetic inequality".

PROPOSITION 3.10. *If $\mathcal{A} \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ and $u \in \mathscr{D}(\mathbb{R}^d)$, then*

$$\int_{\mathbb{R}^d} |\nabla u + \mathrm{i}\mathcal{A}u|^2\,\mathrm{d}x \geqslant \int_{\mathbb{R}^d} \left|\nabla|u|\right|^2\mathrm{d}x. \tag{3.12}$$

The short proof of (3.12) can be found in [133] (see also [138], p. 2). It follows from (3.12), in particular, that if a form $\int |u|^2\rho(\mathrm{d}x)$ for some (signed) measure ρ (or, for a usual potential $V(x)$, the form $\int |u|^2 V\,\mathrm{d}x$) is relatively bounded with certain bound or relatively closed with respect to the form $\int |\nabla u|^2\,\mathrm{d}x$, the same is true with respect to the magnetic Dirichlet form. Thus, if for some electric potential, the self-adjoint extension can be constructed by the quadratic form method for the nonmagnetic Schrödinger operator, it can be constructed for the magnetic Schrödinger operator as well. In fact, a much stronger version of the diamagnetic inequality can be proved (see Section 9.2), and in Section 9.5 we will show its applications in spectral theory. A much more hard task is to find out how the situation *improves* when the magnetic field is being switched on. We discuss some recent results in Section 9.4.

For the Pauli operator, even the weak version of the diamagnetic inequality does not hold. The most direct way to handle the problem of defining the self-adjoint extension here is using the relation (2.6), connecting the Pauli operator with the magnetic Schrödinger operator. Thus, in dimension 2, if we wish to define the Pauli operator with magnetic potential $\mathcal{A}$, magnetic field B and electric field V, it is sufficient to solve this problem for the magnetic Schrödinger operator with electric potentials $V + B$ and $V - B$ (recall that here the magnetic field has only two nonzero components, $\pm B$). In the three-dimensional case one can consider operators with electric field $V + B_\pm$, where $B_\pm$ are the eigenvalues of the matrix $\sigma \cdot B$. A deeper analysis of the interaction of the electric and magnetic fields gives more exact results, and we present some of them in Section 6.4.

For the case when no electric potential is present, there is a more direct way of understanding how the quadratic form method for defining operators works.

For the magnetic Laplacian $-\Delta_{\mathcal{A}}$ on a manifold M, with magnetic potential $\mathcal{A}$ belonging to L^2_{loc}, we can consider the magnetic gradient $\nabla_{\mathcal{A}}$ as an operator acting from some space $\mathfrak{d}_0$ of smooth sections of the trivial linear bundle to L^2_{loc} sections of the bundle of 1-forms. One can show that such an operator is closable, and we denote its closure by $\nabla_{\mathcal{A}}(\mathfrak{d}_0)$. The domain $\mathfrak{d}_0$ can be chosen, generally, in different ways. In particular, we can take $\mathfrak{d}_0$ as $\mathscr{D}$, and this gives us the *minimal operator* $\nabla_{\mathcal{A}}^{\min}$. On the other hand, one might have taken as $\mathfrak{d}_0$ the space of *all* smooth functions $u \in L^2$ on M such that $\nabla_{\mathcal{A}} u \in L^2$, the corresponding closure being the *maximal operator* $\nabla_{\mathcal{A}}^{\max}$. Whatever domain $\mathfrak{d}_0$ is chosen, the process of closing the operator $\nabla_{\mathcal{A}}$ is the same as the process of closing the quadratic form $\mathfrak{h}[u] = \int |\nabla_{\mathcal{A}} u|^2 \,\mathrm{d}x = \langle \nabla_{\mathcal{A}} u, \nabla_{\mathcal{A}} u\rangle$; it is the completion of $\mathfrak{d}_0$ with respect to the same metric. Thus, the operator corresponding to the closure of the quadratic form $\mathfrak{h}[u]$ is the same as the operator $\Delta_{\mathcal{A}}(\mathfrak{d}_0) = \nabla_{\mathcal{A}}(\mathfrak{d}_0)^* \nabla_{\mathcal{A}}(\mathfrak{d}_0)$, and the latter operator is automatically self-adjoint. Without magnetic field, since every function in the maximal domain $\mathbf{H}^1$ can be approximated in $\mathbf{H}^1$ by finitely supported functions, the maximal and minimal quadratic forms and therefore the maximal and minimal operators coincide. A similar approximation approach works also for the magnetic operator in the Euclidean space [29,136]. Here, however, certain additional technicalities occur, since a priori it is not even clear that bounded functions form a dense set in the maximal domain, and this boundedness is important for approximating functions in this maximal domain by cut-offs with compact supports. Here the diamagnetic inequality (see Section 9.2) is crucial. For general Riemannian manifolds the equality of minimal and maximal quadratic forms is still an open question.

The representation (2.8) enables one to use the same reasoning for the Pauli operator in $\mathbb{R}^2$. On some space $\mathfrak{d}_0$ of 2-component smooth functions $u = (u_+, u_-)^\top$ on $\mathbb{R}^2$ consider the first-order operator $Q(u_+, u_-)^\top = (Q_- u_-, Q_+ u_+)^\top$. The closure of this operator, as before, produces the closed operator $Q = Q(\mathfrak{d}_0)$, with domain coinciding with the domain of the closure of the quadratic form $\|Qu\|^2$ defined originally on $\mathfrak{d}_0$. Thus the Pauli operator can be defined as $\mathcal{P}_{\mathcal{A}} = \mathcal{P}_{\mathcal{A}}(\mathfrak{h}_0) = Q^* Q$, which is automatically self-adjoint. There is no diamagnetic inequality for the Pauli operator, so one must not expect that without additional conditions imposed on the magnetic potential, the maximal operator coincides with the minimal one. This is surely not the case for very singular magnetic fields, for example, the Aharonov–Bohm one; see Section 9.3.

In dimension 3 there is no analogy of the factorization (2.8) of the Pauli operator. Probably, the most natural way to construct the self-adjoint operator is to consider it as the square of the self-adjoint Dirac operator $\mathbf{D} = \sigma \cdot (\nabla + \mathrm{i}\mathcal{A})$, thus corresponding to the quadratic form $\int |\sigma \cdot (\nabla + \mathrm{i}\mathcal{A}) u|^2 \,\mathrm{d}x$. In this setting one must use the fact that for $\mathcal{A} \in L^2_{\mathrm{loc}}$ the Dirac operator is essentially self-adjoint on $\mathscr{D}$ (see, e.g., [51,149], wherein it was proved for smooth magnetic field, but the reasoning works for $\mathcal{A} \in L^2_{\mathrm{loc}}$ as well [129]). So, the Pauli operator is the square of the self-adjoint Dirac operator, and this definition coincides with the quadratic form approach. Note that the same reasoning gives a self-adjoint extension for the Pauli operator in dimension 2.

4. Essential self-adjointness

We discuss here the methods for proving self-adjointness of Schrödinger-type operators and present some recent results on this topic.

4.1. *Evolution equation approach*

Having a symmetric densely defined operator H, with domain $\mathfrak{D}(H)$, it is rather unpractical to try to prove essential self-adjointness of H by directly using the definition $\mathfrak{D}(\overline{H}) = \mathfrak{D}(H^*)$, since, generally, it is rather hard to find an explicit description of $\mathfrak{D}(\overline{H})$ and, especially, of $\mathfrak{D}(H^*)$. In the case (we are interested in) when the leading term of H is a second-order elliptic differential operator with smooth coefficients and the lower-order terms are not too singular, locally the domains $\mathfrak{D}(\overline{H})$ and $\mathfrak{D}(H^*)$ coincide, they consist of functions in $\mathbf{H}^2_{\text{loc}}$.

It is the behavior at infinity, or at singular points which may distinguish between these domains, and it is far from being easy to describe explicitly such "boundary conditions". Thus, probably, the technically easiest approach to proving essential self-adjointness is based on the analysis of Cauchy problems for *evolution equations* related to H and H^*, since it is often only local effects that are important here. The following exposition follows essentially the one in [128], where one can find further details.

On the abstract level, the relation between self-adjointness and evolution equations is explained by the following statement, due to Povzner [107] (the abstract version is found in [12], p. 388).

PROPOSITION 4.1. *Let H be a* (*lower*) *semibounded densely defined symmetric operator. Suppose that the abstract wave equation*

$$\psi'' = -H^*\psi, \quad \psi(0) = u_0, \psi'(0) = u_1; u_0, u_1 \in \mathfrak{D}(H), \tag{4.1}$$

has a unique solution in $\mathfrak{D}(H^*)$; here derivatives are understood in the strong sense. Then the operator H is essentially self-adjoint.

The idea of the proof is as follows. If H is not essentially self-adjoint, then it has at least two different self-adjoint extensions, $H^{(1)}$ and $H^{(2)}$, which both are restrictions of H^*. For each extension, the solution of (4.1) exists, and is given by the formula

$$\psi^{(j)}(t) = \cos\sqrt{H^{(j)}}tu_0 + \frac{\sin\sqrt{H^{(j)}}t}{\sqrt{H^{(j)}}}u_1 \tag{4.2}$$

(the choice of the square root does not matter because both functions $\cos(\lambda t)$ and $\sin(\lambda t)/\lambda$ are even). The difference of solutions in (4.2) gives a nonzero solution of (4.1), thus uniqueness is broken.

Proposition 4.1 can be modified in different ways. For instance, it is possible to replace the abstract wave equation (4.1) by the abstract heat equation $\psi' = -H^*\psi$. Even the semiboundedness condition can be dropped, being replaced, in fact, by the equality of deficiency indices of H (otherwise the operator surely cannot be essentially self-adjoint); in this case, uniqueness is required for the abstract Schrödinger evolution equation $\psi' = \mathrm{i}H^*\psi$. The proofs follow the reasoning in Proposition 4.1. Further generalizations allow the derivatives in the evolution equations to be understood in the weak sense, etc.

It is, nevertheless, the wave equation approach which is most efficient for proving essential self-adjointness on $\mathscr{D}(M)$ of second-order elliptic operators on a manifold M. As a matter of fact, in order to prove uniqueness of (4.1), it is not necessary to know H^* since, due to *Holmgren's principle*, this uniqueness follows from the *solvability* of the Cauchy problem for the adjoint equation

$$\psi'' = \overline{H}\psi, \quad \psi(0) = u_0, \psi'(0) = u_1, u_0, u_1 \in \mathfrak{D}(H). \tag{4.3}$$

The latter solvability can be established by means of the *finite propagation speed property* for hyperbolic equations. In fact, if initial conditions for the hyperbolic equation are compactly supported, the solution exists, and the support of the solution is controlled by the eigenvalues of the matrix of leading coefficients of the operator. So, if these coefficients ensure that the support of the solution is compact for all $t > 0$, this solution will surely belong to $\mathfrak{D}(\overline{H})$. This idea is due to Povzner [107]; it was later rediscovered and used in geometrical situations by Chernoff [25].

In particular, if the leading term is the Laplace–Beltrami operator on the manifold M then the propagation of solutions is described by geodesic distance. Therefore, if any geodesic on the manifold can be indefinitely continued – such manifolds are called *geodesically complete* – then the reasoning above establishes essential self-adjointness. Moreover, lower-order terms can be added, as long as they do not destroy two crucial features required by this approach: they do not influence propagation of solutions, and semiboundedness from below must be kept. Shubin has established the following theorem [128].

THEOREM 4.2. *Let M be a complete Riemannian manifold and let H be a second-order symmetric operator of the form*

$$H = -\Delta_g + L, \tag{4.4}$$

where Δ_g is the Laplace–Beltrami operator and L is a differential operator of order not greater than one, with smooth coefficients. Suppose that H is semibounded from below on $\mathscr{D}(M)$. Then H is essentially self-adjoint on $\mathscr{D}(M)$.

Thus, in the case of the standard Schrödinger operator $-\Delta + V$, with $V \in C^\infty$, Theorem 4.2 establishes essential self-adjointness of the minimal operator as soon as it is bounded from below, or, what is equivalent, as soon as

$$-\int_M V|u|^2\,\mathrm{d}x \leqslant \int_M |\nabla u|^2\,\mathrm{d}x + C\int_M |u|^2\,\mathrm{d}x, \quad u \in \mathscr{D}(M). \tag{4.5}$$

Thus, any progress in extending conditions for the inequality (4.5) automatically leads to new self-adjointness conditions. In Section 5 we discuss recent achievements on such estimates.

The local smoothness condition on the potential can be considerably relaxed. In fact, this condition is only needed in order to establish local properties of the domain of the operator, i.e., that $u \in L^2_{\mathrm{loc}}$ and $-\Delta u + Vu \in L^2_{\mathrm{loc}}$ imply $u \in \mathbf{H}^2_{\mathrm{loc}}$; this would justify the reasoning above on local coincidence of domains of $\overline{H}$ and H^*. Following this direction, the regularity conditions on V were gradually weakened up to $V_+ \in L^2_{\mathrm{loc}}$, $V_- \in L^2_{\mathrm{loc}}$ with $p = 2$ for $d \leqslant 3$, $p > 2$ for $d = 4$ and $p = d/2$ for $d \geqslant 5$ (see, for details, [112] for the Schrödinger operator in $\mathbb{R}^d$ and [20] for operators on Riemannian manifolds). The condition on V_+ is sharp, the condition on V_- can be a little bit sharpened, being formulated in terms of local Stummel classes (see, again, [112]). Further weakening of these conditions consists in expressing them not in terms of spaces but rather by means of functional inequalities [20]:

$$\text{for any compact } K \subset M \text{ there exist } a_K < 1 \text{ and } C_K, \text{ such that}$$
$$\int_K |V_-|^2|u|^2\,\mathrm{d}x \leqslant a_K \|\Delta_g u\|^2_{L^2(M)} + C_K\|u\|^2_{L^2(M)}, \quad u \in \mathscr{D}(M). \tag{4.6}$$

However, in practice, all these results are almost sharp: in the example considered in [112], page 172, $V(x) = -\alpha|x|^{-2}$, where the potential just marginally fails to get into the classes above, for $d \geqslant 5$, self-adjointness fails for $\alpha \geqslant \alpha_d$. Here one can satisfy (4.6) for any $a_K \geqslant 1$, by choosing α sufficiently close to α_d.

Results on self-adjointness for semibounded *operators* were preceded by weaker statements, where semibounded *potentials* were involved (see, e.g., [62]).

The condition of the manifold to be complete cannot be dropped, even for the Euclidean space with a nonstandard metric. Examples were constructed (see, e.g., [151]) of elliptic operators of the form $-\sum \partial_j g_{jk}(x)\,\partial_k$ with matrices g_{jk}, which are not essentially self-adjoint on $\mathscr{D}(\mathbb{R}^d)$, due to the fact that the coefficients $g_{jk}(x)$ grow very fast at infinity, thus forcing the propagation speed for solutions of the hyperbolic equation to grow at infinity in such a way that geodesics reach infinity in a finite time. This can, however, happen only in dimensions higher than 2. It was shown by Maz'ya [90] (see also [91]) that in dimensions 1 or 2 this cannot happen, and thus any uniformly elliptic operator in divergence form is essentially self-adjoint on $\mathscr{D}(\mathbb{R}^d)$, $d = 1, 2$.

Returning to the Schrödinger operator in $\mathbb{R}^d$, note that if the potential V is lower semibounded, even the domain of the self-adjoint operator $H^* = \overline{H}$ can be described explicitly, provided some extra conditions are imposed on V, roughly forbidding V to change too fast on a finite distance. So, if, for simplicity, $V \geqslant 1$, $V \in L^\infty_{\mathrm{loc}}$ and

$$\frac{V(x)}{V(y)} \leqslant C, \quad |x - y| \leqslant 1, \tag{4.7}$$

then $\mathfrak{D}(\overline{H}) = \mathbf{H}^2 \cap L^2_{V^2}$; in other words, $-\Delta u + Vu \in L^2$ and $u \in L^2$ imply that both Δu and Vu are in L^2. This property for operators containing several terms is called *separability*. The result above was obtained in [117]. More general (but more complicated) conditions for separability can be found in [92].

4.2. *Nonsemibounded operators*

Suppose that the potential V tends to $-\infty$ at infinity, at least in some directions. This looks like a rather exotic case, from the first glance. However, some important physical models, including the (Stark) Hamiltonian with a constant electric field (thus a linear potential) get into this category. Potentials like $V(x) = -A|x|^2$ arise also in the semiclassical analysis near points of local maximum of the electric potential. In such cases the minimal operator usually turns out to be not semibounded, and Proposition 4.1 cannot be applied, at least directly (see, however, an ingenious trick of Kato, [62]). The Schrödinger equation version of this approach is hard to use as well, since the finite propagation speed property is absent here. Thus the problem of proving essential self-adjointness presents a considerable mathematical challenge, and much work has been done here.

So, consider, first in the Euclidean space, the Schrödinger operator

$$H_V u = -\Delta u + V u, \quad u \in \mathscr{D}\big(\mathbb{R}^d\big), \tag{4.8}$$

with $V \to -\infty$ at infinity. Such an operator is not semibounded, but the involution $u \mapsto \bar{u}$ maps isomorphically the deficiency subspace $\operatorname{Ker}(H_V^* + \mathrm{i}I)$ onto $\operatorname{Ker}(H_V^* - \mathrm{i}I)$, thus H_V has equal deficiency numbers and therefore it has self-adjoint extensions. Following the physical intuition, one must expect that the behavior of integral curves of the complete classical Hamiltonian $H(p,q) = p^2 + V(q)$, $(p,q) \in \mathbb{R}^d \times \mathbb{R}^d$, i.e., solutions of the system $q' = \partial H(p,q)/\partial p$, $p' = -\partial H(p,q)/\partial q$, must determine self-adjointness. If these trajectories can be infinitely continued (the system is *classically complete*), the operator does not need boundary conditions at infinity and therefore must be essentially self-adjoint; if some trajectories reach infinity in a finite time, one has to describe what happens with the system after this time, and this means that some boundary conditions must be specified at infinity, and the choice of these conditions determines the self-adjoint extension. In this case, the intuition turns out to be essentially correct, provided one imposes certain regularity conditions. Without regularity, classical and quantum completeness are not necessarily related even in the one-dimensional case, as discussed in, e.g., [112], Appendix to X.1.

The following approach is used to establish essential self-adjointness of H. This property is equivalent to symmetricity of H^*. Hence, it suffices to prove that, for u, v in the domain of the maximal operator, the equality

$$\langle Hu, v\rangle = \int \big(\nabla u \overline{\nabla v} + V(x) u(x) \overline{v(x)}\big)\, \mathrm{d}x, \tag{4.9}$$

holds, i.e., integration by parts does not produce out-of-integral terms. (Note that one does not expect that integrals of separate terms in (4.9) are finite.) Assuming that local regularity conditions on V are fulfilled, one can hope that for a ball B_R with radius R,

$$\int_{B_R} \Delta u(x) \overline{v(x)}\, \mathrm{d}x + \int_{B_R} \nabla u \overline{\nabla v}\, \mathrm{d}x \to 0, \tag{4.10}$$

as $R \to \infty$, at least along some sequence. This, in fact, can be proved, provided that $V(x)$ can be minorized by a regular function depending only on $r = |x|$ and tending to $-\infty$ not too fast, i.e.,

$$V(x) \geqslant -Q(r), \tag{4.11}$$

where $Q(r)$ is a positive nondecreasing function on the semiaxis, obeying

$$\int_0^\infty Q(r)^{-1/2}\,\mathrm{d}r = \infty. \tag{4.12}$$

This *essential self-adjointness* result by Titchmarsh and Sears (see [13] for a modern exposition), shows that if, e.g., $V(x) = -(1+|x|^2)^{\alpha/2}$ the condition (4.12) is satisfied exactly for $\alpha \leqslant 2$. Separation of variables enables one to study self-adjointness for this operator directly and it turns out that, in fact, if $\alpha > 2$ then the operator is not essentially self-adjoint, so the condition is relatively precise. Also classical completeness takes place exactly for $\alpha \leqslant 2$. More elaborate reasoning enables one to replace (4.11) by a similar estimate expressed in the sense of quadratic forms,

$$\langle Hu, u\rangle \geqslant -\delta\langle \Delta u, u\rangle - \bigl\langle Q(r)u, u\bigr\rangle, \quad u \in \mathscr{D}\bigl(\mathbb{R}^d\bigr), \tag{4.13}$$

for some $\delta \in (0,1)$, thus admitting potentials V which are not even locally bounded from below (see [115]). It is this generalization that establishes self-adjointness for Hamiltonians arising in quantum mechanics, with negative potentials having strong local singularities.

Returning to (4.9), note that, in fact, one cannot guarantee that even under the conditions (4.11) and (4.12), integrals of separate terms in (4.9) are finite. Separability, mentioned at the end of Section 4.1, does not take place even for quadratic forms. What one can prove, nevertheless, and it is an important step in the proof of Titchmarsh–Sears-type theorems, is that the integral $I_Q[u] = \int |\nabla u(x)|^2/Q(2|x|)\,\mathrm{d}x$ is finite for $u \in \mathfrak{D}(\overline{H})$.

This observation turns out to be also essential in finding proper versions of this result for Schrödinger operators on manifolds. In fact, condition (4.12) means also that the Euclidean space $\mathbb{R}^d$, equipped with the modified metric $g_{jk} = \delta_{jk}Q(|x|)^{-1}$ is complete, so finiteness of $I_Q[u]$ is just the finiteness of the Dirichlet integral with respect to this metric.

In the most general setting, essential self-adjointness of nonnecessarily semibounded operators on manifolds was established recently by Oleinik [103].

THEOREM 4.3. *Let M be a complete Riemannian manifold, $V \in L^2_{\mathrm{loc}}$ and let the local condition* (4.6) *be satisfied. Let $Q(x) \geqslant 0$ be a function on M such that $Q^{-1/2}$ is globally Lipschitz on M. Assume that the Schrödinger operator can be estimated from below by $-Q$ in the sense of quadratic forms* (*as in* (4.13)) *and, moreover, assume that*

$$\int Q^{-1/2}\,\mathrm{d}s = \infty \tag{4.14}$$

along any curve in M, going to infinity, i.e., leaving every compact set. Then the minimal operator $H = -\Delta_g + V$ is essentially self-adjoint on $\mathscr{D}(M)$.

Note that the condition (4.14) simply means that the manifold (M, g) is complete with the metric $Q^{-1}g$.

The proof in [103] (see also [20] for further generalizations and general discussion) consists in an elaborate localization: one proves that, for a properly chosen sequence ϕ_n of cut-off Lipschitz functions on M, an approximate integration by parts can be performed for $u, v \in \mathfrak{D}(H^*)$, i.e., the difference

$$\left\langle \phi_n u, H^* v\right\rangle - \left\langle H^* u, \phi_n v\right\rangle$$

can be controlled and tends to zero as the support of ϕ_n extends. This proves that H^* is symmetric. In the process of this localization, the most technically involved step consists in estimating integrals containing $|u|^2$ and $|\nabla u|^2$ with weights via integrals of $|\Delta u|^2$ and $|Hu|^2$.

Note, finally, that the cases of linear or negative quadratic potentials, mentioned in the beginning of this subsection, are dealt with by the most simple versions of Theorem 4.3.

4.3. *Operators with magnetic fields*

For the *magnetic Schrödinger operator* the traditional way of establishing self-adjointness (as well as many other questions) is using inequalities in some way expressing the main property – diamagneticity. The concrete inequality used in self-adjointness studies is the *Kato inequality*, established first in [62] for the Laplacian and then extended to the magnetic Laplacian in [53]. We present the distributional form of Kato's inequality derived in [20]. A distribution ν is called *positive* if $\langle \nu, \phi\rangle \geqslant 0$ for any nonnegative test function ϕ (this definition concerns both the Euclidean space and an arbitrary manifold).

PROPOSITION 4.4 (Kato's inequality). *Let $\mathcal{A}$ be a smooth magnetic potential on the Riemannian manifold M and let $\Delta_{\mathcal{A}}$ be the corresponding magnetic Laplacian defined in* (2.2). *Then, for any $u \in L^1_{\mathrm{loc}}$ such that $\Delta_{\mathcal{A}} u \in L^1_{\mathrm{loc}}$ in the sense of distributions, one has*

$$\Delta|u| \leqslant \Re(\operatorname{sign} u \Delta_{\mathcal{A}} u), \tag{4.15}$$

where Δ is the Laplacian on M and $\operatorname{sign}(u(x))$ *is defined as $u(x)/|u(x)|$ at the points where $u \neq 0$ and zero elsewhere.*

The inequality is proved first for smooth functions u for which one directly establishes $u_\varepsilon \Delta u_\varepsilon \leqslant \Re(u \Delta_{\mathcal{A}} u)$, $u_\varepsilon = (|u|^2 + \varepsilon^2)^{1/2}$, and passes to the limit $\varepsilon \to 0$, and then it extends to any u by means of mollifiers. In [20] an even more general form of the Kato inequality is established, valid for magnetic Laplacians acting on sections of Hermitian vector bundles over M. The conditions on the magnetic potential can be relaxed. The reasoning explained above still works for the continuously differentiable $\mathcal{A}$.

Having (4.15) at disposal, one can follow the proof of essential self-adjointness of the nonmagnetic operator, where, in the process of localization, integrals of the function $|u|$ with different weight were estimated by integrals of Δu. Due to Kato's inequality, these estimates are still valid, with the Laplacian replaced by the magnetic Laplacian. As for estimates of integrals containing the magnetic gradient of u, it is evaluated via the usual gradient, with the help of (3.12). This proves that H^* is symmetric.

This way of reasoning gives, in particular, the following self-adjointness condition obtained by Braverman, Milatovic and Shubin [20].

THEOREM 4.5. *Suppose that the magnetic potential $\mathcal{A}$ on the manifold M is continuously differentiable and the electric potential satisfies the hypotheses of Theorem* 4.3. *Then the operator $H_{\mathcal{A},V}$ is essentially self-adjoint on $\mathscr{D}(M)$.*

The conditions on the magnetic potential in Theorem 4.5 can be relaxed. On a general manifold, it is sufficient to suppose that it is locally Lipschitz. Moreover, in the special case of the Euclidean space, with lower semibounded potential V, essential self-adjointness is proved as soon as the minimal operator is defined on $\mathscr{D}(\mathbb{R}^d)$, i.e., under conditions (3.2) (see [72]).

Kato's inequality can also be used to prove the semigroup form of the diamagnetic inequality (see Theorem 9.2). It will be discussed later in relation to spectral estimates.

Pauli operator. There are no special studies of essential self-adjointness of Pauli operators (with or without electric potential) but a number of results can be obtained from the connection with the magnetic Schrödinger operator; see (2.6). In dimension 2, as well as in dimension 3, provided the magnetic field has constant direction, (2.7) reduces the self-adjointness problem for the Pauli operator $\mathcal{P}_{\mathcal{A}} + V$ to the same problem for two magnetic Schrödinger operators with electric potentials, respectively, $V \pm B$. Thus the results above on the magnetic Schrödinger operator can be used. For general magnetic field one does not have such a decomposition and one has to study magnetic Schrödinger operators with matrix-valued potentials. Such a study was performed in [20], even in a more general setting, for Schrödinger-type operators in sections of Hermitian vector bundles over manifolds. Self-adjointness conditions for Pauli operators can be extracted from [20] (however, no explicit formulations are given there).

5. Quadratic forms estimates

Now we return to the definition of the operators by means of quadratic forms. As it was mentioned above, the quadratic form definition requires considerably less regularity of the electric and magnetic potentials. In fact, they even do not have to be functions; certain classes of measures are allowed. There are several basic questions which we are going to address. First, for a *nonnegative* electric potential V, we discuss conditions under which the form $\int |u|^2 V \, \mathrm{d}x$ is closable with respect to the form $\mathfrak{h}$ of the unperturbed operator H_0. This enables one to use Theorem 3.7 to define the operator $H_0 + V$ by means of quadratic forms.

Another group of questions deals with *nonpositive* potentials. The first problem here lies in finding conditions of boundedness of the quadratic form of the potential with respect to the unperturbed quadratic form, with control over the relative bound. Such estimates are needed already when one studies the essential self-adjointness, see Theorem 4.2. For more singular potentials, this estimate is required by the KLMN theorem. A convenient case, frequent but not universal, is the infinitesimal form-boundedness, where one may avoid controlling constants. Infinitesimal form-boundedness is also closely related to relative compactness which is crucial in the question of finiteness or discreteness of the negative spectrum of the operator $H_0 + V$.

The above questions are classical in spectral theory, going back to the 1930s, but they continue to be of much interest up to today and the latest results, which may be considered as being final for the Schrödinger and magnetic Schrödinger operators, were obtained in the course of the last 3–4 years.

Clearly, the natural next step would be to consider potentials with variable sign. Here the situation is not quite clear yet, however essential progress was achieved lately as well. Until quite recently, the analysis of such potentials was reduced to the consideration of positive and negative parts separately. This led to sufficient but nonnecessary conditions for the crucial inequalities to hold, and thus to conditions only sufficient for semiboundedness of operators, discreteness of the whole or only negative spectrum. Lately new methods have been developed enabling one to take into account possible cancellation of influence of positive and negative parts of the potential. In the process even exact criteria for relative boundedness, infinitesimal boundedness and compactness of the quadratic forms of the perturbation, were found, and this automatically produces new semiboundedness conditions and description of the spectrum for the perturbed operators.

The progress in this field is mainly due to the developing of efficient methods in function theory, especially capacities and potentials. In the next section we present a short review of the necessary facts, for our special case of second-order operators. For details on the general cases, as well as for proofs, and further references, the reader is referred to the book by Maz'ya [91], as well as [4,93] and the recent papers [65,95–97,152].

5.1. *Capacities, potentials and functional classes*

Quite long ago it was noticed, probably, first by Wiener, that the Lebesgue measure is not sufficient to control fine effects for partial differential equations. It turned out that the notion of *capacity* provides one with the adequate instrument. In what follows, when we say that some quantities are equivalent, we mean that their ratio is bounded from above and from below by some constants depending only on the dimension d of the underlying space.

Let F be a compact set in $\mathbb{R}^d$ and let G be an open set containing F. Denote by $\mathfrak{N}(F, G)$ the set of functions $u \in \mathscr{D}(G)$ such that $u \geqslant 1$ on F. Then the *Wiener capacity* of F with respect to G is defined as

$$\operatorname{cap}(F, G) = \inf\left\{ \|\nabla u\|^2_{L^2(G)} \colon u \in \mathfrak{N}(F, G) \right\}. \tag{5.1}$$

In the case when G coincides with $\mathbb{R}^d$, G is omitted from this notation. For $d \geqslant 3$, one always considers $G = \mathbb{R}^d$, since for F being a subset in a cube Q, $\operatorname{cap}(F, \widetilde{Q})$ is equivalent to $\operatorname{cap}(F)$ for the concentric cube $\widetilde{Q}$ with twice as large sidelength.

Wiener capacity for $d \geqslant 3$ is equivalent to *Riesz capacity*, defined by

$$\inf\bigl\{\|f\|_{L^2(\mathbb{R}^d)}\colon\ R * f \geqslant 1 \text{ on } F\bigr\}, \tag{5.2}$$

where the Riesz kernel, $R(|x|) = C_d|x|^{-d+1}$, is the kernel of the integral operator $(-\Delta)^{-1/2}$ in $L^2(\mathbb{R}^d)$.

Another capacity, named after Bessel, is also needed, defined equivalently as

$$\operatorname{Cap}(F) = \inf\bigl\{\|\nabla u\|^2_{L^2(\mathbb{R}^d)} + \|u\|^2\colon\ u \in \mathfrak{N}(F)\bigr\}, \tag{5.3}$$

or

$$\inf\bigl\{\|f\|^2_{L^2(\mathbb{R}^d)}\colon\ G * f \text{ on } F \geqslant 0\bigr\}; \tag{5.4}$$

where the Bessel kernel,

$$B\bigl(|x|\bigr) = C_d|x|^{(1-d)/2}K_{(d-1)/2}\bigl(|x|\bigr), \tag{5.5}$$

K_ν being the modified Bessel functions, is the integral kernel of the operator $(1-\Delta)^{-1/2}$ in $L^2(\mathbb{R}^d)$.

Note that the Riesz and Bessel kernels have the same behavior at zero, while at infinity $B(r)$ decays power-like, and $R(r)$ decays exponentially. Evidently, $\operatorname{cap}(F) \leqslant \operatorname{Cap}(F)$. Bessel capacity is used first of all in the study of the space $\mathbf{H}^1$. Riesz capacity handles the case of the space $\mathbf{h}^1$ (the homogeneous Sobolev space), the closure of $\mathscr{D}(\mathbb{R}^d)$ with respect to the metric $\|\nabla u\|^2$. The latter space is considered only for $d \geqslant 3$; otherwise it is not a space of functions. For such d, of course, $\mathbf{H}^1 \subset \mathbf{h}^1$, and locally the spaces coincide. Among other properties of capacities, we mention the two of most importance for us. First, it is their relation to Lebesgue and Hausdorff measures. One has

$$\operatorname{cap}(F) \geqslant C\bigl(\operatorname{meas}(F)\bigr)^{1-2/d}, \quad d \geqslant 3. \tag{5.6}$$

Furthermore, for $d \geqslant 3$, all sets with finite Hausdorff measure of dimension $d-2$ have zero capacity. For Bessel capacity this statement is correct also for the case of $d = 2$ provided the logarithmic Hausdorff measure of a set is finite. On the other hand, if the set has positive Hausdorff measure of dimension larger than $d-2$, then it has positive capacity.

Another property is related to convergence. We say that a sequence of Lipschitz functions converges to zero quasieverywhere with respect to Riesz or Bessel capacity if it converges everywhere with exception of some set having zero capacity. The property we need says that if a sequence of functions in $\mathscr{D}$ converges to zero in $\mathbf{h}^1$, resp. $\mathbf{H}^1$, then one can select a subsequence converging quasieverywhere with respect to the corresponding capacity.

5.2. *Closability*

In this subsection we will consider perturbation of the Laplace (resp., magnetic Laplace or Pauli) operator by a lower semibounded quadratic form $\mathfrak{g}_\rho[u] = \int |u|^2\, \mathrm{d}\rho(x)$, where ρ is a

Borel measure on $\mathbb{R}^d$. The unperturbed form $\mathfrak{h}$ is closable, so, according to Theorem 3.7, one has to check closability of $\mathfrak{g}_\rho$ with respect to $\mathfrak{h}$, all forms defined on $\mathscr{D}(\mathbb{R}^d)$. We say that the measure ρ is absolutely continuous with respect to Cap if all sets with vanishing Bessel capacity have zero measure ρ.

THEOREM 5.1. *Let the measure ρ be absolutely continuous with respect to* Cap. *Then the form $\mathfrak{g}_\rho$ is closable with respect to the forms of the Laplacian and the magnetic Laplacian*; *for the Laplace operator this condition on the measure is also necessary for closability.*

PROOF. We prove the sufficiency part. For a given magnetic field $\mathcal{A} \in L^2_{\mathrm{loc}}$ we denote by $\mathfrak{h}_\mathcal{A}$ the magnetic quadratic form (3.5) and by $\mathfrak{h}$ the usual Dirichlet form, $\mathfrak{h}[u] = \|\nabla u\|^2$. Let $u_n \in \mathscr{D}$ be a sequence such that $\mathfrak{h}_\mathcal{A}[u_n] + \|u\|^2 \to 0$ and assume it is a Cauchy sequence with respect to the form $\mathfrak{g}_\rho$. Consider the sequence $v_n = |u_n|$. These functions are Lipschitz, and since $||v_n| - |v_m|| \leqslant |v_n - v_m|$, they form a Cauchy sequence for $\mathfrak{g}_\rho$. At the same time, due to (3.12), the sequence v_n converges to zero in $\mathbf{H}^1$. Therefore it contains a subsequence converging to zero quasieverywhere and, due to the conditions imposed on the measure, almost everywhere with respect to measure ρ. Now, since this subsequence is a Cauchy sequence in L^2_ρ, it must converge to zero in this space. The necessity proof is more cumbersome (see [91]). □

Theorem 5.1 enables one to define self-adjoint Schrödinger and magnetic Schrödinger operators with a very wide class of measures as electric potentials. If such a measure is absolutely continuous with respect to Lebesgue measure, this situation (without magnetic field) is taken care of in Example 3.8. Now some singular measures are allowed, in particular, supported on smooth surfaces of co-dimension one and absolutely continuous with respect to the surface measure, or, more generally, on sets having Hausdorff dimension larger than $d - 2$ and absolutely continuous with respect to the corresponding Hausdorff measure. In addition, one may, having a nonsemibounded potential, break it into the positive and negative parts, add the positive part using the theorem, and then add the negative part, using other perturbation theorems.

The reasoning above does not work for the Pauli operator, the main reason for this (as well as for many other questions concerning Pauli operators) being the absence of the diamagnetic inequality for the Pauli quadratic form.

5.3. *Relative boundedness*

As it can be seen above, the key instrument in establishing essential self-adjointness as well as in defining the operator by means of quadratic forms is given by inequalities of the form

$$\left|\mathfrak{g}_\rho[u]\right| = \left|\int |u|^2 \rho(\mathrm{d}x)\right| \leqslant C_1 \mathfrak{h}[u] + C_2 \|u\|^2, \quad u \in \mathscr{D}\big(\mathbb{R}^d\big), \tag{5.7}$$

where $\mathfrak{h}$ is the quadratic form of the unperturbed operator (Laplace, magnetic Laplace, etc., probably, containing some previously included perturbations). In many cases one needs the constant C_1 in (5.7) to be smaller (or not larger) than 1. To trace the constants in inequalities may be a rather tedious job, but the task simplifies considerably if the form $\mathfrak{g}_\rho$ is infinitesimally form-bounded with respect to $\mathfrak{h}$, i.e., (5.7) holds for any positive $C_1 = \varepsilon$, with C_2, of course, depending on ε. For the needs of spectral analysis, one should be able also to find conditions for the form $\mathfrak{g}_\rho$ to be *compact* with respect to $\mathfrak{h}$, meaning that any sequence u_n bounded in L^2 and with respect to $\mathfrak{h}$ contains a subsequence which converges with respect to $\mathfrak{g}_\rho$. Below we present some results concerning these questions.

Consider first the case of a nonnegative measure ρ. The easiest result of the form (5.7) follows directly from Hölder's inequality and Sobolev's embedding theorem. If one assumes that ρ is absolutely continuous with respect to Lebesgue measure, $\rho(\mathrm{d}x) = V(x)\,\mathrm{d}x$ and for some K, the function $(V - K)_+$ belongs to L^r, where $r = d/2$ for $d \geqslant 3$ and $r \geqslant 1$ for $d = 2$, (5.7) holds for an arbitrarily small C_1, i.e., $\mathfrak{g}$ is infinitesimally form-bounded with respect to $\mathfrak{h}$. The proof is essentially the same as in Example 3.8: if the condition above is satisfied then, by choosing a proper K, one can make the L^r norm of $(V - K)_+$ arbitrarily small. The Hölder inequality yields $\mathfrak{g}[u] \leqslant \|V\|_{L^r}\|u\|^2_{L^q}$, $q = 2r/(r-1)$. Finally, the latter norm of u is estimated by $\mathfrak{h}[u] + \|u\|^2$ according to the embedding theorem.

Thus, for example, an arbitrary finite configuration of Coulomb potentials in $\mathbb{R}^d$, $d \geqslant 3$, $V = \sum Z_j/|x - x^{(j)}|$ satisfies the above conditions.

If the function V itself belongs to L^r, $r = d/2$, $d \geqslant 3$, (5.7) holds even without the lower order term $\|u\|^2_{L^2}$:

$$\left|\int |u|^2\rho(\mathrm{d}x)\right| \leqslant C_1\mathfrak{h}[u], \quad u \in \mathscr{D}\big(\mathbb{R}^d\big), \tag{5.8}$$

with $C_1 = c_d\|V\|_{L^{d/2}}$ and the constant c_d depending only on dimension; one just uses the Sobolev inequality (3.11) instead of the embedding theorem.

These results are easy to use, but they do not cover the important case of potentials behaving like r^{-2}, where r is the distance to some point in $\mathbb{R}^d$ or to a submanifold. Although this singularity is essentially stronger than the most interesting Coulomb case, it arises, for example, when one considers singular magnetic fields. At the same time, for V having the order r^{-2}, the inequality of the form (5.7) and even (5.8) may hold, the best known case being the Hardy inequality, viz.

$$\int |x|^{-2}|u|^2\,\mathrm{d}x \leqslant \frac{(d-2)^2}{4}\int |\nabla u|^2\,\mathrm{d}x, \quad d \geqslant 3, u \in \mathbf{h}^1\big(\mathbb{R}^d\big). \tag{5.9}$$

Finding weights (and measures) which can replace $|x|^{-2}$ here is important for many questions in spectral theory, in particular, for obtaining eigenvalue estimates.

The criteria for estimates of the form (5.7) and (5.8) were obtained first in the 1960s by Maz'ya, in terms of capacity [89]. Later, several equivalent criteria were found. One can follow the development in the books [4,91,93] and a series of papers (we mention [95] in the first place). These results deal with a nonnegative measure ρ. Along with capacity, criteria for estimates can be expressed in terms of potentials of the measure. They are defined as $I(\rho) = R * \rho$ (Riesz potential), and $J(\rho) = B * \rho$ (Bessel potential). Of course, certain local and global restrictions on these measures must be imposed in order that the potentials are finite almost everywhere; see the aforementioned references for details and additional boundedness criteria.

THEOREM 5.2 (Relative boundedness for nonnegative measures). *Let ρ be a nonnegative Borel measure on $\mathbb{R}^d$.*

(1) *For $d \geqslant 3$ the following statements are equivalent, with equivalent constants C_1, C_2, C_3:*
  (a) *the inequality $\int |u|^2 \rho(\mathrm{d}x) \leqslant C_1 \|\nabla u\|^2$, $u \in \mathscr{D}(\mathbb{R}^d)$, holds;*
  (b) *for every compact $F \subset \mathbb{R}^d$,*

$$\rho(F) \leqslant C_2 \operatorname{cap}(F); \tag{5.10}$$

  (c) *for every ball B,*

$$\int_B \big(I(\rho_B)\big)^2 \,\mathrm{d}x \leqslant C_3 \rho(B), \tag{5.11}$$

where ρ_B is the restriction of ρ to B.

(2) *For $d \geqslant 2$ the following statements are equivalent, with equivalent constants C_1, C_2, C_3:*
  (a) *the inequality $|\int |u|^2 \rho(\mathrm{d}x)| \leqslant C_1(\|\nabla u\|^2 + \|u\|^2)$, $u \in \mathscr{D}(\mathbb{R}^d)$, holds;*
  (b) *for every compact $F \subset \mathbb{R}^d$,*

$$\rho(F) \leqslant C_2 \operatorname{Cap}(F); \tag{5.12}$$

  (c) *for every ball B,*

$$\int_B \big(J(\rho_B)\big)^2 \,\mathrm{d}x \leqslant C_3 \rho(B). \tag{5.13}$$

REMARK 5.3. The words above, concerning "equivalent constants", require an explanation. This expression means that if one of the equivalent statements is true, with some constant C_j, then another statement is also true with some constant C_k, so that $C_k \leqslant cC_j$ for some c depending only on the dimension. Conversely, if the second condition holds with some C_k, the first one holds with some C'_j, again $C'_j \leqslant cC_k$. The theorem does not declare that $C_j = C'_j$. Thus the statements of the theorem are equivalent as it concerns the inequalities involved, but still there is a gap in the size of constants. This remark concerns also all further results where equivalent constants are present.

Thus, if for a given measure ρ in (5.12) or (5.13) the constants C_2 or C_3 are sufficiently small, so that the constant C_1 is smaller than 1, the form $\int |u|^2 \rho(\mathrm{d}x)$ is bounded with respect to the form of the Laplacian with relative bound smaller than 1, and therefore one can apply the KLMN theorem to define the Schrödinger operator with "potential" $-\rho$.

Since capacitory or potential terms are rather implicit, it is fairly hard to apply the above criteria directly. One can, however, extract more transparent *sufficient* conditions, using estimates for capacity and potentials. For example, the estimate (5.6) of capacity from below by Lebesgue measure, gives a sufficient condition $\rho(F) \leqslant C_4 \operatorname{Cap}(F)$ for relative boundedness.

In particular, let $\rho(\mathrm{d}x) = V(x)\,\mathrm{d}x$, $V \geqslant 0$. Denote by $M(t) = M(t, V)$ the set $\{x\colon\ V(x) > t\}$ and set $m(t) = \operatorname{meas}(M(t))$. Then for a fixed $\gamma = \operatorname{meas}(F)$, the largest possible value of $\rho(F)$ is obtained when $F = M(s)$, where $s = \sup\{t\colon\ m(t) \geqslant \gamma\}$. An easy calculation shows that if $m(t) \leqslant C_6 t^{-d/2}$ then

$$\rho\bigl(M(s)\bigr) = -\int_s^\infty t\,\mathrm{d}m(t) \leqslant C_7\bigl(\operatorname{meas}\bigl(M(s)\bigr)\bigr)^{1-2/d},$$

and therefore, $\rho(F) \leqslant C_7 \operatorname{meas}(F)^{1-2/d}$ for any compact F. This, together with (5.6), gives the following sufficient condition for boundedness [91].

COROLLARY 5.4. *Let* $\rho(\mathrm{d}x) = V\,\mathrm{d}x$, $V \geqslant 0$, $d \geqslant 3$, *and* $\operatorname{meas}(M(t, V)) \leqslant C_6 t^{-d/2}$. *Then the estimate* (5.8) *holds with a constant equivalent to* C_6.

(Note that the functions V satisfying the above condition are said to belong to the "weak $L^{d/2}$" space, denoted $L_w^{d/2}$.)

When applying the KLMN theorem, and generally, establishing semiboundedness, we need not (5.8) but only (5.7) with $C_1 < 1$. Therefore the following condition is more convenient to use, putting restrictions only on "large" values of V but not requiring any decay at infinity. It follows automatically from Corollary 5.4.

COROLLARY 5.5. *Let* $\rho(\mathrm{d}x) = V\,\mathrm{d}x$, $V \geqslant 0$, $d \geqslant 3$, *and assume that* $\operatorname{meas}(M(t, V)) \leqslant C_6 t^{-d/2}$ *for* $t > t_0$. *Then* (5.7) *holds, with* C_1 *equivalent to* C_6 *and some* C_2 *depending on* t_0. *If the resulting* C_1 *is not greater than* 1, *the quadratic form* $\|\nabla u\|^2 - \int V|u|^2\,\mathrm{d}x$ *is lower semibounded on* $\mathscr{D}(\mathbb{R}^d)$. *If* $C_1 < 1$, *this quadratic form defines a self-adjoint operator in* $L^2(\mathbb{R}^d)$.

The general theorem can also be customized for measures supported on submanifolds or fractal sets, using the relation of capacity to Hausdorff measure.

Due to the weak diamagnetic inequality (3.12), the conditions above immediately imply relative boundedness and, resp., semiboundedness of the forms for the magnetic Schrödinger operator.

As it was mentioned above, if one can establish infinitesimal form-boundedness of the potential perturbation, one can handle the problem of defining the operator more comfortably, since one does not need to trace the values of the constants. As for the conditions of Corollary 5.5, infinitesimal form-boundedness obviously follows if one replaces there the inequality $\operatorname{meas}(M(t, V)) \leqslant C_6 t^{-d/2}$ by $\operatorname{meas}(M(t, V)) = \mathrm{o}(t^{-d/2})$, $t \to \infty$. A recent criterion [97], similar to Theorem 5.2, can also be expressed in terms of capacities (for other terms, see [97], Theorem 4.1).

THEOREM 5.6. *For a nonnegative measure ρ on $\mathbb{R}^d, d \geqslant 2$, infinitesimal form-boundedness of $\int |u|^2 \rho(\mathrm{d}x)$ with respect to $\|\nabla u\|^2$ is equivalent to*

$$\lim_{\delta\to 0} \sup\left\{\frac{\rho(F)}{\mathrm{Cap}(F)}\colon F \subset \mathbb{R}^d, \mathrm{diam}(F) \leqslant \delta\right\} = 0. \tag{5.14}$$

5.4. *Nonsign-definite weights*

In the case of nonsign-definite weights, relative boundedness conditions can, of course, be obtained from the sign-definite case. In fact, if we represent the measure ρ as the difference of two nonnegative measures, $\rho = \rho_+ - \rho_-$, and the measure ρ_+ is closable with respect to the form $\|\nabla u\|^2$, then the boundedness with proper relative bound (infinitesimal boundedness) of the form $\int |u|^2 \rho_-(\mathrm{d}x)$ with respect to $\|\nabla u\|^2$ implies the same with respect to a larger form $\|\nabla u\|^2 + \int |u|^2 \rho_+(\mathrm{d}x)$, and this, as before, leads to semiboundedness and closedness of the whole perturbed form. This, however, does not take into account an improvement of situation which hopefully may happen if the influence of ρ_+ and of ρ_- cancel each other. It was only in 2002 that a new approach was developed capable of taking into account such cancellation. As a result, necessary and sufficient conditions were found for form-boundedness and infinitesimal form-boundedness [96].

THEOREM 5.7. *Let ρ be a Borel measure on $\mathbb{R}^d$.*

(1) *For $d \geqslant 3$ the following are equivalent, with equivalent constants*:

(a) *the inequality $|\int |u|^2 \rho(\mathrm{d}x)| \leqslant C_1 \|\nabla u\|^2$, $u \in \mathscr{D}(\mathbb{R}^d)$ holds*;

(b) *there exists a vector field $\boldsymbol{\Gamma} = (\Gamma_1, \dots, \Gamma_d) \in L^2_{\mathrm{loc}}$ such that the quadratic form $\mathfrak{g}_{\boldsymbol{\Gamma}}[u] = \int (\sum |\Gamma_j|^2)|u|^2\,\mathrm{d}x$ satisfies the inequality $\mathfrak{g}_{\boldsymbol{\Gamma}}[u] \leqslant C_2\|\nabla u\|^2$, $u \in \mathscr{D}(\mathbb{R}^d)$, and $\rho = \mathrm{div}\,\boldsymbol{\Gamma}$ in the sense of distributions.*

(2) *The form $\int |u|^2 \rho(\mathrm{d}x)$ is bounded with equivalent constants (infinitesimally bounded) with respect to $\|\nabla u\|^2$, $d \geqslant 1$, if and only if there exist a vector field $\boldsymbol{\Gamma} = (\Gamma_1, \dots, \Gamma_d) \in L^2_{\mathrm{loc}}$ and a function $\Gamma_0 \in L^1_{\mathrm{loc}}$ such that the quadratic form $\int (|\Gamma_0| + \sum |\Gamma_j|^2)|u|^2\,\mathrm{d}x$ is bounded (infinitesimally bounded) with respect to $\|\nabla u\|^2$ and $\rho = \mathrm{div}\,\boldsymbol{\Gamma} + \Gamma_0$ in the sense of distributions.*

PROOF. In the expression

$$\int |u|^2 \,\mathrm{div}\,\boldsymbol{\Gamma}\,\mathrm{d}x = \int u\bar{u}\,\mathrm{div}\,\boldsymbol{\Gamma}\,\mathrm{d}x, \quad u \in \mathscr{D}\big(\mathbb{R}^d\big),$$

we integrate by parts, obtaining

$$\left|\int |u|^2 \,\mathrm{div}\,\boldsymbol{\Gamma}\,\mathrm{d}x\right| = 2\left|\int \Re(u\nabla u)\Gamma\,\mathrm{d}x\right| \leqslant 2\|\nabla u\|\left(\int |u|^2|\Gamma|^2\,\mathrm{d}x\right)^{1/2},$$

and this explains the sufficiency part of the theorem. The "necessary" part is by far more delicate. It turns out that one can even get explicit expressions for Γ_0 and $\boldsymbol{\Gamma}$:

$$\boldsymbol{\Gamma} = -\nabla(1-\Delta)^{-1}\rho, \qquad \Gamma_0 = (1-\Delta)^{-1}\rho,$$

where $(1-\Delta)^{-1}\rho$ is understood in the sense of distributions or as the Bessel potential of order 2. □

In this way, the case of nonsign-definite potential is reduced to the case of a nonnegative potential, which is taken care of by Theorems 5.2 and 5.6.

Again, by means of the weak diamagnetic inequality (3.12), the above theorem gives *sufficient* conditions for boundedness (infinitesimal boundedness) for the case when a magnetic field is present. Necessary conditions for that case are unknown at the moment of writing.

5.5. *Relative compactness*

In the study of *spectral properties* of the Schrödinger operator the key role is played by compactness theorems. As it will be shown in the next section, questions of discreteness of the whole or negative spectrum as well as finiteness of the negative spectrum can be reduced to the relative compactness of corresponding forms.

Generally, having a closable nonnegative form $\mathfrak{h}$ in the Hilbert space, we say that the form $\mathfrak{g}$ is *compact* with respect to $\mathfrak{h}$ if any sequence u_n, such that $\mathfrak{h}[u_n]$ is bounded, contains a subsequence, convergent in the $\mathfrak{g}$-norm.

In other way, this property is convenient to describe by means of the *Birman–Schwinger operator* $T_{\mathfrak{g},\mathfrak{h}}$ defined by the quadratic form $\mathfrak{g}[u]$ in the Hilbert space $\mathfrak{D}(\mathfrak{h})$, equipped with the norm $\mathfrak{h}[u] + \|u\|^2$ (see also Section A.10). In more details, using the sesquilinear forms corresponding to our quadratic forms, this means

$$\mathfrak{h}[T_{\mathfrak{g},\mathfrak{h}}u, v] = \big\langle (H+1)^{1/2}T_{\mathfrak{g},\mathfrak{h}}u, (H+1)^{1/2}v\big\rangle = \mathfrak{g}[u, v], \tag{5.15}$$

where H is the self-adjoint operator corresponding to the form $\mathfrak{h}$. Compactness of the form $\mathfrak{g}$ with respect to $\mathfrak{h}$ means that the operator $T_{\mathfrak{g},\mathfrak{h}}$ is compact. If we return to our basic Hilbert space, setting $(H+1)^{1/2}u = \phi$, $(H+1)^{1/2}v = \psi$ in (5.15), we find that the operator $T_{\mathfrak{g},\mathfrak{h}}$ is unitary equivalent to the operator $K_{\mathfrak{g},\mathfrak{h}}$ there, defined by the relation

$$\langle K_{\mathfrak{g},\mathfrak{h}}\phi, \psi\rangle = \mathfrak{g}\big[(H+1)^{-1/2}\phi, (H+1)^{-1/2}\psi\big]. \tag{5.16}$$

The operator $K_{\mathfrak{g},\mathfrak{h}}$ is just another useful variant of the Birman–Schwinger operator which is applied time after time in spectral analysis; see also Sections 6.2, 7.1 and A.10.

In the particular case when $H = -\Delta$ and $\mathfrak{g} = \mathfrak{g}_\rho$, the operator $K_{\mathfrak{g},\mathfrak{h}}$, according to (5.16), is the integral operator with kernel

$$K_{\mathfrak{g},\mathfrak{h}}(x, y) = \int B\big(|x-z|\big)B\big(|y-z|\big)\rho(\mathrm{d}z), \tag{5.17}$$

where B is the Bessel kernel (5.5), the integral kernel for the operator $(-\Delta+1)^{-1/2}$. When $\rho = V\,\mathrm{d}x$, the operator $K_{\mathfrak{g},\mathfrak{h}}$ can also be written as $(H+1)^{-1/2}V(H+1)^{-1/2}$.

A number of spectral properties of the Schrödinger operator can be expressed in terms of the Birman–Schwinger operator. In particular, relative compactness of the form $\mathfrak{g}_\rho$ is simply compactness of $T_{\mathfrak{g},\mathfrak{h}}$ or $K_{\mathfrak{g},\mathfrak{h}}$.

Unlike boundedness, compactness conditions for forms in the whole space consist of two parts. Local conditions assure that the operator of the type (5.15) or (5.16) defined by the restriction of the measure ρ to any bounded ball is compact. Additionally, one has to require some sort of decay at infinity, so that the operator defined by the restriction of the measure ρ to the exterior of the ball decays in the norm as the ball extends; here the boundedness conditions described in Sections 5.3 and 5.4 play a key role. This implements the basic property of compact operators: the norm limit of a sequence of compact operators is compact.

The easiest compactness conditions follow from the Sobolev–Rellich embedding theorems (see, e.g., [32]). Let, for example, $\rho(\mathrm{d}x) = V(x)\,\mathrm{d}x$, $V \in L^\infty$ and $V \to 0$ as $|x| \to \infty$. In the ball B_R, Hölder's inequality gives $|\int_{B_R} |u|^2 V\,\mathrm{d}x| \leqslant \|u\|^2_{L^q(B_R)}\|V\|_{L^{q^*}(B_R)}$ with some $q^* > d/2$, $(2q)^{-1} + q^{*-1} = 1$. Since $q^{-1} > d^{-1} - 2^{-1}$, the Sobolev–Rellich embedding theorem assures compactness of the embedding of $\mathbf{H}^1(B_R)$ into $L^q(B_R)$, and therefore grants compactness of the form $\int_{B_R} |u|^2 V\,\mathrm{d}x$ in $\mathbf{H}^1(B_R)$. Outside the ball, the supremum of V goes to zero as $R \to \infty$, and therefore, $\sup_u |\int_{\complement B_R} |u|^2 V\,\mathrm{d}x|/\|u\|^2$ goes to zero as $R \to \infty$; this demonstrates the general scheme. One can see that even some singularities of V are allowed in a compact set, as long as the step involving Hölder's inequality goes through.

Looking at this example, one might get a feeling that relative compactness must be closely related to infinitesimal form-boundedness. This is partly correct. If the form $\mathfrak{g}$ is compact with respect to $\mathfrak{h}$, it is infinitesimal form-bounded. The converse is not true: even the form $\int |u|^2\,\mathrm{d}x$ is not compact in $\mathbf{H}^1(\mathbb{R}^d)$. To show this, one can take a function u_0 supported in the unit cube and then construct a sequence u_j consisting of the shifted copies of u placed in some other, disjoint cubes. Surely, the sequence u_j is not compact with respect to $\|u\|_{L^2}$. This demonstrates that one cannot drop the requirement of decay of the potential at infinity.

For a nonnegative measure ρ, the necessary and sufficient compactness conditions in terms of capacity were first found by Maz'ya [89], and later developments (see [4,91,93]) are related to finding equivalent formulations, in particular, in terms of potentials, similar to Theorem 5.2. We present here the initial formulation from [89]. It uses the function of sets defined by

$$\pi(F) = \pi(F;\rho) = \frac{\rho(F)}{\mathrm{Cap}(F)},$$

provided the Bessel capacity is positive, and $\pi(F) = 0$ otherwise.

THEOREM 5.8. *The conditions*

$$\lim_{\delta\to 0}\sup\bigl\{\pi(F)\colon\ \mathrm{diam}(F) \leqslant \delta\bigr\} = 0 \tag{5.18}$$

and

$$\limsup_{R\to\infty}\bigl\{\pi(F):\ F\subset \complement B_R, \operatorname{diam}(F)\leqslant 1\bigr\}=0 \tag{5.19}$$

are necessary and sufficient for compactness of the form $\int |u|^2\rho(\mathrm{d}x)$ *in the Sobolev space* $\mathbf{H}^1(\mathbb{R}^d)$, $d\geqslant 3$.

The compactness conditions for $d=2$ are considerably more cumbersome, see [4,93]. Recalling the discussion preceding Theorem 5.8, note that (5.18) is a local condition, restricting the local singularities of ρ, while (5.19) is a condition of decay at infinity. To understand why the theorem is likely to be correct (not proving it, of course), one can just try constructing a counterexample, a sequence, bounded in $\mathbf{H}^1$, but not compact with respect to the form $\mathfrak{g}$. On the one hand, one might try taking some function u_0 with compact support and then construct a sequence u_n of shifted functions. Supports of such functions run away towards infinity and the global condition (5.19) forces u_n to 0 thus failing our effort. Another attempt might consist in, again, starting with a function u_0 with compact support and constructing a sequence of functions u_n with disjoint supports, but not running away. To do this, one takes $u_n(x)=2^{(d-2)/2}u(2^n x)$, so that $\|u_n\|_{L^2}$ are bounded and separated from zero. The size of the supports of u_n decays very fast therefore one can shift them together, so that the supports are disjoint, and fit this infinite sequence into some fixed ball in $\mathbb{R}^d$. However, this time, the condition (5.18) forces $\mathfrak{g}[u_n]$ to zero, and again we have convergence. The actual proof, both of necessity and sufficiency, is based upon these considerations; see [89] for details.

Using estimates from below for the capacity, say, via Lebesgue or Hausdorff measure, one can derive more transparent sufficient compactness conditions. For example, following Corollary 5.5, one obtains the following result as a consequence of Theorem 5.8.

COROLLARY 5.9. *Suppose that* $\rho(\mathrm{d}x)=V\,\mathrm{d}x$ *with a function* $V\geqslant 0$ *which belongs locally, i.e., on any ball, to the weak class* $L_w^{d/2}$. *For any ball* B_R, *let* $\operatorname{meas}(\{x\in B_R, V(x)\geqslant t\})=\mathrm{o}(t^{-d/2})$ *as* $t\to\infty$, *and for some functions* $s(R), t(R)\to 0$ *as* $\mathbb{R}\to\infty$, $\operatorname{meas}(\{x\in \complement B_R, V(x)\geqslant t\}\leqslant s(R)t^{-d/2})$ *for* $t>t(R)^{-1}$. *Then the form* $\int |u|^2V\,\mathrm{d}x$ *is compact in* $\mathbf{H}^1(\mathbb{R}^d)$, $d\geqslant 3$.

Similar to Theorem 5.7, one obtains a compactness criterion for the case of a signed measure (see [96]).

THEOREM 5.10. *Let* ρ *be a measure on* $\mathbb{R}^d$, $d\geqslant 3$. *The form* $\mathfrak{g}[u]=\int |u|^2\,\mathrm{d}x$ *is compact in* $\mathbf{H}^1$ *if and only if there exists a vector-valued function* $\boldsymbol{\Gamma}\in L^2_{\mathrm{loc}}$ *and a function* $\Gamma\in L^1_{\mathrm{loc}}$ *such that* $\rho=\operatorname{div}\boldsymbol{\Gamma}+\Gamma_0$ *and the form* $\int(|\boldsymbol{\Gamma}|^2+|\Gamma_0|)|u|^2\,\mathrm{d}x$ *is compact in* $\mathbf{H}^1$.

The proof of the sufficiency part goes in the same way as in Theorem 5.7. Necessity is by far more intricate. It turns out that one can use $\boldsymbol{\Gamma}=-\nabla(-\Delta)^{-1}\rho$ and $\Gamma_0=(-\Delta)^{-1}\rho$.

An attempt to derive sufficient conditions for compactness of the form $\mathfrak{g}$ with respect to the magnetic Dirichlet form $\mathfrak{h}_{\mathcal{A}}[u]=\int |\nabla u+\mathrm{i}\mathcal{A}u|^2\,\mathrm{d}x$ from the above results along the lines of reasoning for boundedness or infinitesimal boundedness, fails. In fact, supposing that the "nonmagnetic conditions" are satisfied, we take a sequence u_n such that $\mathfrak{h}_{\mathcal{A}}[u_n]$ is bounded. From the weak diamagnetic inequality (3.12) it follows that the sequence $v_n = |u_n|$ is bounded in $\mathbf{H}^1$, and according to the theorems just stated, it is compact with respect to $\mathfrak{g}$. However, it is the sequence v_n that is compact due to this reasoning, not u_n, as we wish. Hence, to obtain compactness in this case, we need some more instruments.

Recall that we are interested in compactness of the Birman–Schwinger operator in (5.16), $K_{\mathfrak{g}_\rho,\mathfrak{h}_{\mathcal{A}}}$, where $\mathfrak{h}_{\mathcal{A}}[u] = \|(\nabla + \mathrm{i}\mathcal{A})u\|^2$ is the quadratic form of the operator $-\Delta_{\mathcal{A}}$, see Section 2. Suppose that the measure ρ is nonnegative and compare $K_{\mathfrak{g}_\rho,\mathfrak{h}_{\mathcal{A}}}$ with operator $K_{\mathfrak{g}_\rho,\mathfrak{h}}$ corresponding to the nonmagnetic Laplacian; from Theorem 5.8 we already know compactness conditions for the latter. However, $K_{\mathfrak{g}_\rho,\mathfrak{h}}$ possesses one more important property, it is *positivity preserving* (see also Section 7.5). In fact, $(-\Delta + 1)^{-1/2}$ is an integral operator with *positive kernel*; this can be seen from the explicit formula for the kernel in terms of modified Bessel functions, or from the representation

$$(-\Delta + 1)^{-1/2} = C \int_0^\infty \exp\bigl(t(\Delta - 1)\bigr) t^{-1/2}\,\mathrm{d}t \tag{5.20}$$

and the Poisson formula for the heat kernel (integral kernel for $\exp(t\Delta)$). So, for any nonnegative functions u and v the expression $\langle K_{\mathfrak{g}_\rho,\mathfrak{h}} u, v\rangle = \int (-\Delta + 1)^{-1/2} u \times (-\Delta + 1)^{-1/2} \bar{v} \rho(\mathrm{d}x)$ is nonnegative, which implies that $K_{\mathfrak{g}_\rho,\mathfrak{h}} u \geqslant 0$ a.e. for any $u \geqslant 0$. This property is called *positivity preserving*. Now we use the *strong* diamagnetic inequality in Theorem 9.2 (see Section 9.2), which, after comparing (5.20) and similar representation for $(-\Delta_{\mathcal{A}} + 1)^{-1/2}$, gives that the integral kernel of $(-\Delta_{\mathcal{A}} + 1)^{-1/2}$ (generally, a nonreal one) is majorized by the one of $(-\Delta + 1)^{-1/2}$, $|(-\Delta_{\mathcal{A}} + 1)^{-1/2}(x, y)| \leqslant (-\Delta + 1)^{-1/2}(x, y)$ a.e. This majoration leads us to $|\langle K_{\mathfrak{g}_\rho,\mathfrak{h}_{\mathcal{A}}} u, v\rangle| \leqslant \langle K_{\mathfrak{g}_\rho,\mathfrak{h}} |u|, |v|\rangle$ and therefore

$$\bigl|K_{\mathfrak{g}_\rho,\mathfrak{h}_{\mathcal{A}}} u(x)\bigr| \leqslant K_{\mathfrak{g}_\rho,\mathfrak{h}} |u|(x) \quad \text{a.e.} \tag{5.21}$$

For operators satisfying a relation like (5.21) one says that $K_{\mathfrak{g}_\rho,\mathfrak{h}_{\mathcal{A}}}$ is *dominated* by $K_{\mathfrak{g}_\rho,\mathfrak{h}}$. It is an important question, which properties of the dominating operator are inherited by the dominated one. About compactness such a property was established by Pitt [106]. This gives us a sufficient compactness condition for the magnetic quadratic form.

COROLLARY 5.11. *Let $\mathcal{A} \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$, let the measure ρ be nonnegative, and let the form $\mathfrak{g}[u]$ be compact with respect to $\mathfrak{h}[u] = \|\nabla u\|^2$. Then $\mathfrak{g}[u]$ is compact with respect to $\mathfrak{h}_{\mathcal{A}}[u] = \|\nabla u + \mathrm{i}\mathcal{A}u\|^2$. In particular, this is the case if the hypotheses of Theorem* 5.8, $d \geqslant 3$, *are fulfilled.*

Note that the approach above does not apply to the case of a nonsign-definite measure ρ, where nothing but separate compactness of the forms corresponding to the positive and negative parts of the measure ρ is known to grant compactness of the whole $\mathfrak{g}_\rho$.

We discuss now a somewhat different case, compactness of the form $\mathfrak{g} = \mathfrak{g}_\rho$ in the space $\mathbf{h}^1$, the closure of $\mathscr{D}(\mathbb{R}^d)$ with respect to the metric $\|\nabla u\|^2$, $d \geqslant 3$. This space locally coincides with $\mathbf{H}^1$ but the conditions on the behavior at infinity are weaker. Indeed, functions in $\mathbf{h}^1$ do not have to belong to L^2; according to the Hardy inequality one can only be sure that they belong to L^2 with weight $|x|^{-2}$. Therefore the local conditions of compactness of the form $\mathfrak{g}_\rho$, are the same here as for $\mathbf{H}^1$, but the global conditions are more restrictive: one has to control the quantity $\pi(F)$ not only over the sets with diameter not bigger than 1, as in (5.19), but over all compact sets. The following result can be found in [91] (see also [89]).

THEOREM 5.12. *Let ρ be a nonnegative measure. The form $\mathfrak{g}_\rho$ is compact in $\mathbf{h}^1$ if and only if the condition* (5.18) *is satisfied and*

$$\lim_{\delta\to 0}\sup\bigl\{\rho(F):\ \operatorname{diam}(F)\leqslant\delta\bigr\}=0, \tag{5.22}$$

$$\lim_{R\to\infty}\sup\bigl\{\rho(F):\ F\subset\complement B_R\bigr\}=0. \tag{5.23}$$

There is also an analogy of Theorem 5.10 for this situation (see [96]).

THEOREM 5.13. *Let ρ be a nonsign-definite measure. To ensure compactness of the form $\mathfrak{g}_\rho$ in $\mathbf{h}^1$ it is necessary and sufficient that there exists a vector-function $\boldsymbol{\Gamma}\in L^2_{\mathrm{loc}}$ such that $\rho=\operatorname{div}\boldsymbol{\Gamma}$ and that the form $\int|\boldsymbol{\Gamma}|^2|u|^2\,\mathrm{d}x$ is compact in $\mathbf{h}^1$.*

The "sufficiency" part of Theorem 5.12 carries over to the magnetic case in the same way as Theorem 5.8.

6. Qualitative spectral analysis

Having defined a Schrödinger operator, $H=H_0+V$, the next question one tries to answer is about the qualitative structure of its spectrum.

There are at least two physically motivated situations, where discrete spectrum comes into consideration. If the electric potential vanishes at infinity, one must usually expect that the Schrödinger operator has essential spectrum on the positive semiaxis, with some negative eigenvalues, finitely or infinitely many. On the other hand, the electric potential may grow at infinity, similar to the harmonic oscillator, and here it is natural to expect that the whole spectrum is discrete, consisting of a sequence of eigenvalues tending to infinity. (Spectral properties in the case of potentials tending to $-\infty$, see Section 4.2, are completely unknown for $d\geqslant 2$.)

We are thus going to look for answers to the following questions.

1. Is the operator nonnegative (the negative spectrum is empty)?
2. Is the negative spectrum finite?
3. Is the negative spectrum discrete?
4. Is the whole spectrum discrete?

In quantum mechanical applications one usually considers the Schrödinger operator with the square of Planck's constant in front of the Laplacian. It is important also to find conditions for the answers to the above questions to be the same for the whole family $H_\hbar=-\hbar^2\Delta-V$.

As soon as the existence of some discrete spectrum has been established, additional questions concern how to describe it quantitatively, finding estimates or even asymptotics. These questions are addressed in the sequel. If the essential spectrum is nonempty, the vast field of *scattering theory* emerges; we do not discuss this topic in the present chapter.

6.1. *Positivity*

The first question seems to be the easiest one. As a direct consequence of Theorem 5.7, one gets the following result.

THEOREM 6.1. *Let the measure ρ satisfy one of the conditions of part one of Theorem* 5.7 *so that the constant C_1 there is not greater that* 1. *Then the operator defined by the form $\|\nabla u\|^2 - \mathfrak{g}_\rho[u]$ is nonnegative and thus its negative spectrum is empty. For a nonnegative ρ the conditions of part one of Theorem* 5.2 *with proper C_1 are also necessary for this.*

REMARK 6.2. As it was explained in Remark 5.3, there is a gap, controlled by a factor depending only on dimension, in the constants appearing in the (necessary and sufficient) conditions.

Discreteness and finiteness for all $\hbar$. We proceed to the case when there may be some negative spectrum.

Starting, probably, from Friedrichs' paper [44], a relation between spectral properties of the Schrödinger operator and compactness properties of some embeddings was established (although embedding theorems were not invented by that time). It is in the fundamental paper by Birman [14] that this relation was emphasized and understood, and one may say that a major part of modern spectral analysis of Schrödinger operators, both qualitative and quantitative, is based upon this paper, its ideas and methods.

We formulate the results from [14] (see, mostly, Theorem 1.4 therein) in a form which is convenient for our purpose.

THEOREM 6.3. *Let $\mathfrak{h}$ and $\mathfrak{g}$ be quadratic forms in a Hilbert space $\mathcal{H}$, defined initially on $\mathfrak{D}^0$, such that $\mathfrak{h}$ is positive and closable, and $\mathfrak{g}$ is closable with respect to $\mathfrak{h}$. Assume, moreover, that the conditions of* (*the KLMN*) *Theorem* 3.7 *are satisfied for $\hbar^2\mathfrak{h} - \mathfrak{g}$. Let $H(\hbar)$ be the self-adjoint operator defined by the quadratic form $\hbar^2\mathfrak{h} - \mathfrak{g}$. Denote by $\mathfrak{D}$, resp. $\mathfrak{D}_1$, the closures of $\mathfrak{D}^0$ with respect to norms $\mathfrak{h}[u]$, resp. $\mathfrak{h}[u] + \|u\|^2$. Then the following statements are true*:

(1) *For the negative spectrum of $H(\hbar)$ to be finite* for all $\hbar > 0$, *it is sufficient, and for nonnegative $\mathfrak{g}$ also necessary, that the form $\mathfrak{g}$ is compact in the space $\mathfrak{D}$.*

(2) *For the negative spectrum of $H(\hbar)$ to be discrete* for all $\hbar > 0$, *it is sufficient, and for nonnegative $\mathfrak{g}$ also necessary, that the form $\mathfrak{g}$ is compact in the space $\mathfrak{D}_1$.*

Note that the requirement of closability of $\mathfrak{g}$ with respect to $\mathfrak{h}^0$ is redundant: it is satisfied automatically if the compactness conditions are fulfilled.

REMARK 6.4. Since the operator $\hbar^{-2}H(\hbar)$ corresponds to the quadratic form $\mathfrak{h} - \hbar^{-2}\mathfrak{g}$, where the parameter $\hbar^{-2}$ stands before the perturbation and plays the role of a coupling constant, the statements of Theorem 6.3 hold also for this situation.

Applying Theorem 6.3 to compactness conditions obtained for particular quadratic forms in the preceding section (Theorems 5.12 and 5.13), we get concrete conditions for finiteness, resp. discreteness, of the spectrum for all $\hbar > 0$ for the nonmagnetic or magnetic Schrödinger operators.

THEOREM 6.5. *For the Schrödinger operator $H_\rho = -\hbar^2\Delta - \rho$ defined by means of quadratic forms, one has that*:

(1) *For the negative spectrum to be finite for all $\hbar > 0$, it is sufficient that the conditions of Theorem* 5.13 *are satisfied*; *for a nonnegative measure ρ, the conditions of Theorem* 5.12 *are necessary and sufficient.*
(2) *For the negative spectrum to be discrete for all $\hbar > 0$, it is sufficient that the conditions of Theorem* 5.10 *are satisfied*; *for a nonnegative measure ρ, the conditions of Theorem* 5.8 *are necessary and sufficient.*
(3) *The sufficiency statements above for nonnegative measures hold for the magnetic Schrödinger operator with magnetic potential $\mathcal{A} \in L^2_{\mathrm{loc}}$.*

The particular sufficient compactness conditions from Section 5.5 lead immediately to more transparent conditions for discreteness or finiteness of the negative spectrum. We present just two examples, more can be found in [13,91,113].

COROLLARY 6.6. *If $V \in L^{d/2}(\mathbb{R}^d)$, $d \geqslant 3$, then the Schrödinger operator $-\hbar^2\Delta - V$ has finite spectrum for all $\hbar > 0$. If $d \geqslant 3$, $V = V_1 + V_2$, $V_1 \in L^{d/2}(\mathbb{R}^d)$, $V_2 \in L^\infty(\mathbb{R}^d)$, $V_2 \to 0$ at infinity, then $-\hbar^2\Delta - V$ has discrete negative spectrum for all $\hbar$.*

6.2. *Birman–Schwinger principle*

The finiteness and discreteness conditions for a *fixed* $\hbar$ (we set $\hbar = 1$) are more intricate, but still can be obtained, not directly from Theorem 6.3 but rather from the more universal and crucial *Birman–Schwinger principle* (see also Section A.10), concealed in [14] as Lemma 1.3, being up to now the main instrument in the spectral analysis of singular operators (later, Schwinger proposed this principle in a considerably less general form [124]).

THEOREM 6.7 (Birman–Schwinger principle). *Under the conditions of Theorem* 6.3, *the number of negative eigenvalues of the operator $H(\hbar)$ equals the number of eigenvalues in $(\hbar, \infty)$ of the operator $T_{\mathfrak{h},\mathfrak{g}}$ (see* (5.15)) *or $K_{\mathfrak{g},\mathfrak{h}}$ (see* (5.16)) *defined by the quadratic form $\mathfrak{g}$ in the space $\mathfrak{D}$, provided one of these numbers is finite. Otherwise, for both operators the dimension of the spectral subspace in the corresponding interval is infinite.*

One obtains Theorem 6.3 as a particular case – directly for finiteness (for a compact operator, the number of eigenvalues above $\hbar$ is finite for any $\hbar > 0$), and with $\mathfrak{d}_1$ in place of $\mathfrak{d}$ for the discreteness.

6.3. *Finiteness and discreteness of the spectrum*

For a fixed $\hbar$ (again we set $\hbar = 1$) it is more convenient to apply Theorem 6.7 directly. The mechanism of establishing finiteness of the discrete spectrum of $T_{\mathfrak{h},\mathfrak{g}}$ above $\hbar = 1$ for $\mathfrak{g} = \int |u|^2 \rho(\mathrm{d}x)$ is the following. For a fixed R, split the measure ρ into two parts, ρ_R, supported in the ball B_R, and $\tilde{\rho}_R$, supported outside the ball. Correspondingly, the quadratic form $\mathfrak{g}$ splits into two parts, $\mathfrak{g} = \mathfrak{g}_R + \tilde{\mathfrak{g}}_R$, as well as the operator $T_{\mathfrak{h},\mathfrak{g}} = T_R + \widetilde{T}_R$. If local compactness conditions are satisfied, the spectrum of T_R above 1 is finite. So, one has to suppose that the norm of $\widetilde{T}_R$ (or, even more generally, the norm of the positive part of $\widetilde{T}_R$) is less than 1. Then the spectrum of $\widetilde{T}_\rho$ above 1 is finite. Furthermore, discreteness of the spectrum of T for fixed $\hbar = 1$ is equivalent to finiteness of the spectrum of the operator $T - \varepsilon$ for any $\varepsilon > 0$. This line of reasoning gives also necessary conditions for finiteness, resp. discreteness, of the negative spectrum.

Implementing this strategy to particular forms corresponding to Schrödinger operators, we arrive at the following description of the spectrum. Less general conditions were found in [14,89,91].

THEOREM 6.8. *Let $d \geqslant 3$ and let the measure ρ satisfy the condition* (5.18).

(1) *If for sufficiently large R the restriction of ρ to the exterior of the ball B_R satisfies the conditions of the first part of Theorem* 5.7 *so that the constant C_1 in the resulting estimate is not larger than* 1, *then the negative spectrum of the operator $-\Delta - \rho$ is finite.*

(2) *For $\varepsilon > 0$, define a measure ρ_ε, $\rho_\varepsilon(F) = \sup\{(\rho(e) - \varepsilon\,\mathrm{meas}(e))_+, e \subset F\}$ (it is the positive part of $(\rho - \varepsilon\,\mathrm{meas})$). Suppose that for any $\varepsilon > 0$, the restriction of ρ_ε to the exterior of a sufficiently large ball B_R, $R = R(\varepsilon)$, satisfies the conditions of the first part of Theorem* 5.2 *so that the constant C_1 in this theorem is less than* 1. *Then the negative spectrum of the operator $-\Delta - \rho$ is discrete.*

(3) *The statements above hold also for the magnetic Schrödinger operator with magnetic potential $\mathcal{A} \in L^2_{\mathrm{loc}}$.*

If the measure ρ is nonnegative, one can even give exact values of the constants involved (see [89,91]).

THEOREM 6.9. *Let $d \geqslant 3$ and suppose that the condition* (5.18) *is fulfilled. Denote by $S(\rho)$ the quantity*

$$\lim_{\delta\to 0} \limsup_{R\to\infty} \bigl\{\pi(F)\colon\ F \subset \complement B_R, \mathrm{diam}(F) \leqslant \delta\bigr\}. \tag{6.1}$$

For discreteness of the negative spectrum of the Schrödinger operator $-\Delta - \rho$ the condition $S(\rho) \leqslant 1/4$ is sufficient, and $S(\rho) \leqslant 1$ is necessary. If one drops $\mathrm{diam}(F) \leqslant \delta$ *from* (6.1), *the same conditions are sufficient, resp., necessary, for finiteness of the negative spectrum.*

We give an example of the finiteness condition in elementary terms. This condition follows from the estimates expressed in terms of the capacity, taking also into account the relation of the capacity and Lebesgue measure (5.6).

COROLLARY 6.10. *Let $V \geqslant 0$, $d \geqslant 3$. Suppose that $V = V_1 + V_2$, where $V_1 \in L^{d/2}(\mathbb{R}^d)$ and $V_2 \leqslant \frac{c}{4}(|x|+1)^{-2}$. If $c < (d-2)^2$, then the negative spectrum of the operator $-\Delta - V$ in $L^2(\mathbb{R}^d)$ is finite.*

The condition here is exact in the sense that if one sets $c = (d-2)^2$, the result may break down, i.e., finiteness does not hold for all values of the coupling constant. The authors failed to find examples where the negative spectrum is discrete but not for all values of the coupling constant (with local compactness conditions still holding).

6.3.1. *Discreteness of the whole spectrum.* Physical intuition suggests that if the potential $V(x)$ tends to $+\infty$ as $|x| \to \infty$, one must expect that the whole spectrum of the Schrödinger operator $-\Delta + V$ is discrete: conservation of energy does not allow a particle to escape to infinity. As it often happens, the intuition turns out to be essentially correct.

Here, again, the Birman–Schwinger principle enables one to reduce the problem to compactness of certain embeddings. Let H_0 be the unperturbed operator associated with the quadratic form $\mathfrak{h}_0$ (e.g., $H_0 = -\Delta$) and for the measure ρ the self-adjoint, semibounded operator $H_\rho = H_0 + \rho$ corresponds to the quadratic form $\mathfrak{h}_0 + \mathfrak{g}_\rho$ defined initially on $\mathscr{D}(\mathbb{R}^d)$. Choose s so that $\mathfrak{h}_s[u] = \mathfrak{h}_0[u] + \mathfrak{g}_\rho[u] + s\|u\|^2 \geqslant \|u\|^2$. Consider the space $\mathfrak{D}$ defined as the closure of $\mathscr{D}(\mathbb{R}^d)$ in the norm $\mathfrak{h}_s[u]$.

PROPOSITION 6.11. *The following statements are equivalent*:

(1) *The spectrum of $H_\rho = H_0 + \rho$ is discrete.*

(2) *The form $\|u\|^2$ is compact in the space $\mathfrak{D}$* (*or, in other words, the embedding of $\mathfrak{D}$ into $L^2(\mathbb{R}^d)$ is compact*).

We present the proof which demonstrates the beauty of the Birman–Schwinger principle.

PROOF OF PROPOSITION 6.11. Without loss of generality we may set $s = 0$. Discreteness of the spectrum of the operator means that for any $\lambda > 0$ the spectrum of the operator H_ρ below λ is finite. This is equivalent to the negative spectrum of the operator $H_\rho - \lambda$ being finite. The operator $H_\rho - \lambda$ is defined by the quadratic form $\mathfrak{h}_0[u] + \mathfrak{g}_\rho[u] - \lambda\|u\|^2$. Now, according to the Birman–Schwinger principle, the number of its negative eigenvalues is equal to the number of eigenvalues above λ^{-1} of the operator T defined by the form $\|u\|^2$ in the space with norm $\mathfrak{h}_0[u] + \mathfrak{g}_\rho[u]$, i.e., in $\mathfrak{D}$. The quadratic form of T, i.e., $\|u\|^2$ is positive, therefore, T is a positive self-adjoint operator. Finiteness of its spectrum above any $t = \lambda^{-1}$ means exactly that the operator is compact. □

Having Proposition 6.11 at disposal, one can further obtain different discreteness conditions, just by finding conditions for the compactness of the above embedding.

The mechanism of proving this compactness is usually the following. Take R large enough and split the form $\|u\|^2$ into two terms, $\|u\|^2 = \mathfrak{n}_R[u] + \tilde{\mathfrak{n}}_R[u]$, where the first form corresponds to integration over the ball B_R, and the second one to integration over its complement. In the case of H_0 being the Laplacian, the first form is compact in $\mathbf{H}^1$, due to the Sobolev–Rellich embedding theorem. So it remains to show that the form $\tilde{\mathfrak{n}}_R[u]$ satisfies

$$\tilde{\mathfrak{n}}_R[u] \leqslant \varepsilon(R)\mathfrak{h}_0[u] \quad \text{with } \varepsilon(R) \to 0 \text{ as } R \to \infty, \tag{6.2}$$

and it is here the additional analytical work begins. As soon as (6.2) is fulfilled, this gives a norm approximation of the operator T (generated by the form $\|u\|^2$) by compact operators, so T is compact. The most simple condition for this is just to require that $\rho(\mathrm{d}x) = V(x)\,\mathrm{d}x$ and V tends to $+\infty$ at infinity. Then, if $V \geqslant \varepsilon^{-1}$ for $|x| > R$, (6.2) follows from

$$\int_{|x|>R} |u|^2\,\mathrm{d}x \leqslant \varepsilon \int_{|x|>R} V|u|^2\,\mathrm{d}x \leqslant \varepsilon\left(\mathfrak{h}_0[u] + \int V|u|^2\,\mathrm{d}x\right).$$

This gives the following sufficient condition, originally obtained by Friedrichs [44].

THEOREM 6.12. *If $V(x) \to +\infty$ as $|x| \to \infty$, then the spectrum of $-\Delta + V$ is discrete.*

Theorem 6.12 was later substantially improved. In [102] it was shown that for $V \geqslant 0$, in dimension 1, the condition of V going to $+\infty$ can be replaced by the requirement for the integral of V over intervals of every fixed size to tend to $+\infty$, when the interval moves to infinity, and this property is necessary and sufficient for discreteness of the spectrum. In higher dimensions, this condition is still necessary (with intervals replaced by cubes or balls), but not sufficient. The necessary and sufficient condition, found by Molchanov [102] is as follows.

The set F in the cube $Q = Q_d$ with sidelength δ is called *negligible* if $\operatorname{cap}(F, \widetilde{Q}) \leqslant \gamma \operatorname{cap}(Q, \widetilde{Q})$ for some $\gamma < 1$, where $\widetilde{Q}$ is a concentric cube, twice as large, and $\operatorname{cap}(F, \widetilde{Q})$ is the Wiener capacity (see Section 5.1). For $d \geqslant 3$ one can replace $\operatorname{cap}(F, \widetilde{Q})$, resp. $\operatorname{cap}(Q, \widetilde{Q})$, by the Bessel capacity $\operatorname{Cap}(F)$, resp. $\operatorname{cap}(Q)$. Introduce the *Molchanov functional*

$$\mathcal{M}(V, Q) = \mathcal{M}(V, Q; \gamma) = \inf\left\{\int_{Q\setminus F} V(x)\,\mathrm{d}x\right\}, \tag{6.3}$$

where infimum is taken over all negligible sets in the cube Q. So $\mathcal{M}(V, Q)$ can be small if V is large only on a small, nonessential set, although the integral of V over the whole cube might be large. For the case of a measure, $\mathcal{M}(V, \rho)$ is defined by (6.3), with the integral replaced by $\rho(Q \setminus F)$.

THEOREM 6.13 (Molchanov's criterion). *There exists γ such that the condition*

$$\mathcal{M}(V, Q_\delta; \gamma) \to +\infty \quad \text{as } Q_\delta \to \infty, \tag{6.4}$$

for any $\delta > 0$ is necessary and sufficient for discreteness of the spectrum of the operator $-\Delta + V$ in $L^2(\mathbb{R}^d)$.

In other words, the discreteness condition declares that not only the integral of V over the whole cube must be large, but also the integral over complements of all negligible sets. So, if V is supported in a negligible set in any cube (or just in a sequence of cubes going to infinity), it cannot assure discreteness of the spectrum, no matter how large V is.

Although Molchanov found some concrete value of the constant γ, it remained unknown, for more than 50 years, how close to zero this constant can be, in other words, how small the negligible sets are allowed to be. Formally, the smaller γ is, the larger $\mathcal{M}$ is and therefore the less restrictive are the conditions on V in (6.4). This question is especially important if one replaces the function V by a measure ρ, which may be supported on some small, probably, negligible sets. This problem was solved only recently, and the result by Maz'ya and Shubin was quite unexpected [94]. The negligibility characteristic, γ, can be taken arbitrarily close to 0.

THEOREM 6.14. *Let ρ be a nonnegative measure such that the operator $-\Delta + \rho$ can be defined via quadratic forms. Then, for any function $\gamma : (0, \infty) \to (0, 1)$, the necessary and sufficient condition for discreteness of the spectrum of $-\Delta + \rho$ consists in*

$$\mathcal{M}\big(\rho, Q_\delta; \gamma(\delta)\big) \to +\infty \quad \text{as } Q_\delta \to \infty \quad \forall \delta > 0. \tag{6.5}$$

6.4. *Discreteness of spectrum for magnetic operators*

Due to the diamagnetic inequality, the condition (6.5) is sufficient also for discreteness of spectrum of the magnetic Schrödinger operator; the reasoning goes in the same way as in Corollary 5.11. However, this condition is far from being necessary. It was predicted first by physicists, and then justified mathematically that the magnetic Laplacian $-\Delta_{\mathcal{A}} = -(\nabla + \mathrm{i}\mathcal{A})^2$ can have discrete spectrum even without any electric potential. This phenomenon is called a *magnetic bottle*. The physical explanation is that the magnetic field, acting orthogonally to the velocity of the particle, forces the particle to turn, and if the magnetic field grows at infinity, this may prevent the particle from escaping to infinity. In fact, if $|B(x)| \to \infty$ at infinity, then in dimension $d = 2$ the spectrum of the magnetic Laplacian is discrete. In dimension $d \geqslant 3$ this is not sufficient since the magnetic field $\mathbf{B}(x)$, being a vector now, may change its direction, and therefore it may fail to prevent a particle from escaping (it was Dufresnoy [31] who constructed the first example). One may require that the direction of the field varies sufficiently slowly, and this implies discreteness of the spectrum. In the presence of the electric potential V the situation becomes even more complicated, since it is the cooperation of both fields that determines the spectrum, and it is possible, that acting together, electric and magnetic fields grant discreteness of the spectrum while separately they fail to do this.

Until recently the study of this problem gave some sufficient conditions, far from necessary ones, as well as spectacular examples demonstrating various pathologies. Lately, however, discreteness conditions, necessary and sufficient, were found by Kondratiev, Maz'ya and Shubin [65,66].

As usual for operators with magnetic fields, the case $d = 2$ is more clear. We present here the sufficient condition from [66], nicely illustrating interaction of magnetic and electric fields. To avoid the influence of other factors, we assume that the magnetic potential $\mathcal{A}$ belongs to C^1 and V is locally bounded.

THEOREM 6.15. *For the magnetic Schrödinger operator in $L^2(\mathbb{R}^2)$ the effective potential V_δ^{eff} is defined as*

$$V_\delta^{\mathrm{eff}}(x) = V(x) + \delta B(x).$$

Suppose that for some $\delta \in [-1, 1]$, $V_\delta^{\mathrm{eff}}(x) \to +\infty$ as $|x| \to \infty$. Then the operator $H_{\mathcal{A},V} = -(\nabla + \mathrm{i}\mathcal{A})^2 + V$ is essentially self-adjoint on $\mathscr{D}(\mathbb{R}^2)$ and has discrete spectrum.

In less elementary terms, it is even sufficient that $V_\delta^{\mathrm{eff}}(x)$ satisfies Molchanov's condition (6.4).

The conditions of Theorem 6.15 are not necessary for discreteness of the spectrum but they are exact in the sense that there are examples of the conclusion failing if the conditions are satisfied only for some δ outside the interval $[-1, 1]$.

In higher dimensions, not only the values of the electric and magnetic fields contribute to the effective potential but also a characteristic of the direction of the magnetic field; the *smoothened direction* of the magnetic field $\mathbf{B}(x) = \mathrm{d}\mathcal{A}(x)$ is defined as $\nu_{jk}(x) = \chi(|\mathcal{B}|(x))B_{jk}(x)/|\mathbf{B}(x)|$, where χ is a Lipschitz function on $[0, \infty)$, $\chi(r) = 0$, $r > 1/2$, $\chi(r) = 1, r > 1$, $\chi(r) = 2r - 1, r \in [1/2, 1]$. Kondratiev and Shubin obtained the following result [66].

THEOREM 6.16. *Suppose that B_{jk} are Lipschitz functions, and for some positive measurable function $X(x)$, the inequality $\sum_k |\partial_k \nu_{jk}| \leqslant X(x)$ holds for all $x \in \mathbb{R}^d$. For $\varepsilon > 0$ and $\delta \in [0, 1)$, define the effective potential*

$$V_{\delta,\varepsilon}^{\mathrm{eff}}(x) = V(x) + \frac{\delta}{d - 1 + \varepsilon}\big|\mathbf{B}(x)\big| - \frac{\delta d}{4\varepsilon(d - 1 + \varepsilon)}X^2(x).$$

If for some δ, ε the effective potential satisfies the Molchanov condition (6.4) (*in particular, if $V_{\delta,\varepsilon}^{\mathrm{eff}}(x) \to +\infty$ at infinity*), *then the operator $H_{\mathcal{A},V}$ is essentially self-adjoint, semibounded and has discrete spectrum.*

Several more transparent conditions following from Theorem 6.16 are given in [66]. We present here just one of them.

COROLLARY 6.17. *Suppose that $|\operatorname{grad} \mathbf{B}(x)|(1 + |\mathbf{B}(x)|)^{-3/2} \to 0$ as $|x| \to \infty$. If the effective potential $V(x) + \frac{\delta}{d-1}|B|$ satisfies the Molchanov condition* (6.4)*, in particular, if it tends to $+\infty$ at infinity, then the conclusion of Theorem* 6.16 *hold.*

Theorem 6.16 and other results in [66] are carried over also to the magnetic Schrödinger operators on Riemannian manifolds with bounded geometry.

The most recent result in this field, established by Kondratiev, Maz'ya and Shubin [65], finally gives the necessary and sufficient conditions for discreteness of the spectrum of the magnetic Schrödinger operator. These criteria also involve local characteristics of the magnetic and electric fields, so that their sum has to tend to infinity at infinity. However, these characteristics are more elaborate than in Theorem 6.16. For the description of the electric potential V, the Molchanov functional $\mathcal{M}(V, Q; \gamma)$ (see (6.3)) is used. To describe the magnetic field, its local energy $\mu_0(Q) = \mu_0(Q, \mathcal{A})$ in the cube Q is introduced as

$$\mu_0(Q) = \inf_u \left\{ \frac{\int_Q |\nabla u + \mathrm{i}\mathcal{A} u|^2 \, \mathrm{d}x}{\int_Q |u|^2 \, \mathrm{d}x} \right\}, \tag{6.6}$$

where infimum is taken over Lipschitz functions in Q. Thus $\mu_0(Q)$ is the lowest eigenvalue of the Neumann problem for the magnetic Laplacian in Q. The discreteness conditions require that some combinations of these two characteristics grow at infinity. To describe these combinations, we consider a positive function $g(\delta)$, $\delta \in (0, \delta_0)$ such that $g(\delta) \geqslant \delta^2$ and $g(\delta) \to 0$ as $\delta \to 0$. Let also the function $f : [0, \infty) \to (0, \infty)$ be continuous, decreasing and, moreover, $f(t) \leqslant (1 + \log(1 + t))^{-1}$, $d = 2$, and $f(t) \leqslant (1 + t)^{(2-d)/2}$, $d > 2$.

THEOREM 6.18. *The spectrum of the magnetic Schrödinger operator $-(\nabla + \mathrm{i}\mathcal{A})^2 + V$ is discrete if and only if for some δ_0 and some pair of functions f and g as above, and for every $\delta \leqslant \delta_0$,*

$$\mu_0(Q_\delta) + \delta^{-d}\mathcal{M}\big(V, Q_\delta; \gamma(\delta)\big) \to +\infty \quad \text{as } Q_\delta \to \infty, \tag{6.7}$$

where $\gamma(\delta) = \gamma(\delta; f, g) = c_d \delta^2 f(\delta^2 \mu_0(Q_\delta)) g(\delta)^{-1}$.

A more simple, sufficient but not necessary discreteness condition is that for some fixed γ and every $\delta \leqslant \delta_0$,

$$\mu_0(Q_\delta) + \delta^{-d}\mathcal{M}(V, Q_d; \gamma) \to +\infty \quad \text{as } Q_\delta \to \infty.$$

7. CLR estimates

In this section we begin the study of quantitative characteristics of the discrete spectrum of Schrödinger-type operators. The first quantity to be considered is the number of negative eigenvalues, and here due to tradition we consider $H_V = -\Delta - V$.

The first bounds on $N(0; H_V)$ having the correct strong coupling behavior (see the Introduction) are due to Simon [130], however, they are not expressed in terms of the *phase-space volume*; see the right-hand side of (1.1). The breakthrough was independently achieved in the 1970s by Cwikel [28], Lieb [76] and Rozenblum [116], who gave three very different proofs of the following inequality, now known as the *CLR estimate*.

THEOREM 7.1 (CLR estimate). *If $d \geqslant 3$ and $V_+ \in L^{d/2}(\mathbb{R}^d)$, then the number of negative eigenvalues of the operator $-\Delta - V$ satisfies*

$$N(0; H_V) \leqslant c_d \int_{\mathbb{R}^d} V_+(x)^{d/2} \, \mathrm{d}x \tag{7.1}$$

for a suitable constant c_d.

We make here an important observation concerning (7.1) and all the following spectral estimates. Although the eigenvalues of the operator depend on the potential as a whole, only its positive part is present in the estimate. In other words, the influence of the negative part of V, which may decrease the number of negative eigenvalues, is not taken into account. At present the methods in qualitative spectral analysis do not allow one to evaluate the interaction of the positive and negative parts of the potential.

The three original methods for proving (7.1), as well as later developments, of which we especially note the proofs by Li and Yau [75] and Fefferman [42] are based upon different mathematical machinery, admit different generalizations, and give different values of the constant c_d. Lieb's approach yields the best constant but the question of the optimal constant in (7.1) remains unanswered; to be precise, one would like to determine

$$R_{0,d} := \sup_V \frac{N(0; H_V)}{L^c_{0,d} \int_{\mathbb{R}^d} V_+^{d/2}\,\mathrm{d}x} \geqslant 1. \tag{7.2}$$

Other problems we address here are generalizations to Schödinger operators with magnetic fields, with variable coefficients, and Schrödinger operators with operator-valued potentials.

In recent years the methods of Cwikel, Lieb, and also Li and Yau, have inspired significant progress, which we present in subsequent sections. Thus it is natural to discuss their methods here. Before doing so it is necessary to take another look at the Birman–Schwinger principle (cf. Section 6.2) and, in particular, the *Birman–Schwinger operator* already used for the needs of the qualitative analysis in Section 5.5.

7.1. *Birman–Schwinger operator*

We are interested in estimating the number of negative eigenvalues of the Schrödinger operator $-\Delta - V$, where $V : \mathbb{R}^d \to \mathbb{R}$ is a measurable function and $-\Delta$ is the negative Laplacian. The Laplacian $-\Delta$ corresponds to the closed quadratic form

$$\mathfrak{h}[u] = \int_{\mathbb{R}^d} \left|\nabla u(x)\right|^2 \mathrm{d}x, \quad u \in \mathfrak{D}(\mathfrak{h}) = \mathbf{H}^1\left(\mathbb{R}^d\right).$$

Its spectrum is absolutely continuous and coincides with the semiaxis $[0, \infty)$.

EXAMPLE 7.2. If $u \in \mathscr{D}(\mathbb{R}^3)$, then the resolvent $(-\Delta + E^2)^{-1}$, $E > 0$, can be expressed explicitly as

$$\left[(-\Delta + E)^{-1} u\right](x) = (4\pi)^{-1} \int_{\mathbb{R}^3} \frac{\mathrm{e}^{-\sqrt{E}|x-y|}}{|x-y|} u(y)\,\mathrm{d}y. \tag{7.3}$$

The quadratic form associated with the potential is given by $\mathfrak{g}[u] = \mathfrak{g}_V[u] = \int_{\mathbb{R}^d} V(x) \times |u(x)|^2\,\mathrm{d}x$. Assume first that $V \geqslant 0$. In order to estimate the eigenvalues below $-E$ of

$-\Delta - V$, according to Glazman's lemma (see Section A.9) and the Birman–Schwinger principle, we should study subspaces where the ratio

$$\frac{\mathfrak{g}[u]}{\mathfrak{h}[u] + E\|u\|^2} = \frac{\int_{\mathbb{R}^d} V(x)|u(x)|^2\,\mathrm{d}x}{\int_{\mathbb{R}^d} |\nabla u(x)|^2\,\mathrm{d}x + E\int_{\mathbb{R}^d} |u(x)|^2\,\mathrm{d}x}$$

is smaller than 1. Setting here $v = (-\Delta + E)^{1/2}u$ and thus $u = (-\Delta + E)^{-1/2}v$ we obtain

$$\frac{\mathfrak{g}[u]}{\mathfrak{h}[u] + E\|u\|^2} = \frac{\int_{\mathbb{R}^d} v(-\Delta + E)^{-1/2}V(-\Delta + E)^{-1/2}\bar{v}\,\mathrm{d}x}{\int_{\mathbb{R}^d} |v(x)|^2\,\mathrm{d}x}.$$

It follows therefore that the eigenvalues of the operator $T_{\mathfrak{g},\mathfrak{h}+E}$ defined by $\mathfrak{g}$ with respect to the norm $(\mathfrak{h}[\cdot] + E\|\cdot\|^2)^{1/2}$ are identical to the eigenvalues of the operator

$$\widetilde{K}_E(V) = K_{\mathfrak{g},\mathfrak{h}+E} = (-\Delta + E)^{-1/2}V(-\Delta + E)^{-1/2} = X^*X \tag{7.4}$$

on $L^2(\mathbb{R}^d)$, where $X = X_E(V) = V^{1/2}(-\Delta + E)^{-1/2}$. This operator $\widetilde{K}_E(V)$ has the same nonzero eigenvalues as

$$K_E(V) = V^{1/2}(-\Delta + E)^{-1}V^{1/2} = XX^*; \tag{7.5}$$

both operators are called *Birman–Schwinger operators*, and it depends on the particular problem, which of them is more convenient to use.

On $\mathbb{R}^3$, for suitable V, $K_E(V)$ is a Hilbert–Schmidt operator. Since $(-\Delta + E)^{-1}$ has the integral kernel $(4\pi|x-y|)^{-1}\exp(-\sqrt{E}|x-y|)$ (cf. (7.3)),

$$\begin{aligned} &N(-E; H_V) \\ &\quad\leqslant \operatorname{tr}\big(K_E(V)K_E(V)^*\big) \\ &\quad= \frac{1}{(4\pi)^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} V(x)V(y)|x-y|^{-2}\exp\big(-2\sqrt{E}|x-y|\big)\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{7.6}$$

For $E \geqslant 0$ and $\alpha \in [0,1]$ we define the generalized Birman–Schwinger operator

$$\begin{aligned} K_{\alpha E}(V) &\equiv \big(V(x) - (1-\alpha)E\big)_+^{1/2}(-\Delta + \alpha E)^{-1}\big(V(x) - (1-\alpha)E\big)_+^{1/2} \\ &\equiv V_\alpha^{1/2}(-\Delta + \alpha E)^{-1}V_\alpha^{1/2}, \end{aligned} \tag{7.7}$$

where $V_\alpha \equiv (V(x) - (1-\alpha)E)_+$ denotes the positive part of the function $V(x) - (1-\alpha)E$. Lieb and Thirring introduced the following modification of the Birman–Schwinger principle [85].

THEOREM 7.3. *The number $N(-E; H_V)$ of eigenvalues of $-\Delta - V$ in $(-\infty, -E)$ is bounded above by the number of eigenvalues of the generalized Birman–Schwinger operator $K_{\alpha E}(V)$ in $(1, \infty)$ counting multiplicities.*

7.2. *Cwikel's method*

This method, having its roots in harmonic analysis, handles a class of operators for which $X_E(V)$, defined immediately after (7.4), is a particular case. Such operators are defined by means of the Fourier transform in the following way.

If $\mathscr{F}$ denotes the Fourier transform and $g \in L^\infty(\mathbb{R}^d)$ then $\phi \mapsto \mathscr{F}^{-1}(g\mathscr{F}\phi)$ defines a bounded operator on $L^2(\mathbb{R}^d)$; we shall denote it by $g(-\mathrm{i}\nabla)$. The weak L^p space consists of all the functions g for which

$$\|g\|^*_{L^p_w(\mathbb{R}^d)} := \sup_{t>0} t \cdot \bigl(\mathrm{meas}\bigl(\bigl|g(\cdot)\bigr| > t\bigr)\bigr)^{1/p} < \infty; \tag{7.8}$$

we already encountered it in Section 5.3.

Assuming that $f \in L^p(\mathbb{R}^d)$, $\mathscr{F}g \in L^{p'}_w(\mathbb{R}^d)$ with $1/p + 1/p' = 1$ and $2 < p < \infty$, we can define

$$Z_{f,g}\psi = fg(-\mathrm{i}\nabla)\psi, \quad \psi \in \mathscr{S}\bigl(\mathbb{R}^d\bigr), \tag{7.9}$$

which extends to a bounded operator on $L^2(\mathbb{R}^d)$. Cwikel proved the following result [28].

THEOREM 7.4 (Cwikel's inequality). *For $2 < p < \infty$, let $f \in L^p(\mathbb{R}^d)$, $\mathscr{F}g \in L^{p'}_w(\mathbb{R}^d)$ with $1/p + 1/p' = 1$. Then $Z_{f,g}$ is a compact operator on $L^2(\mathbb{R}^d)$ and, in fact, it belongs to the weak trace ideal $\mathfrak{S}^p_w$. In particular, its singular values μ_k obey*[3]

$$k^{1/p}|\mu_k| \leqslant c_{p,d}\|f\|_{L^p}\|g\|^*_{L^p_w}, \quad k = 1, 2, \ldots, \tag{7.10}$$

where $c_{p,d}$ is a constant depending only on p and d.

In particular,

$$c_{p,d} \leqslant \frac{p}{2}\left(\frac{4}{p/2 - 1}\right)^{1-2/p} \omega_d^{1/p} \pi^{d/p}.$$

For $d \geqslant 3$, Theorem 7.4 can be applied to our operator $X_0(V) = Z_{f,g}$ with $p = d$ and the functions f, g chosen as $g(x) = \frac{1}{2}\pi^{-(d+1)/2}\Gamma(\frac{d-1}{2})|x|^{1-d}$ with $\|g\|^*_{L^{d-1}_w} = \frac{1}{2}\pi^{-(d+1)/2}\Gamma(\frac{d-1}{2})$ and $f(x) = V(x)$. The Birman–Schwinger principle gives then that

$$N(0; H_V) \leqslant c^d_{d,d} 2^{-d} \pi^{-(d+1)/2d} \Gamma\left(\frac{d-1}{2}\right)^d \int_{\mathbb{R}^d} V^{d/2}\,\mathrm{d}x \tag{7.11}$$

which yields the constant $\frac{27}{2\pi^2}$ if $d = 3$ (which is about 17 times greater than the best possible value; cf. Lieb's approach). The estimate (7.11), in conjunction with monotonicity

[3]Finiteness of $\|g\|^*_{L^p_w}$ follows from the condition $\mathscr{F}g \in L^{p'}_w$.

(cf. Section A.9, Lemma A.4), proves (7.1) for a general V of variable sign; with the constant appearing in (7.11).

The analysis of the proof of Theorem 7.4, made in [18], shows that, in fact, a much more general theorem holds: one can replace the Fourier transform in the definition of the operator $Z_{f,g}$ by any integral operator with a bounded kernel, which is bounded in L^2. This generalization might lead to obtaining CLR-type estimates on some noncompact manifolds different from $\mathbb{R}^d$, however this possibility has not been explored yet.

If one tries to apply (7.10) to the Schrödinger operator in dimension $d = 2$, one has to take $p = d = 2$ which is not allowed. A weaker version of Theorem 7.4 for this special case was found by Weidl [154].

7.3. *Lieb's approach*

E. Lieb applies completely different tools in order to derive the phase-space bound on $N(-E; H_V)$; in particular, the Wiener integral representation of functions F of the Birman–Schwinger operator $K_E(V)$ given in (7.5).

If $\mathrm{d}\mu_{x,y,t}$ denotes the conditional Wiener measure on continuous paths $\omega(\tau)$, $0 \leqslant \tau \leqslant t$, in $\mathbb{R}^d$ with $\omega(0) = x$ and $\omega(t) = y$, then the Green function for the heat semigroup $\mathrm{e}^{t\Delta}$ can be represented as

$$(4\pi t)^{-d/2} \exp\left(-\frac{|x-y|^2}{4t}\right) = \int \mathrm{d}\mu_{x,y,t}(\omega). \tag{7.12}$$

The Feynman–Kac formula [112], Theorem X.68, generalizes (7.12) to the semigroup $\mathrm{e}^{t(\Delta+V)}$. Using this, Lieb found the following expression for the trace of a function of the Birman–Schwinger operator [76].

THEOREM 7.5. *Let $V \geqslant 0$ and $V \in L^p(\mathbb{R}^d) + L^q(\mathbb{R}^d)$, with $p = d/2$ (if $d \geqslant 3$), $p > 1$ ($d = 2$), $p = 1$ ($d = 1$) and $p < q < \infty$. Let f be a nonnegative lower semicontinuous function on $[0, \infty)$ satisfying $f(0) = 0$ and $x^r f(x) \to 0$ as $x \to \infty$ for some $r < \infty$. Define F by*

$$F(x) = \int_0^\infty f(xy)\frac{\mathrm{e}^{-y}}{y}\,\mathrm{d}y. \tag{7.13}$$

Then

$$\operatorname{tr} F(K_E) = \int_0^\infty \frac{\mathrm{d}t}{t}\mathrm{e}^{-|E|t} \int_{\mathbb{R}^d}\int \mathrm{d}\mu_{x,x,t}\, f\left(\int_0^t V\big(\omega(s)\big)\,\mathrm{d}s\right). \tag{7.14}$$

To use (7.14), one supposes additionally that f is a convex function. Then Jensen's inequality applied to the last term in Lieb's trace formula, yields that

$$f\left(\int_0^t V\big(\omega(s)\big)\,\mathrm{d}s\right) \leqslant \frac{1}{t}\int_0^t f\big(tV\big(\omega(s)\big)\big)\,\mathrm{d}s. \tag{7.15}$$

Fubini's theorem allows one to change the order of integrations, which implies that

$$\operatorname{tr} F(K_E) \leqslant \int_0^\infty \frac{\mathrm{d}t}{t} \mathrm{e}^{-Et} \int \mathrm{d}\mu_{0,0;t} \int_0^t \frac{\mathrm{d}s}{t} \int_{\mathbb{R}^d} \mathrm{d}x\, f\big(V\big(\omega(s)+x\big)\big). \tag{7.16}$$

Since $\mathrm{d}x$ is translation invariant, one may ignore the ω-term for any field path $t \to \omega(t)$. The latter fact allows us to carry out the s-integration and the integration with respect to $\mathrm{d}\mu_{0,0,t}$ using the representation (7.12). In this way we deduce Lieb's result [76,77].

COROLLARY 7.6. *Let f be convex and satisfy the conditions of Theorem* 7.5. *Then*

$$\operatorname{tr} F(K_E) \leqslant (4\pi)^{-d/2} \int_0^\infty \frac{\mathrm{d}t}{t} t^{-d/2} \mathrm{e}^{-Et} \int_{\mathbb{R}^d} \mathrm{d}x\, f\big(tV(x)\big). \tag{7.17}$$

The Birman–Schwinger principle and the monotonicity of F yield

$$\begin{aligned} N(-E; H_V) &= n\big(1; K_E(V)\big) && \text{(B–S principle)} \\ &\leqslant F(1)^{-1} \sum_j F(\nu_j) && (\nu_j \text{ are eigenvalues of } K_E(V)) \\ &= F(1)^{-1} \operatorname{tr} F\big(K_E(V)\big), \end{aligned}$$

which is valid also in the limit as E tends to zero. Finally, the change of variable $\tau = tV(x)$ yields the CLR estimate.

COROLLARY 7.7. *Let f be convex and satisfy the conditions of Theorem* 7.5. *Set* $Z_d(f) = (\int_0^\infty f(s)s^{-d/2-1}\,\mathrm{d}s)(\int_0^\infty f(s)s^{-1}\mathrm{e}^{-s}\,\mathrm{d}s)^{-1}$. *Then*

$$N(0; H_V) \leqslant (4\pi)^{-d/2} Z_d(f) \int_{\mathbb{R}^d} V_+(x)^{d/2}\,\mathrm{d}x \tag{7.18}$$

or, in the notations of (7.2),

$$R_{0,d} \leqslant Z_d(f)\Gamma\left(\frac{d}{2}+1\right). \tag{7.19}$$

Hence, to find the best (for this method) constant in (7.1), one has to evaluate

$$c_d^s \equiv \inf\left\{\Gamma\left(\frac{d}{2}+1\right) Z_d(f) f \not\equiv 0,\, f(0)=0,\, f \text{ nonnegative and convex}\right\}. \tag{7.20}$$

With the particular choice

$$f(s) = \begin{cases} 0, & s \leqslant a, \\ s-a, & a \leqslant s, \end{cases}$$

for some $a > 0$ (recall that $d > 2$), we find that

$$c_d^s \leqslant \min_{x>0} \frac{\Gamma(\frac{d-2}{2})}{x^{d/2}\Gamma(-1, x)}, \tag{7.21}$$

where $\Gamma(z, x) = \int_x^\infty t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}t$ denotes the incomplete Γ function. If $d = 3$ one gets $c_3^s \leqslant 0.116$.

The following formal computation hints that the above choice of f is, in fact, optimal. Indeed, rewrite the right-hand side of (7.19) as

$$\Phi_{l,m}(f) = \frac{\int_0^\infty l''(s) f(s)\,\mathrm{d}s}{\int_0^\infty m''(s) f(s)\,\mathrm{d}s}$$

with $l(s) = \Gamma(\frac{d-2}{2})s^{1-d/2}$ and

$$m(s) = \int_s^\infty \int_t^\infty \frac{\mathrm{e}^{-t'}}{t'}\,\mathrm{d}t'\,\mathrm{d}t = s\Gamma(-1, s).$$

Then integration by parts yields

$$\Phi_{l,m}(f) = \frac{\int_0^\infty l(s) f''(s)}{\int_0^\infty m(s) f''(s)} \geqslant \min_s \frac{l(s)}{m(s)}$$

which is attained if $f''(s) \simeq \delta(s - x_{\min})$ where $x_{\min}$ is the position of the minimum of $l(s)/m(s)$.

7.4. *The proof by Li and Yau*

To the Schrödinger operator one can associate another semigroup, also useful in the study of eigenvalues. Suppose that $V > 0$ and consider the operator $K_E^{-1} = V^{-1/2}(-\Delta + E)V^{-1/2}$; the inverse to the Birman–Schwinger operator (7.5). The semigroup $\mathrm{e}^{-tK_E^{-1}}$ consists, under suitable conditions, of trace class operators. Moreover, $\varphi(t) := \operatorname{tr}\mathrm{e}^{-tK_E^{-1}} = \sum \mathrm{e}^{-\lambda_j^{-1}(K_E)}$, where $\lambda_j(K_E)$ are the eigenvalues of K_E. Another way to understand $\varphi(t)$ is taking the trace of the Green function of the parabolic equation

$$\frac{1}{V(x)}\Delta f = \frac{\partial f}{\partial t}.$$

Considering this equation first on a bounded domain $\Omega \subseteq \mathbb{R}^d$ with Dirichlet conditions and using positivity of the aforementioned Green function and the Sobolev inequality (3.11), Li and Yau established the estimate $\varphi(t) \leqslant c_d t^{-d/2} \int_\Omega V^{d/2}\,\mathrm{d}x$, where the constant c_d depends only on the dimension d. By choosing $t = d/(4\lambda_j)$, they obtained the following result [75].

THEOREM 7.8. *Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geqslant 3$. Let λ_j denote the jth eigenvalue of the boundary value problem*

$$\begin{cases} -\Delta f(x) = \lambda V(x) f(x), \\ f|_{\partial\Omega} = 0. \end{cases} \tag{7.22}$$

Then

$$j K_{d,2^*}^{-d} \mathrm{e}^{1-d/2} \leqslant \lambda_j^{d/2} \int_\Omega V(x)^{d/2}\, \mathrm{d}x, \tag{7.23}$$

where $K_{d,2^*}$ denotes the best constant appearing in the Sobolev inequality (3.11), *namely*

$$K_{d,2^*} = \frac{1}{\sqrt{\pi d(d-2)}} \left(\frac{\Gamma(d)}{\Gamma(d/2)} \right)^{1/d}. \tag{7.24}$$

Let λ_N be the largest eigenvalue less than or equal to 1. Then Theorem 7.8 yields

$$\begin{aligned} \int_\Omega V^{d/2}\, \mathrm{d}x &\geqslant \lambda_N^{d/2} \int_\Omega V^{d/2}\, \mathrm{d}x \\ &\geqslant N K_{d,2^*}^{-d} \mathrm{e}^{1-1/2} \\ &\geqslant N(0; -\Delta - V) K_{d,2^*}^{-d} \mathrm{e}^{1-1/2}, \end{aligned}$$

where the last inequality follows from the Birman–Schwinger principle. Some technical steps, justified by Li and Yau, make it possible to take the limit "$\Omega \to \mathbb{R}^d$" and also to dismiss the condition $V > 0$. In this way, one arrives at the CLR estimate.

THEOREM 7.9. *The CLR estimate* (7.1) *holds and the constant is given by $c_d = \mathrm{e}^{(d-2)/2} K_{d,2^*}^d$, where $K_{d,2^*}$ denotes the constant* (7.24).

Further analysis of the proof, performed by Levin and Solomyak [74], has shown that it is only the positivity of the heat semigroup and the Sobolev inequality, but not the differential nature of the Laplacian, that are essential for this approach. This enabled the authors of [74] to carry over the CLR estimate to certain operators with variable coefficients.

7.5. *CLR estimate for positivity preserving semigroups*

In both approaches above the semigroups have an important positivity property. It turns out that it is this property that is crucial for eigenvalue estimates.

Throughout this subsection Ω is a space with a σ-finite measure μ. Moreover, L^p will denote the space $L^p(\Omega, \mu)$, $1 \leqslant p \leqslant \infty$.

Positivity. There are two natural notions of positivity for operators acting in the Hilbert space $L^2(\Omega)$. First, an operator S is *nonnegative* if $\langle Su, u\rangle \geqslant 0$ for all u in the domain $\mathfrak{D}(S)$ of S; this is the positivity in the operator sense. Second, S is *positivity preserving* (P.P.) if $(Su)(x) \geqslant 0$ almost everywhere (a.e.) for any a.e. positive function $u \in \mathfrak{D}(S)$; this notion is related to the lattice structure of the L^2 space.

Positivity preserving semigroups. Any self-adjoint nonnegative operator T in L^2 generates a strongly continuous contractive semigroup $Q(t) = Q_T(t) = \mathrm{e}^{-tT}$, $0 \leqslant t < \infty$. The class of positivity preserving or, shortly, *positive* semigroups is defined by the property $Q(t)u \geqslant 0$ a.e. for any nonnegative function $u \in L^2$. Generators of positive semigroups can be characterized in terms of their quadratic forms according to the following criterion [113], Theorem XIII.50.

THEOREM 7.10 (First Beurling–Deny criterion). *Let T be a nonnegative self-adjoint operator on $L^2(\Omega, \mathrm{d}\mu)$. Extend $\langle\psi, T\psi\rangle$ to all of L^2 by setting it equal to infinity if $\psi \notin \mathfrak{Q}(T)$. Then the following are equivalent*:

(i) $Q(t) = \mathrm{e}^{-tT}$ *is positivity preserving for all* $t > 0$.

(ii) $\langle|\psi|, T|\psi|\rangle \leqslant \langle\psi, T\psi\rangle$ *for all* $\psi \in L^2$.

(iii) $Q(t)$ *is reality preserving and*

$$\langle\psi_+, T\psi_+\rangle \leqslant \langle\psi, T\psi\rangle$$

for all real-valued $\psi \in L^2$. *Here* $\psi_+ := \max\{\psi(x), 0\}$.

(iv) $Q(t)$ *is reality preserving and*

$$\langle\psi_+, T\psi_+\rangle + \langle\psi_-, T\psi_-\rangle \leqslant \langle\psi, T\psi\rangle$$

for all real-valued $\psi \in L^2$. *Here* $\psi_- = \psi_+ - \psi$.

Another (related) notion is the one of $(2,\infty)$-*bounded semigroup* defined as a self-adjoint contractive semigroup $Q(t) = \mathrm{e}^{-tT}$ in L^2, which for each $t > 0$ is bounded as acting from L^2 to L^∞. Such operators turn out to be integral operators, see, e.g., [60], Section XI.1, or [8], Theorem 1.3. Denoting the integral kernel of e^{-tT} by $Q(t; x, y) = Q_T(t; x, y)$, $(2,\infty)$-boundedness yields that

$$M_T(t) := \operatorname*{ess\,sup}_x \int_\Omega \left| Q_T\left(\frac{t}{2}; x, y\right)\right|^2 \mathrm{d}y = \left\|\mathrm{e}^{-(t/2)T}\right\|^2_{L^2\to L^\infty} < \infty \tag{7.25}$$

(for brevity, we write $\mathrm{d}y$ instead of $\mu(\mathrm{d}y)$).

The kernel $Q(t; x, y)$ is defined almost everywhere on $\Omega \times \Omega$ for any $t > 0$. One can re-define the kernel on a set of measure zero for each $t > 0$ in such a way that it becomes measurable in all variables (see [8], Lemma 2.2) and symmetric: $Q(t; x, y) = \overline{Q(t; y, x)}$ a.e. We always suppose that this has been done.

By duality, $Q(t)$ is also bounded as an operator from L^1 to L^2. The semigroup property $Q(t_1)Q(t_2) = Q(t_1 + t_2)$ shows that $Q(t)$ acts from L^1 to L^∞ and is factorized through L^2.

This makes it possible to define the value of the kernel on the diagonal $y = x$ for almost all x,

$$Q(t; x, x) = \int_\Omega Q(t_1; x, y) Q(t_2; y, x)\, \mathrm{d}y, \quad t_1, t_2 > 0 \text{ and } t_1 + t_2 = t. \tag{7.26}$$

The resulting function is well defined as an element of L^∞ which does not depend on the particular choice of t_1 and t_2. Thus, (7.25) can be rewritten as

$$\begin{aligned} M_T(t) &:= \operatorname*{ess\,sup}_x \int_\Omega Q_T\left(\frac{t}{2}; x, y\right) Q_T\left(\frac{t}{2}; y, x\right) \mathrm{d}y \\ &= \operatorname*{ess\,sup}_x Q(t; x, x). \end{aligned} \tag{7.27}$$

So we see that

$$\left|Q_T(t; x, y)\right| \leqslant M_T(t) \qquad \text{a.e. on } \mathbb{R}_+ \times \Omega \times \Omega, \tag{7.28}$$

$$0 \leqslant Q_T(t; x, x) \leqslant M_T(t) \quad \text{a.e. on } \mathbb{R}_+ \times \Omega. \tag{7.29}$$

The same semigroup property, together with contractivity in L^2, imply that $M_T(t)$ is nonincreasing on $(0, \infty)$. We usually require

$$\int_a^\infty M_T(t)\, \mathrm{d}t < \infty, \quad a > 0. \tag{7.30}$$

We will write $T \in \mathcal{P}$ if the self-adjoint operator T generates a semigroup which is both positive and $(2, \infty)$-bounded. For such T, the kernel $Q_T(t; x, y)$ is nonnegative:

$$\begin{aligned} &Q_T(t; x, y) \geqslant 0 \quad \text{a.e. on } \mathbb{R}_+ \times \Omega \times \Omega, \\ &Q_T(t; x, x) \geqslant 0 \quad \text{a.e. on } \mathbb{R}_+ \times \Omega. \end{aligned} \tag{7.31}$$

If $T \in \mathcal{P}$ and $T \geqslant \gamma$ with some $\gamma \geqslant 0$, then for any $r \geqslant -\gamma$ the operator $T_r = T + r$ also belongs to $\mathcal{P}$. The corresponding semigroup is $Q_{T_r}(t) = \mathrm{e}^{-rt} Q_T(t)$, thus $M_{T_r}(t) = \mathrm{e}^{-rt} M_T(t)$. It follows that $M_{T_r}(t)$ with $r > -\gamma$ decays at infinity exponentially.

CLR estimate for generators of positivity preserving semigroups. Let T be a nonnegative self-adjoint operator in L^2. Suppose that a given measurable function $V \geqslant 0$ (more rigorously, the operator of multiplication by V) is form-bounded with respect to T, with a bound smaller than 1. Then the self-adjoint bounded from below operator $T - V$ is defined by the method of quadratic forms (see Section 3.2).

Denote by $\mathfrak{G}$ the class of all continuous functions G on $[0, \infty)$, growing at infinity no faster than a polynomial and such that $z^{-1}G(z)$ is integrable at zero; the latter assumption implies $G(0) = 0$. We already encountered this class in Section 7.3. In addition, set

$$g_0 = \int_0^\infty z^{-1} G(z) \mathrm{e}^{-z}\, \mathrm{d}z. \tag{7.32}$$

Rozenblum and Solomyak [121], Theorem 2.1, have established the following CLR estimate for a positive semigroup perturbed by a potential V.

THEOREM 7.11. *Let $T \in \mathcal{P}$ be such that $M_T(t)$ satisfies* (7.30) *and $M_T(t) = \mathrm{O}(t^{-\alpha})$ at zero, with some $\alpha > 0$. Fix a nonnegative convex function $0 \not\equiv G \in \mathfrak{G}$, polynomially growing at infinity and such that $G(z) = 0$ near $z = 0$. Then*

$$N(0; T - V) \leqslant \frac{1}{g_0} \int_0^\infty \frac{\mathrm{d}t}{t} \int_\Omega M_T(t) G\bigl(tV(x)\bigr)\,\mathrm{d}x, \tag{7.33}$$

as long as the expression on the right-hand side is finite.

Their method is an "abstract version" of Lieb's approach. The latter was based upon the path integral technique. Instead of this formalism, fairly hard to be carried over from the Laplacian to other operators, the authors of [121] use the operator analysis only. As a consequence, their approach applies to a quite general situation; requiring from Ω only the structure of measure space.

REMARK 7.12. 1. The finiteness of (7.33) assures that the operator $T - V$ is bounded from below.

2. It follows from convexity that $G(z)$ grows at infinity at least as a constant times z. Therefore, the condition (7.30) is necessary in order that the estimate (7.33) be meaningful.

Since the function G appears in (7.33) as a parameter, one has a *family* of estimates and, consequently, it is possible to optimize in the parameter G. The idea of a parametric estimate of the type (7.33) is due to Lieb [76], who proved it for the Schrödinger operator. He also suggested to use the function $G_a(z) = (z - a)_+$, $a > 0$, where a is a scalar parameter. Optimization of G can be performed as described in Section 7.3.

EXAMPLE 7.13 (Schrödinger operator). Let $\Omega = \mathbb{R}^d$ with the Lebesgue measure, $T = -\Delta$. The kernel of e^{-tT} equals $Q(t; x, x) = (2\pi)^{-d/2} t^{-d/2}$ on the diagonal. So, (7.30) dictates $d \geqslant 3$. Following Lieb [76] (see also [138], p. 96), select $G(z) = (z - a)_+$, with some positive constant a to be chosen. Then (7.33) gives

$$N(0; -\Delta - V) \leqslant (2\pi)^{-d/2} g_0^{-1} \int_0^\infty \int_{\mathbb{R}^d} t^{-d/2-1} G\bigl(tV(x)\bigr)\,\mathrm{d}x\,\mathrm{d}t, \tag{7.34}$$

where, according to (7.32), $g_0 = \int_a^\infty (1 - az^{-1}) \mathrm{e}^{-z}\,\mathrm{d}z$. After the change of variables we come to the CLR estimate

$$N(0; -\Delta - V) \leqslant C(G) \int_{\mathbb{R}^d} V(x)^{d/2}\,\mathrm{d}x, \tag{7.35}$$

where

$$C(G) = (2\pi)^{-d/2} g_0^{-1} \int_a^\infty (t - a) t^{-d/2-1}\,\mathrm{d}t.$$

Then one should optimize in a. For $d = 3$, it turns out that the best constant in (7.35) is obtained when $a = 0.25$ which gives $C(G) = 0.1156$, the constant found by Lieb. It was explained in Section 7.3 that this choice of G is optimal.

The approach above applies to many operators of Mathematical Physics, e.g., the relativistic Schrödinger operator, the discrete Schrödinger operator, sub-Laplacian on a nilpotent Lie group, and nonisotropic operators, the only decisive restriction being the condition of positivity [121].

7.6. *CLR estimate for operator-valued potentials*

Some problems in quantum physics can be reduced to Schrödinger-type operators acting not on scalar-valued functions but on finite, and even infinite-dimensional vector-valued functions. Some of such problems arise when considering the *Born–Oppenheimer approximation* for systems containing both heavy and light particles (see, e.g., [43]), others are obtained from the scalar systems by separation of variables. A natural question is how to obtain eigenvalue estimates for such operators. Even for operators acting on functions with a finite-dimensional space of values, it is important to have eigenvalue estimates not depending on the dimension of this latter space.

Hundertmark [55] has shown that Cwikel's proof of the CLR estimate can be adapted to the following operator-valued setting.

Let $\mathcal{G}$ be a Hilbert space with norm $\|\cdot\|_{\mathcal{G}}$, scalar product $\langle\cdot,\cdot\rangle_{\mathcal{G}}$, and let $\mathbf{1}_{\mathcal{G}}$ be the identity operator on $\mathcal{G}$. Let $L^q(\mathbb{R}^d, \mathfrak{S}^r(\mathcal{G}))$ be the space of operator-valued functions f whose norm

$$\|f\|_{q,r}^q = \|f\|_{L^q(\mathbb{R}^d,\mathfrak{S}^r(\mathcal{G}))}^q := \int_{\mathbb{R}^d} \operatorname{tr}_{\mathcal{G}}\big(\big|f(x)\big|^r\big)^{q/r}\,\mathrm{d}x$$

is finite; $\mathfrak{S}^r$ is defined in Section A.8.

Hundertmark obtained the following result.

THEOREM 7.14 (Cwikel's inequality, operator-valued case). *Let f and g be operator-valued functions on an auxiliary Hilbert space $\mathcal{G}$. Assume that $f \in L^q(\mathbb{R}^d, \mathfrak{S}^q(\mathcal{G}))$ and $g \in L^q_w(\mathbb{R}^d, \mathcal{B}(\mathcal{G})))$ for some $q > 2$. Then $f(x)g(-\mathrm{i}\nabla)$ is a compact operator on $L^2(\mathbb{R}^d, \mathcal{G})$. In fact, it belongs to the weak operator ideal $\mathfrak{S}^q_w(L^2(\mathbb{R}^d, \mathcal{G}))$ and, moreover,*

$$\big\|f(x)g(-\mathrm{i}\nabla)\big\|_{q,w}^* := \sup_{k\geqslant 1} k^{1/q}\mu_k\big(f(x)g(-\mathrm{i}\nabla)\big) \leqslant K_q\|f\|_{q,q}\|g\|_{q,w}^*, \tag{7.36}$$

where the constant K_q is given by

$$K_q = (2\pi)^{-d/q}\frac{q}{2}\left(\frac{8}{q-2}\right)^{1-2/q}\left(1+\frac{2}{q-2}\right)^{1/q}. \tag{7.37}$$

As in the scalar-valued case, the CLR estimate is an immediate consequence of Theorem 7.14; in the operator-valued setting it takes the following form.

THEOREM 7.15. *Let $\mathcal{G}$ be some auxiliary Hilbert space and let V be a potential with $V_+ \in L^{d/2}(\mathbb{R}^d, \mathfrak{S}^{d/2}(\mathcal{G}))$. Then the operator $-\Delta \otimes \mathbf{1}_{\mathcal{G}} - V$ has a finite number N of negative eigenvalues and the following estimate holds*

$$N(0; -\Delta \otimes \mathbf{1}_{\mathcal{G}} - V) \leqslant C_d \int_{\mathbb{R}^d} \operatorname{tr}_{\mathcal{G}} V_+^{d/2} \, \mathrm{d}x. \tag{7.38}$$

7.7. *Generalizations to variable coefficients*

A natural generalization of the CLR estimate, with the Laplacian replaced by the Laplace–Beltrami operator (2.1) on a manifold M with a metric g, would have the form

$$N(0; -\Delta_g - V) \leqslant C \int_M V_+(x)^{d/2} \mu_g(\mathrm{d}x), \quad d \geqslant 3, \tag{7.39}$$

where $\mu_g(\mathrm{d}x)$ is the Riemannian volume,

$$\mu_g(\mathrm{d}x) = (\det g)^{-1/2} \, \mathrm{d}x, \tag{7.40}$$

in local coordinates. There is no hope that this estimate holds for all manifolds, in particular, it surely fails for compact manifolds, for which even the smallest positive V produces at least one negative eigenvalue.

However, in the case of the Euclidean space with a nonstandard metric, certain results can be obtained. In principle, all methods proving the usual CLR estimate, might be used to handle (7.39), with exception of Cwikel's one, the latter being too closely related to the Euclidean structure of $\mathbb{R}^d$. The initial approach, originating in [16], used by Rozenblum to prove the CLR estimate (see [116], and modern presentations of this method in [99] or in [119]) can be applied to handle this problem. The idea of the approach lies in constructing a subspace $\mathcal{L}$ of proper co-dimension, on which the ratio of quadratic forms corresponding to the two terms in $-\Delta_g - V$ is less than 1,

$$\int_{\mathbb{R}^d} V(x) \bigl|u(x)\bigr|^2 \mu_g(\mathrm{d}x) < \int_{\mathbb{R}^d} |\nabla_g u|^2 \mu_g(\mathrm{d}x), \quad u \in \mathcal{L} \setminus \{0\}, \tag{7.41}$$

where $|\nabla_g u|^2 = |\sum g^{jk} \partial_j u \, \partial_k u|$. As soon as such a subspace is constructed, the co-dimension of $\mathcal{L}$ gives, according to Glazman's lemma (see Section A.9), an estimate from above for the number of eigenvalues above 1 of the operator defined by quadratic forms in (7.41). The Birman–Schwinger principle then immediately leads to estimates for $N(0; -\Delta_g - V)$. Hence, the problem lies in constructing such subspaces. In fact, it is a problem in approximation theory. One can understand $\mathcal{L}$ as the orthogonal complement to a finite-dimensional subspace $\mathcal{K}$ (it is the dimension of this $\mathcal{K}$ that is the co-dimension of $\mathcal{L}$).

Thus the estimate (7.41) simply measures how good *arbitrary* functions with controlled $\int |\nabla_g u|^2 \mu_g(\mathrm{d}x)$ can be approximated by functions in $\mathcal{K}$ in the metrics of the space L^2 with weight V. In [116] $\mathcal{K}$ was taken as a linear span of characteristic functions of a certain system of cubes, so in this case the approximation method was piecewise constant. A concrete realization of this idea involves obtaining Poincaré-type estimates in cubes,

$$\int_Q V(x)\big|u(x) - u_Q\big|^2 \mu_g(\mathrm{d}x) \leqslant C(V, g, Q) \int_Q |\nabla_g u|^2 \mu_g(\mathrm{d}x), \tag{7.42}$$

where u_Q is the mean value of u over the cube Q. Combinatorial–geometric considerations then provide us with a system Ξ of cubes, covering some compact part Ω in $\mathbb{R}^d$ with finite multiplicity, so that the constant $C(V, g, Q)$ has the order 1 for $Q \in \Xi$. Outside Ω, V is "small": $\int_{\mathbb{R}^d \setminus \Omega} V|u|^2 \mu_g(\mathrm{d}x) \leqslant \varepsilon \int_{\mathbb{R}^d \setminus \Omega} |\nabla_g u|^2 \mu_g(\mathrm{d}x)$. After this construction is done, one takes as $\mathcal{K}$ the space spanned by the characteristic functions of the cubes $Q \in \Xi$. In the original realization in [116] (see also [99,119]), the Hölder inequality and the Sobolev embedding theorems were used to establish (7.42). Applied to our concrete situation, one obtains the estimate

$$\begin{aligned} &N(0; -\Delta_g - V) \\ &\quad \leqslant C\left(\int_{\mathbb{R}^d} V_+(x)^{pd/2}\,\mathrm{d}x\right)^{1/p} \left(\int_{\mathbb{R}^d} \omega(x)^{-qd/2}\,\mathrm{d}x\right)^{1/q}, \quad d \geqslant 3, \end{aligned} \tag{7.43}$$

for arbitrary $p, q \geqslant 1$, $p^{-1} + q^{-1} = 1$, where $\omega(x)$ is the smallest eigenvalue of the matrix $g_{jk}(x)$. Thus possible cancellation of singularities of g and V at finite points or at infinity cannot be taken into account. Another shortcoming in (7.43) is that it involves the smallest eigenvalue of the matrix g_{jk} and not the determinant, as one would like to have, thus weakening the result for nonisotropic metrics.

Fefferman in [42] initiated the use of advanced methods of harmonic analysis, more exactly, the theory of singular integrals, to establish inequalities of the form (7.42) with better control over constants. He has also improved the combinatorial–geometrical considerations. Further developments (by Chanillo and Wheeden [23] and Tachizawa [145]) led, in particular, to estimates of the form

$$N(0; -\Delta_g - V) \leqslant C \int_{\mathbb{R}^d} V_+(x)^{d/2} \omega(x)^{-d/2}\,\mathrm{d}x, \quad d \geqslant 3, \tag{7.44}$$

under some mild conditions imposed on $\omega(x)$: this function must belong to the A_2 class of Mackenhoupt and satisfy a certain reverse Hölder inequality – see [23] for details. Say, functions having power-like singularities satisfy these conditions. In particular, if the metric is isotropic, that is, the matrix $g_{jk}(x)$ is a multiple of the identity matrix, (7.44) turns into (7.39).

We note here that this approach generalizes successfully to operators of higher order; this involves piecewise polynomial, instead of piecewise constant, approximation. Such a generalization is impossible for the methods using the heat kernel analysis: they rely heavily on positivity (or positive domination) of the kernel, which always fail for higher-order operators.

The recent progress in eigenvalue estimates on general manifolds is related to new results in the heat kernel analysis. On the one hand, for a manifold (M, g) with a nonnegative Ricci curvature, the heat kernel $Q(t, x, x)$ is estimated in terms of the volume of geodesic balls $B(x, t)$ with centrum at x and radius $t^{1/2}$ (see [24] and literature therein),

$$Q(t, x, x) \leqslant C\left|B\left(x, t^{1/2}\right)\right|^{-1}. \tag{7.45}$$

Setting this estimate into Theorem 7.11, one obtains an eigenvalue estimate for the Schrödinger operator. This estimate, generally, does not have CLR-form, since one does not have at present any estimates of volumes of balls in terms of the metric tensor g.

Another recent development, generalizing the method of Li and Yau, can be found in the work by Grigor'yan and Yau [50]. Being customized to the case of a geometrical operator (the work, actually, deals with an abstract setting of operators defined by Dirichlet forms) the result of Grigor'yan and Yau is the following [50], Theorem 4.3.

Let μ be a measure on M, absolutely continuous with respect to the Riemannian measure, and let $a(x)$ be a positively definite matrix depending continuously on $x \in M$.

THEOREM 7.16. *Suppose that for some $b > 0$, $p > 1$ and for any open set $\Omega \subset M$, the estimate*

$$\int_\Omega |\nabla_g u|^2 \mu_g(\mathrm{d}x) \geqslant b\mu(\Omega)^{-1/p} \int_\Omega |u|^2 \mu_g(\mathrm{d}x) \tag{7.46}$$

holds for any $u \in C_0^\infty(\Omega)$. Then, for a nonnegative function V,

$$N\left(0; -\frac{1}{m}\nabla_g^*\left(a^2\nabla_g\right) - V\right) \leqslant cb^{-p} \int_M V^p \mu_g(\mathrm{d}x), \quad m := \frac{\mathrm{d}\mu}{\mathrm{d}\mu_g}. \tag{7.47}$$

The main ingredient in the proof is the fact that the condition (7.46) implies rather detailed heat kernel estimates for the operator $-\frac{1}{m}\nabla_g^*(a^2\nabla_g)$.

For the case of the Euclidean space, (7.46) is obtained easily from the Sobolev inequality and the Hölder inequality and this produces again the estimate (7.43). Establishing (7.46) in more general situations will lead to a progress in proving (7.44).

8. Lieb–Thirring inequalities

For $\gamma \geqslant 0$, we consider the sum

$$S_{\gamma,d}(V; \hbar) \equiv \sum_{E_j \leqslant 0} |E_j|^\gamma, \tag{8.1}$$

where $E_j = E_j(V; \hbar)$ denote the negative eigenvalues of $H_V = -\hbar^2\Delta - V$. Obviously $S_0(V; \hbar) = N(0; H_V)$. By generalizing the reasoning leading to (1.1), one expects that bounds on the *moments* $S_{\gamma,d}(V;\hbar)$ expressed in terms of the *phase-space volume* exist, i.e.,

$$S_{\gamma,d}(V;\hbar) \leqslant R_{\gamma,d} S^c_{\gamma,d}(V;\hbar), \tag{8.2}$$

where $R_{\gamma,d}$ are (optimal) constants and

$$\begin{aligned} S^c_{\gamma,d}(V;\hbar) &:= (2\pi\hbar)^{-d} \iint_{\{(x,\xi)\,|\,H(x,\xi)\leqslant 0\}} \bigl|H(x,\xi)\bigr|^\gamma \,\mathrm{d}x\,\mathrm{d}\xi \\ &= L^c_{\gamma,d}\hbar^{-d} \int_{\mathbb{R}^d} V_+(x)^{\gamma+d/2}\,\mathrm{d}x \end{aligned} \tag{8.3}$$

with

$$L^c_{\gamma,d} = \frac{\Gamma(\gamma+1)}{2^d \pi^{d/2}\Gamma(\gamma+1+d/2)} \tag{8.4}$$

being the *classical constant*; when $\hbar = 1$ we set $S_{\gamma,d}(V) := S_{\gamma,d}(V;1)$. The word "classical" is explained by the relation to the eigenvalue asymptotics (1.3). Using Lemma A.5 (in Section A.9) we find that

$$\begin{aligned} S_{\gamma,d}(V;\hbar) &= \gamma \int_0^\infty E^{\gamma-1} N\bigl(0; H_{-E+V}(\hbar)\bigr)\,\mathrm{d}E \\ &= \gamma \int_0^\infty E^{\gamma-1} N(0; H_{g(-E+V)})\,\mathrm{d}E, \quad g = \hbar^2. \end{aligned} \tag{8.5}$$

Substituting instead of $N(0; H_{g(-E+V)})$ its asymptotical expression (1.3), we arrive at

$$S_{\gamma,d}(V;\hbar) = \bigl(1+\mathrm{o}(1)\bigr) S^c_{\gamma,d}(V;\hbar) \quad \text{as } \hbar \to 0. \tag{8.6}$$

In particular, the question arises, for which values of γ and d does the inequality $S_{\gamma,d}(V;\hbar) \leqslant S^c_{\gamma,d}(V;\hbar)$ hold for all V?

8.1. *Admissible values of γ and results*

It is well known that any arbitrary small attractive potential well has at least one bound state in low dimensions $d = 1, 2$ [14,132]. Thus, the quantity $S_{0,d}(V)$, being the number of negative eigenvalues, is a positive integer for any nontrivial nonnegative V. On the other hand, the phase-space volume $S^c_{0,d}(V)$ can be made arbitrary small and, consequently, we cannot estimate $S_{0,d}(V)$ from above by $S^c_{0,d}(V)$. Moreover, we can rule out the case $0 < \gamma < 1/2$, $d = 1$, because the unique weakly coupled negative eigenvalue obeys [132]

$$\bigl(-E_1(V;\hbar)\bigr)^{1/2} = \bigl(2^{-1}+\mathrm{o}(1)\bigr)\hbar^{-1} \int_{\mathbb{R}} V(x)\,\mathrm{d}x \quad \text{as } \hbar \to \infty, \tag{8.7}$$

which shows that $S_{\gamma,1}(V;\hbar) = \mathrm{O}(\hbar^{-2\gamma})$ for large $\hbar$, while $S^c_{\gamma,1}(V;\hbar) = \mathrm{O}(\hbar^{-1})$.

In 1975, Lieb and Thirring proved the following result for the cases $\gamma > 1/2$, $d = 1$, and $\gamma > 0$, $d \geqslant 2$ [84,85].

THEOREM 8.1 (Lieb–Thirring inequality). *Let $E_1 \leqslant E_2 \leqslant \cdots < 0$ denote the negative eigenvalues of $H_V = -\hbar^2\Delta - V$ in $L^2(\mathbb{R}^d)$. Then the inequality*

$$S_{\gamma,d}(V;\hbar) \leqslant L_{\gamma,d}\hbar^d \int_{\mathbb{R}^d} V_+(x)^{\gamma+d/2}\,\mathrm{d}x \tag{8.8}$$

holds for suitable constants $L_{\gamma,d}$ if and only if

$$\begin{aligned} &\gamma \geqslant 1/2 \quad \text{for } d = 1,\\ &\gamma > 0 \quad\ \ \text{for } d = 2,\\ &\gamma \geqslant 0 \quad\ \ \text{for } d \geqslant 3. \end{aligned} \tag{8.9}$$

Note that the right-hand side of (8.8) has the correct order of the semiclassical Weyl-type asymptotics (8.6). In comparison to the latter, however, the inequality (8.8) is *uniform* in $\hbar > 0$. This enables one to obtain information on the negative spectrum of Schrödinger operators from the classical considerations also in the *nonasymptotical* regime.

Lieb and Thirring applied (8.8) to give a proof of *stability of matter* within nonrelativistic quantum mechanics [78,84,85]. Their proof of (8.8) proceed by establishing an upper bound to $N(-E;H_V)$ using their generalization of the Birman–Schwinger principle, Theorem 7.3, in combination with the bound (7.6). The result then follows from Lemma A.5 in Section A.9.

For values of d, where the CLR estimate is valid, the Lieb–Thirring inequality can easily be derived from it:

$$\begin{aligned} S_{\gamma,d}(V) &= \gamma^{-1}\int_0^\infty \#\bigl\{E_j(V) < -t\bigr\} t^{\gamma-1}\,\mathrm{d}t\\ &= \gamma^{-1}\int_0^\infty S_{0,d}(V-t)t^{\gamma-1}\,\mathrm{d}t\\ &\leqslant R_{0,d}\gamma^{-1}\int_0^\infty S^c_{0,d}(V-t)t^{\gamma-1}\,\mathrm{d}t\\ &= R_{0,d}\gamma^{-1}\int_0^\infty \left\{\iint_{H(\xi,x)<-t} \frac{\mathrm{d}x\,\mathrm{d}\xi}{(2\pi)^d}\right\} t^{\gamma-1}\,\mathrm{d}t\\ &= R_{0,d}S^c_{\gamma,d}(V). \end{aligned} \tag{8.10}$$

Even here it does not give the optimal constant. For $d \leqslant 2$, however, the latter approach does not work and progress is thus not related to (8.10).

The case $\gamma = 1/2$ and $d = 1$ was treated by Weidl [153]. In fact, Weidl obtains a *two-sided* estimate

$$S^c_{1/2,1}(V) \leqslant S_{1/2,1}(V) \leqslant 2S^c_{1/2,1}(V), \quad V \geqslant 0, V \in L^1(\mathbb{R}). \tag{8.11}$$

The optimal constant in the upper estimate was established in [57] and it seems that the optimal lower estimate in (8.11) was first noted in [49]. By comparing the weak and the strong coupling constant behavior one easily sees that $\gamma = 1/2$ and $d = 1$ is the only case in the Lieb–Thirring scale, where one can express such two-sided estimate via the phase-space volume.

It is worth to mention that the lower estimate in (8.11) implies that (see [49])

$$S_{0,2}(V) \geqslant S^c_{0,2}(V) \tag{8.12}$$

for nonnegative *spherical symmetric* V in two dimensions. Furthermore, for $d = 2$ the negative spectrum of H_V is infinite for *any* nonnegative potential $V \in L^1_{\mathrm{loc}} \backslash L^1$, i.e., when $S^c_{0,2}(V)$ is infinite [153]. An interesting open problem is to find out in which way (8.12) can be extended to a larger class of potentials. This calculation can be also successfully repeated, with the Laplacian replaced by any other operator, for which the CLR estimate is known. In particular, if applying here the estimate (7.44) for the Laplacian on $\mathbb{R}^d$ with a non-Euclidean metric, we obtain

$$S_{\gamma,d}(g, V) \leqslant C \int_{\mathbb{R}^d} V_+(x)^{d/2+\gamma} \omega(x)^{-d/2}\, \mathrm{d}x, \quad d \geqslant 3,$$

where $S_{\gamma,d}(g, V)$ is the Lieb–Thirring sum for the operator $-\Delta_g + V$. Results of this kind were recently obtained by Tachizawa for certain degenerate elliptic operators [145].

8.2. *Optimal values of the constants $L_{\gamma,d}$*

An important, and still partially open problem is to determine the optimal value of the constant $L_{\gamma,d}$, especially to determine those cases in which $L_{\gamma,d} = L^c_{\gamma,d}$, in other words, when $R_{\gamma,d} \equiv L_{\gamma,d}/L^c_{\gamma,d} = 1$. In fact, in the study of the stability of matter the numerical value of the constant $L_{\gamma,d}$ is compared with fundamental physical constants, so exact knowledge of $L_{\gamma,d}$ may decide the fate of the Universe.

By an argument similar to the one leading to (8.10), Aizenman and Lieb showed that $R_{\gamma,d}$ is a monotonically nonincreasing function of γ [7].

THEOREM 8.2. *If $R_{\gamma,d}$ is finite for some d and some $\gamma \geqslant 0$, then*

$$R_{\gamma',d} \leqslant R_{\gamma,d} \quad \textit{for all } \gamma' \geqslant \gamma. \tag{8.13}$$

The results in dimension $d = 1$ are the most detailed ones. Optimal constants were obtained already in [7,85], wherein it was shown that

$$R_{\gamma,1} = 1 \quad \text{for all } \gamma \geqslant 3/2. \tag{8.14}$$

Since the asymptotical behavior (8.6) entails

$$R_{\gamma,d} \geqslant 1 \tag{8.15}$$

for admissible pairs γ and d, the constants (8.14) must be sharp.

In the case $\gamma = 1/2$, Hundertmark, Lieb and Thomas [57] showed that

$$R_{1/2,1} = 2 \tag{8.16}$$

by means of a monotonicity principle for partial eigenvalue moments of the modified Birman–Schwinger operator (7.7); the constant (8.16) reflects the weak coupling limit behavior (8.7). Furthermore, if $V(x) = \delta(x)$ then H_δ has the unique negative eigenvalue $E_1(\delta) = -1/4$. Up to translation and scaling this is the only potential for which the constant (8.16) is achieved [57].

In the case $1/2 < \gamma < 3/2$, an analysis of the lowest bound state shows that

$$R_{\gamma,1} = \sup_{V \in L^{\gamma+1/2}} \frac{S_{\gamma,1}(V)}{S^c_{\gamma,1}(V)} \geqslant \sup_{V \in L^{\gamma+1/2}} \frac{(-\lambda_1(V))^\gamma}{S^c_{\gamma,1}(V)} = 2\left(\frac{\gamma - 1/2}{\gamma + 1/2}\right)^{\gamma-1/2}. \tag{8.17}$$

The maximizing potential is

$$V(x) = \left(\gamma^2 - \frac{1}{4}\right) \cosh^{-2} x.$$

Lieb and Thirring conjectured in [85] that $R_{\gamma,1}$ is actually equal to the numerical factor on the right-hand side of (8.17). The result (8.16) in combination with (8.13) implies at least $R_{\gamma,1} \leqslant 2$. But the sharp values of $R_{\gamma,1}$ remain unknown.

Another case where the Lieb–Thirring inequalities can be tested by direct calculation is the harmonic oscillator.

THEOREM 8.3. *For the harmonic oscillator* $V_a = \Lambda - \sum_{k=1}^d a_k^2 x_k^2$, $a = (a_1, \dots, a_d)$, $a_k > 0$ *and* $\Lambda \geqslant 0$, *the inequality*

$$S_{\gamma,d}(V_a) \leqslant S^c_{\gamma,d}(V_a) \tag{8.18}$$

holds for all $\gamma \geqslant 1$ *and all* $d \in \mathbb{N}$.

For the dimension $d = 1$ this can be shown by direct computation in conjunction with the Aizenman–Lieb argument. The reasoning in higher dimensions $d > 1$ is more involved and has been carried out by De la Bretéche [30] and Laptev [68].

The harmonic oscillator provides also illuminating counterexamples. Indeed, one can show that if $\gamma < 1$ then for certain parameters Λ and a the inequality $S_{\gamma,d}(V_a) > S^c_{\gamma,d}(V_a)$ is valid [52], and hence

$$R_{\gamma,d} > 1 \quad \text{for all } \gamma \leqslant 1, d \in \mathbb{N}. \tag{8.19}$$

Further analysis, see [77,85], shows that for $d = 2$ the inequality (8.19) holds for all $\gamma < \gamma_0 \approx 1.16$ and it also gives some explicit upper bounds for the constants $R_{\gamma,d}$.

Until recently the following Lieb–Thirring conjecture posed in [85] was completely open.

CONJECTURE 8.4. In any dimension d there exists a finite critical value $\gamma_{\mathrm{cr}}(d)$ such that $R_{\gamma,d} = 1$ for all $\gamma \geqslant \gamma_{\mathrm{cr}}(d)$. One expects that $\gamma_{\mathrm{cr}}(d) = 1$ for $d \geqslant 3$.

One of the most important cases for applications is $\gamma = 1$, $d = 3$, for which the latter conjecture would imply, via a duality argument, that the kinetic energy of fermions is bounded below by the Thomas–Fermi ansatz for the kinetic energy, which in turn has certain consequences for the energy of large Coulomb systems [77,84], a topic of much current interest. The latest essential progress here is related to considering Lieb–Thirring inequalities for operator-valued potentials.

Lieb–Thirring inequalities for operator-valued potentials. In a breakthrough paper Laptev and Weidl [69] showed that, at first sight, a purely technical extension of the Lieb–Thirring inequality from scalar-valued to operator-valued potentials first suggested by Laptev [67] is a key in establishing at least a part of the Lieb–Thirring Conjecture 8.4.

Their setting is as in Section 7.6: let $\mathcal{G}$ be a separable Hilbert space, let $\mathbf{1}_{\mathcal{G}}$ be the identity operator on $\mathcal{G}$ and let V be a function on $\mathbb{R}^d$ with values being self-adjoint operators $V(x)$ in $\mathcal{G}$.

Let $\{E_j(V;\hbar)\}$ denote the negative eigenvalues of the operator $H_{V,\hbar} = -\hbar^2\Delta \otimes \mathbf{1}_{\mathcal{G}} - V(x)$ acting on $L^2(\mathbb{R}^d) \otimes \mathcal{G}$. Moreover, let

$$S_{\gamma,d}(V;\hbar) = \mathrm{tr}_{L^2(\mathbb{R}^d)\otimes\mathcal{G}}\big[H_-(V;\hbar)\big]^\gamma = \sum_j \big|E_j(V;\hbar)\big|^\gamma$$

and

$$S^c_{\gamma,d}(V;\hbar) = \iint \mathrm{tr}_{\mathcal{G}}\big[H_-(\xi,x)\big]^\gamma \frac{\mathrm{d}x\,\mathrm{d}\xi}{(2\pi\hbar)^d} = L^c_{\gamma,d}\hbar^{-d}\int \mathrm{tr}_{\mathcal{G}}\, V_+(x)^{\gamma+d/2}\,\mathrm{d}x,$$

where $H(\xi,x) = |\xi|^2 \otimes \mathbf{1}_{\mathcal{G}} - V(x)$ and $\mathrm{tr}_{\mathcal{G}}[H_-(\xi,x)]^\gamma$ is the Lieb–Thirring sum for the operator $H(\xi,x)$. Define

$$\widetilde{R}_{\gamma,d} = \sup_V \frac{S_{\gamma,d}(V;\hbar)}{L^c_{\gamma,d}\hbar^{-d}\int \mathrm{tr}_{\mathcal{G}}[V_+(x)]^{\gamma+d/2}\,\mathrm{d}x}.$$

Laptev and Weidl obtained the following generalized Lieb–Thirring inequality [69].

THEOREM 8.5 (Lieb–Thirring inequalities, operator-valued case). *The inequality*

$$S_{\gamma,d}(V;\hbar) \leqslant \widetilde{R}_{\gamma,d} S^c_{\gamma,d}(V;\hbar) \tag{8.20}$$

holds with

$$\widetilde{R}_{\gamma,d} = R_{\gamma,d} = 1 \tag{8.21}$$

whenever $\gamma \geqslant 3/2$ *and* $d \in \mathbb{N}$.

The latter confirms the first part of the Lieb–Thirring conjecture with $\gamma_{\mathrm{cr}} \leqslant 3/2$. The crucial observation made by Laptev and Weidl was that this extension of the Lieb–Thirring inequality enables one to perform an inductive proof for $R_{3/2,d} = 1$ as long as one has the a priori information $R_{\gamma,1} = 1$ for operator-valued potentials.

For the physically most important case $\gamma = 1$ and $d = 3$, the best-known estimate was $R_{1,3} \leqslant 5.24$ [19] until Hundertmark, Laptev and Weidl [56] obtained the following improvement by using arguments similar to the ones in [69] in combination with ideas in [57].

THEOREM 8.6. *The inequalities*

$$R_{\gamma,d} \leqslant \widetilde{R}_{\gamma,d} \leqslant \begin{cases} 4 & \text{for } 1/2 \leqslant \gamma < 1, \\ 2 & \text{for } 1 \leqslant \gamma < 3/2, \end{cases} \tag{8.22}$$

are valid for all $d \in \mathbb{N}$. *In particular, if* $d = 1$ *then*

$$R_{1/2,1} = \widetilde{R}_{1/2,1} = 2 \quad \text{for } \gamma = 1/2, \tag{8.23}$$

$$R_{\gamma,1} \leqslant \widetilde{R}_{\gamma,1} \leqslant 2 \qquad \text{for } 1/2 < \gamma < 3/2, \tag{8.24}$$

where $\widetilde{R}_{\gamma,d}$ *is defined in* (8.20) *and its relation to* $R_{\gamma,d}$ *is given in* (8.21) *provided* $\gamma \geqslant 3/2$.

Unlike the scalar-valued case, the range of parameters γ and d for which (8.20) holds is not known. Theorem 8.6 shows only that these inequalities are true for $\gamma \geqslant 1/2$ and all $d \in \mathbb{N}$. This shortcoming arises from the very way the Lieb–Thirring inequalities are shown for operator-valued potentials. First, the inequality is justified for $d = 1$ (for this case, an elementary proof based on the Darboux transformation was later found by Benguria and Loss [11]). Then the induction on dimension is used with the one-dimensional result applied both as a base of induction and on each step. This approach gives good estimates for the coefficients $R_{\gamma,d}$ in the Lieb–Thirring inequality, for instance, they do not depend on the dimension. Moments for $\gamma < 1/2$, however, cannot be estimated via this technique, since the one-dimensional estimate fails already in the scalar-valued case.

REMARK 8.7. The method of [85] extends to systems and verifies (8.20) for $\gamma > 0$ if $d \geqslant 2$ and for $\gamma > 1/2$ if $d = 1$ with the same upper estimates on the constants $\widetilde{R}_{\gamma,d}$ as are obtained for $R_{\gamma,d}$ therein. The validity of (8.20) for $\gamma = 0$ and $d \geqslant 3$ has been established by Hundertmark [55]; see Section 7.6.

Based upon the aforementioned results, Laptev and Weidl [70] formulated the following conjecture, which remains open.

CONJECTURE 8.8. The inequality (8.20) holds for all pairs γ, d for which (8.2) holds, and the optimal values of the constants $R_{\gamma,d}$ and $\widetilde{R}_{\gamma,d}$ coincide.

9. Diamagnetic properties and spectral estimates

We pass here to the quantitative spectral analysis of operators with magnetic fields. The key role here is played by different monotonicities arising for such operators.

9.1. *Domination*

Given two operators S and T in $L^2(\Omega)$ such that T is positivity preserving, one says that S is *positively dominated* by T (abbreviated as P.D. or, in symbols, $S \preccurlyeq T$) if

$$\big|(Su)(x)\big| \leqslant \big(T|u|\big)(x), \quad u \in L^2(\Omega), \tag{9.1}$$

almost everywhere. If S, T are integral operators with kernels $S(x,y)$, $T(x,y)$, then this is equivalent to the inequality $|S(x,y)| \leqslant T(x,y)$ for almost all $(x,y) \in \Omega \times \Omega$.[4]

It is natural to pose the question: which properties of T are inherited by S? Boundedness is clearly inherited and Pitt [106] showed that compactness is carried over as well; we already used it in Section 5.5.

Domination, however, does not imply that all eigenvalues of S are smaller than the corresponding eigenvalues of T as the following example shows.

EXAMPLE 9.1. Consider

$$S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Here S has eigenvalues $1, 1$ and T has eigenvalues $\sqrt{2}, 0$.

Hence, by measuring the "quality" of an operator's compactness in terms of the rate of decay of its s-numbers (or eigenvalues, if the operator is self-adjoint), one must not expect that $S \preccurlyeq T$ implies the inequality $n(\lambda, S) \leqslant n(\lambda, T)$ for the distribution functions for the s-numbers of S and T (see Section A.8).

Compactness can also be "measured" in terms of Schatten ideals (see Section A.8). It is easy to show (see, e.g., [137]) that if the operator T belongs to the Hilbert–Schmidt ideal $\mathfrak{S}^2$ then the same holds for S. This statement is valid also for the classes $\mathfrak{S}^p$ consisting of compact operators for which the sequence of s-numbers belongs to l^p, provided p is an even integer [137]. However, this does not remain true for all $\mathfrak{S}^p$, $p \geqslant 2$, as Simon conjectured in [137]. Indeed, Peller [105] showed that for any p which is not an even integer, there exist operators $S \preccurlyeq T$ such that $T \in \mathfrak{S}^p$ but $S \notin \mathfrak{S}^p$.

If we replace here $\mathfrak{S}^p$ by the class $\mathfrak{S}^p_w$ (see Section A.8), then even for $p = 2$ (and $p = 4$) such examples are found [26].

[4] For additional notions of domination, we refer to Section 9.4.

9.2. *Diamagnetic inequality*

The study of domination (as defined above) was partially motivated by the spectral analysis of Schrödinger operators with magnetic fields; see Section 2.

One of the fundamental facts for Schrödinger operators with (regular) magnetic fields is the so-called *diamagnetic inequality*.

THEOREM 9.2 (Diamagnetic inequality). *If $\mathcal{A} \in L^2_{\rm loc}$ then the semigroup* $\exp(t\Delta)$ *dominates the semigroup* $\exp(t\Delta_{\mathcal{A}})$, $t > 0$,

$$e^{t\Delta_{\mathcal{A}}} \preccurlyeq e^{t\Delta}. \tag{9.2}$$

The inequality hints that the overall effect of introducing the magnetic field is to "push up the spectrum" and, consequently, it implies that the free energy of a system of spinless distinguishable particles or of spinless bosons within a magnetic field obeying the condition is not smaller than without a magnetic field [133].

In combination with the Trotter–Kato–Masuda formula (see Theorem 9.3) it implies that the resolvent (or, which is equivalent, the heat semigroup) of the *perturbed* Schrödinger operator $H_{\mathcal{A},V} = -\Delta_{\mathcal{A}} - V$ with electric potential $-V$ is dominated by that of $H_{0,V}$. It might thus be very tempting to derive individual eigenvalue estimates for the magnetic operator from the nonmagnetic ones, using the diamagnetic inequality only. This would, in particular, lead automatically to estimates not depending on the magnetic field, which agrees with the physical intuition. The example constructed in [9], however, shows that even in this concrete situation switching on the magnetic field may move some eigenvalues in the "wrong" direction, therefore domination does not generally imply inequalities for eigenvalues. Obtaining eigenvalue estimates for the magnetic Schrödinger operator therefore became a separate problem, and many methods used in the nonmagnetic situation did not work in the magnetic case.

The diamagnetic inequality (9.2) was originally conjectured by Simon [134] as a particular case of a general conjecture on when $e^{-tS} \preccurlyeq e^{-tT}$. The general conjecture was proven independently by Hess, Schrader and Uhlenbrock [53] and Simon [135] providing a purely functional analytic proof by relating it to Kato's (distributional) inequality (see Proposition 4.4).

There are several ways to establish the diamagnetic inequality. The proof we give here (first proposed in [136]) is not the simplest one but, probably, the most enlightening. It is based on an important abstract result from operator theory [64].

THEOREM 9.3 (Trotter–Kato–Masuda formula). *Let $S_j \geqslant 0$, $j = 1, \ldots, k$, be self-adjoint operators so that the sum $S = S_1 + \cdots + S_k$ is defined in the sense of quadratic forms. Then, as a strong limit,*

$$\exp(-tS) = s - \lim_{n\to\infty}\left[\exp\left(-\frac{tS_1}{n}\right)\exp\left(-\frac{tS_2}{n}\right)\cdots\exp\left(-\frac{tS_k}{n}\right)\right]^n.$$

Now we are able to prove Theorem 9.2. For j fixed, define

$$\psi_j(x) = \psi_j(x_1, \dots, x_j, \dots, x_d) = \int_0^{x_j} A_j(x_1, \dots, \xi, \dots, x_d)\, \mathrm{d}\xi.$$

Then

$$\mathrm{e}^{-\mathrm{i}\psi_j}\, \partial_j \mathrm{e}^{\mathrm{i}\psi_j} = \tilde{\partial}_j = \partial_j + \mathrm{i}A_j. \tag{9.3}$$

This is a formal relation, however under the condition $\psi_j \in L^1_{\mathrm{loc}}$, the equality (9.3) holds in the rigorous sense: the maximal operators on the left-hand side, resp. right-hand side, of (9.3) are self-adjoint and have the same domain. Thus the operator $-\Delta_{\mathcal{A}}$ can be represented as a sum (in the sense of quadratic forms) of operators $-\tilde{\partial}_j^2$. Therefore we may apply Theorem 9.3 to express the semigroup generated by $-\Delta_{\mathcal{A}}$,

$$\mathrm{e}^{t\Delta_{\mathcal{A}}} = \lim_{n\to\infty} \left(\mathrm{e}^{-\mathrm{i}\psi_1} U_1\left(\frac{t}{n}\right) \mathrm{e}^{\mathrm{i}(\psi_1 - \psi_2)} U_2\left(\frac{t}{n}\right) \cdots U_d\left(\frac{t}{n}\right) \mathrm{e}^{\mathrm{i}\psi_d} \right)^n, \tag{9.4}$$

where $U_j(t) = \mathrm{e}^{t\partial_j^2}$. Now note that each $U_j(t)$ is an integral operator with positive kernel, and the expression in (9.4) is a composition of several (rather many) such operators and multiplications by functions having absolute value 1. If we replace these exponents by their absolute value, in other words delete them, the value of the integral can only increase, and this gives $|\mathrm{e}^{t\Delta_{\mathcal{A}}}(x, y)| \leqslant \mathrm{e}^{t\Delta}(x, y)$ as we need.

9.3. *Semigroup domination criterion*

A different way of establishing domination, found in the 1970s, is based on the Kato inequality in Proposition 4.4. A development of this method, enabling one to consider quadratic forms rather than operators, was proposed by Ouhabaz [104]; this technique permits one to handle more singular cases than $\mathcal{A} \in L^2_{\mathrm{loc}}$.

Let $\mathfrak{s}$ be a sesquilinear form which satisfies

$$\mathfrak{D}(\mathfrak{s}) \text{ is dense in } L^2\big(\mathbb{R}^d\big), \tag{9.5}$$

$$\Re\mathfrak{s}[u, u] \geqslant 0 \quad \forall u \in \mathfrak{D}(\mathfrak{s}), \tag{9.6}$$

$$\big|\mathfrak{s}[u, v]\big| \leqslant C\|u\|_{\mathfrak{s}}\|v\|_{\mathfrak{s}} \quad \forall u, v \in \mathfrak{D}(\mathfrak{s}), \tag{9.7}$$

where C is a constant and $\|u\|_{\mathfrak{s}} = \sqrt{\Re\mathfrak{s}[u, u] + \|u\|^2}$, and moreover,

$$\big\langle \mathfrak{D}(\mathfrak{s}), \|\cdot\|_{\mathfrak{s}} \big\rangle \text{ is a complete space.} \tag{9.8}$$

DEFINITION 9.4. Let $\mathcal{K}$ and $\mathcal{L}$ be two subspaces of $L^2(\mathbb{R}^d)$. We shall say that $\mathcal{K}$ is an ideal of $\mathcal{L}$ if the following two assertions are fulfilled:

(1) $u \in \mathcal{K}$ implies $|u| \in \mathcal{L}$.

(2) If $u \in \mathcal{K}$ and $v \in \mathcal{L}$ such that $|v| \leqslant |u|$ then $v \cdot \operatorname{sign} u \in \mathcal{K}$.

Let $\mathfrak{s}$ and $\mathfrak{t}$ be two sesquilinear forms both of which satisfy (9.5)–(9.8). The semigroups associated to corresponding self-adjoint operators S, T will be denoted by e^{-tS} and e^{-tT}, respectively. The following result was established by Ouhabaz [104], Theorem 3.3 and its Corollary.

THEOREM 9.5. *Assume that the semigroup* e^{-tT} *is positive. The following assertions are equivalent*:

(1) $\mathrm{e}^{-tS} \preccurlyeq \mathrm{e}^{-tT}$ *for all* $t \geqslant 0$ *and all* $f \in \mathcal{H}$.

(2) $\mathfrak{D}(\mathfrak{s})$ *is an ideal of* $\mathfrak{D}(\mathfrak{t})$ *and*

$$\Re\mathfrak{s}\big[u, |v| \operatorname{sign} u\big] \geqslant \mathfrak{t}\big[|u|, |v|\big] \tag{9.9}$$

for all $(u, v) \in \mathfrak{D}(\mathfrak{s}) \times \mathfrak{D}(\mathfrak{t})$ *such that* $|v| \leqslant |u|$.

(3) $\mathfrak{D}(\mathfrak{s})$ *is an ideal of* $\mathfrak{D}(\mathfrak{t})$ *and*

$$\Re\mathfrak{s}[u, v] \geqslant \mathfrak{t}\big[|u|, |v|\big] \tag{9.10}$$

for all $u, v \in \mathfrak{D}(\mathfrak{s})$ *such that* $u \cdot \bar{v} \geqslant 0$.

In particular, one gets

$$S \in \mathcal{PD}(T) \quad \Longrightarrow \quad \big\{\langle Su, u\rangle \geqslant \mathfrak{t}\big[|u|, |u|\big] \text{ for any } u \in \mathfrak{D}(S)\big\}. \tag{9.11}$$

By means of Ouhabaz' criterion, Melgaard, Ouhabaz and Rozenblum have shown that the diamagnetic inequality is valid for the Friedrichs extension of the Aharonov–Bohm Hamiltonian [98].

THEOREM 9.6. *Let* $\mathcal{A}_{\mathrm{AB}}$ *be the Aharonov–Bohm vector potential given in* (3.7). *The Aharonov–Bohm Hamiltonian* $-\Delta_{\mathrm{AB}} = -(\nabla + \mathrm{i}\mathcal{A}_{\mathrm{AB}})^2$ *in* $L^2(\mathbb{R}^2)$ *obeys*

$$\mathrm{e}^{t\Delta_{\mathrm{AB}}} \preccurlyeq \mathrm{e}^{t\Delta}.$$

Note that the result does not follow directly from the known diamagnetic inequality since the components of the Aharonov–Bohm vector potential do not belong to L^2_{loc}; this latter condition is crucial in all previously existing proofs of the diamagnetic inequality. The proof in [98] carries over to an arbitrary system of Aharonov–Bohm magnetic fields placed at the points of some discrete set in the plane, e.g., a lattice (see also Section 9.5).

9.4. *Diamagnetic and paramagnetic monotonicity*

A drawback of the diamagnetic inequality is that it only allows one to compare the two extreme cases, namely with or without magnetic field. It does not provide control when a magnetic field is varied (pointwise) from a given nonzero value to another nonzero value. In this section we shall see that *diamagnetic monotonicity* and, in the case of Pauli operators, *paramagnetic monotonicity*, in this broader sense are much more difficult to obtain and they are even wrong in general.

Due to the nature of the questions (and answers) addressed here, it is convenient to change slightly our notation for magnetic Schrödinger operators and Pauli operators. Throughout we emphasize the dependence of the magnetic field by setting

$$H_{\mathbf{B},V} = -\Delta_{\mathcal{A}} + V \quad \text{and}$$
$$\mathcal{P}_{\mathbf{B},V} = \left[\sigma \cdot (\nabla + \mathrm{i}\mathcal{A})\right]^2 + V,$$

where $\mathbf{B} = \mathrm{d}\mathcal{A}$ is the (2-form) magnetic field and V is the electric potential. The *pointwise strength* of the magnetic field is defined as $\lfloor \mathbf{B}(x) \rfloor := \operatorname{tr}|\mathscr{B}(x)| = \sum_{n=1}^{[d/2]} |\beta_n(x)|$, where $\mathscr{B}(x)$ is considered as a real skew-symmetric matrix with entries $\{\langle \mathbf{B}(x)e_j, e_k\rangle\}_{j,k=1}^d = \{B_{jk}(x)\}$ (for any orthonormal basis $\{e_k\}_{k=1}^d$) with eigenvalues $\pm\mathrm{i}\beta_n(x)$ (in odd dimensions there is an extra zero eigenvalue). Here $[\cdot]$ denotes the integer part. Furthermore, the *local energy norm* of the magnetic is defined as $\|\mathbf{B}(x)\|_2 = (\operatorname{tr}|\mathscr{B}(x)|^2)^{1/2} = (2\sum_{n=1} |\beta_n(x)]^2)^{1/2}$. Note that $\lfloor \mathbf{B}(x) \rfloor = \|\mathbf{B}(x)\|_2$ for $d = 2, 3$.

For potentials V_1, V_2, being just real-valued functions, domination is easily introduced as a (partial) ordering: we say that V_1 is *smaller* than V_2 if $V_1(x) \leqslant V_2(x)$ holds for all $x \in \mathbb{R}^d$.

Domination for magnetic fields, on the other hand, can be ordered in three ways, at least. First, we say that $\mathbf{B}_1$ is smaller than $\mathbf{B}_2$ in the sense of *pointwise strength* if $\lfloor \mathbf{B}_1(x) \rfloor \leqslant \lfloor \mathbf{B}_2(x) \rfloor$ for all $x \in \mathbb{R}^d$. Second, $\mathbf{B}_1$ is smaller than $\mathbf{B}_2$ in the sense of *local energy norm* if $\|\mathbf{B}_1(x)\|_2 \leqslant \|\mathbf{B}_2(x)\|_2$ for all $x \in \mathbb{R}^d$. Third, $\mathbf{B}_1$ is smaller than $\mathbf{B}_2$ as a 2-form if $|\mathscr{B}_1(x)| \leqslant |\mathscr{B}_2(x)|$ as nonnegative matrices for all x. Observe that all three types of domination coincide in $d = 2$ and in $d = 3$ provided the magnetic fields have constant direction. The first two notions coincide in $d = 3$ for any field.

We list next some possibilities to say that an operator T dominates S defined by means of various (partial) orderings of self-adjoint operators (assuming that both are semibounded, defined on $L^2(\mathbb{R}^d)$ and the kernels below exist).[5]

(D.1) $S \leqslant T$. (Operator sense.)

(D.2) $|\exp(-tT)(x,y)| \leqslant |\exp(-tS)(x,y)|$ $\forall x, y \in \mathbb{R}^d$, $t > 0$. (Heat kernel pointwise.)

(D.3) $\exp(-tT)(x,x) \leqslant \exp(-tS)(x,x)$ $\forall x, y \in \mathbb{R}^d$, $t > 0$. (Heat kernel diagonal pointwise.)

(D.4) $\operatorname{tr}\exp(-tT) \leqslant \operatorname{tr}\exp(-tS)$ $\forall t > 0$. (Trace of heat kernel.)

(D.5) $\inf\sigma(S) \leqslant \inf\sigma(T)$. (Infimum of spectrum.)

(D.6) $\|\exp(-tT)\|_{p,q} \leqslant \|\exp(-tS)\|_{p,q}$, $1 \leqslant p \leqslant q \leqslant \infty$. (Heat kernel $L^p \to L^q$-norm.)

[5]For the Pauli operator one always consider the partial trace $\operatorname{tr}_{\mathbb{C}^2}$ with respect to the spin variables to reduce it to an operator on $L^2(\mathbb{R}^d)$.

Evidently these definitions are related. Some of them imply some of the others, e.g., (D.2) $\Rightarrow$ (D.3) $\Rightarrow$ (D.4) $\Rightarrow$ (D.5). Within Physics, (D.1), (D.4) and (D.5) are the most significant.

If the magnetic field equals zero, then the following classical result says, roughly, that the energy increases as the electric potential increases; the result is most easily shown by means of the Trotter–Kato formula.

PROPOSITION 9.7. *In any dimension, if $V_1 \leqslant V_2$ pointwise, $H_2 = -\Delta + V_2$ dominates $H_1 = -\Delta + V_1$ in the sense of* (D.1)–(D.6).

The search for analogous statements in the case where the magnetic field increases may be guided by the following loosely formulated conjectures.

CONJECTURE 9.8. (i) Diamagnetic principle: The Schrödinger operator "increases" as the magnetic field increases pointwise.

(ii) Paramagnetic principle: The Pauli operator "decreases" as the magnetic field increases pointwise.

Several arguments support these conjectures, the simplest one based upon explicit formulas for the diagonal element of the corresponding heat kernel for a constant magnetic field and zero potential. We will see below that both statements are true in the "short time" limit for the heat kernel limit (i.e., for small t) and, equivalently, in the semiclassical limit $\hbar \to 0$. For the Schrödinger case, it is even true in the strong field limit where $\mathbf{B} \mapsto \mu\mathbf{B}$ and $\mu \to \infty$. It turns out, however, that if no limiting procedure is involved then none of these conjectures are correct unlike the nonmagnetic case.

Magnetic Schrödinger operators. If both $\mathbf{B}_1$ and $\mathbf{B}_2$ are *constant* and $V = 0$, then the heat kernels can be compared explicitly, since they are given by Mehler's formula (see [138]), and one easily sees that domination in the sense of (D.2) (hence also (D.3)–(D.5)) follows if the fields are ordered in the sense of 2-forms. Due to the lack of positivity, however, the domination (D.6) is not trivial. Nevertheless, Loss and Thaller [86] managed to show that the domination (D.6) follows under certain conditions on p, q and $\mathbf{B}_1$.

Next we turn to the case of nonconstant fields $\mathbf{B}_1$ and $\mathbf{B}_2$ obeying that $\mathbf{B}_1$ is "smaller" than $\mathbf{B}_2$ in some of the aforementioned senses. An obvious simplification is to introduce a coupling parameter in the magnetic field, viz. $H_{\mu\mathbf{B},0} = -\Delta_{\mu\mathcal{A}} = (\nabla + \mathrm{i}\mu\mathcal{A})^2$, and consider the weak and large field limits ($\mu \to 0$, $\mu \to \infty$).

We consider first the short time case with $\|\mathbf{B}(x)\|_2$ determining the ordering and we impose the constraint $t\mu \to 0$. In particular, this covers the semiclassical limit with $t =: \beta\hbar^2$, $\mu = \hbar^{-1}$, where β is the "inverse temperature".

The following result, obtained by Erdös, shows that Conjecture 9.8(i) is valid in the short time (large temperature) limit. In other words, domination (D.3) and (D.4) are valid [35], Corollary II.2.

THEOREM 9.9. *If the magnetic fields* $\mathbf{B}_1$ *and* $\mathbf{B}_2$ *with finite* C^1*-norm satisfy* $\|\mathbf{B}_1(x)\|_2 \leqslant \|\mathbf{B}_2(x)\|_2$, V *is semibounded and continuously differentiable, then for any* $x \in \mathbb{R}^d$,

$$\begin{aligned} &\mathrm{e}^{-tH_{\mu\mathbf{B}_1,V}}(x,x) - \mathrm{e}^{-tH_{\mu\mathbf{B}_2,V}}(x,x) \\ &\quad \geqslant C^*(x)(4\pi t)^{-d/2}\mathrm{e}^{-tV(x)}t^2\mu^2\big(\big\|\mathbf{B}_2(x)\big\|_2^2 - \big\|\mathbf{B}_1(x)\big\|_2^2\big) > 0 \end{aligned} \tag{9.12}$$

for some $C^*(x)$ *and for small enough* $t\mu \leqslant \varepsilon_0$, $t \leqslant \varepsilon_0$, *where* ε_0, $C^*(x) > 0$ *depend on the* C^1*-norms of* $\mathbf{B}_1$ *and* $\mathbf{B}_2$, *on* $V_0 = \min_{y\in\mathbb{R}^d} V(y)$ *and, moreover,* $C^*(x)$ *depends on* $\max_{y:\,|y-x|\leqslant 1} |\nabla V(y)|$. *If, in addition,* $\int \exp(-tV(x))\,\mathrm{d}x$ *is finite for all* $t > 0$ *and the* C^1*-norm of* V *is finite, then the domination* (D.4) *is valid as well for small enough* t.

Next we consider the large field limit $\mu \to \infty$ for which Erdös obtained the following result [35], Theorem II.8.

THEOREM 9.10. *Suppose that* $\mathbf{B}_1$ *and* $\mathbf{B}_2$ *are two* C^1 *fields obeying* $0 < \lfloor \mathbf{B}_1(x) \rfloor \leqslant \lfloor \mathbf{B}_2(x) \rfloor$ *and let* V *be a smooth semibounded potential. Then the domination* (D.4)–(D.6) *are true for the operators* $H_n = H_{\mu\mathbf{B}_n,V}$, $n = 1, 2$, *in the* $\mu \to \infty$ *limit*; *for* (D.6) *one needs the additional requirement that* $\exp(-t(-\Delta + V))$ *is in trace class for all* $t > 0$.

Next the question arises whether these conjectures are true without any limiting procedure. It suffices to consider dimension two. We define the *ground-state energy*,

$$E_0(B,V) := \inf \sigma(H_{B,V}) = \inf_{\|\psi\|=1} \langle \psi, H_{B,V}\psi \rangle.$$

The following counter-intuitive examples proposed by Erdös demonstrate that the general form of the Conjecture 9.8(i) is wrong [35].

EXAMPLE 9.11. There exist a potential V (in fact, centrally symmetric such that $|x| \mapsto V(|x|)$ is decreasing outside some compact set) and two constant magnetic fields such that $0 < B_1 < B_2$ but $E_0(B_2, V) < E_0(B_1, V)$.

EXAMPLE 9.12. There exist two nonconstant centrally symmetric magnetic fields obeying $0 \leqslant B_1(x) \leqslant B_2(x)$ for all $x \in \mathbb{R}^2$ but $E_0(B_2, 0) < E_0(B_1, 0)$.

These counterexamples, in particular, show that the domination in the sense of (D.5) cannot be true, which excludes also domination in the sense of (D.2)–(D.4).

Nevertheless, as shown by Leschke, Ruder and Warzel [73], Theorem 1, one can restore domination in the sense of (D.5) if the ground-state wave function in the Poincaré gauge is real. Indeed, in Examples 9.11 and 9.12 the ground-state wave function ψ_0 of H_{B_2,V_2}, in the *Poincaré gauge*,

$$\mathcal{A}(x) := (-x_2, x_1) \int_0^1 \zeta B_2(\zeta x)\,\mathrm{d}\zeta, \tag{9.13}$$

is an eigenfunction of the canonical angular-momentum[6] $L_3 := q_1 p_2 - q_2 p_1$ operator with a nonzero eigenvalue. Thus ψ_0 is not real-valued, which motivates the additional assumption in the following result [73], Theorem 1.

THEOREM 9.13. *Let the Schrödinger operator H_{B_2,V_2} in the Poincaré gauge* (9.13) *possess a real-valued ground-state wave function, i.e., $E_0(B_2, V_2) = \langle\psi_0, H_{B_2,V_2}\psi_0\rangle$ for some ψ_0 in the domain of H_{B_2,V_2} with $\|\psi_0\| = 1$. Then the pointwise inequalities $|B_1(x)| \leqslant |B_2(x)|$ and $V_1(x) \leqslant V_2(x)$ for all $x \in \mathbb{R}^2$ imply the inequality*

$$E_0(B_1, V_1) \leqslant E_0(B_2, V_2)$$

for the corresponding ground-state energies.

As mentioned above, Loss and Thaller studied domination in the sense of (D.6) associated with a nonconstant field $B(x)$ which is globally bounded from below by a positive constant. In particular, they have shown the following inequality [86], Theorem 1.3.

THEOREM 9.14. *Suppose that $B(x) \geqslant B_0 > 0$ is continuous. Then the magnetic heat kernel satisfies the estimate*

$$\left|e^{-tH_B}(x, y)\right| \leqslant \frac{B_0}{4\pi \sinh(B_0 t/2)} e^{-(x-y)^2/2t}. \tag{9.14}$$

Erdös showed that (9.14) is basically the best "off-diagonal" upper bound one can hope for, unless $B : \mathbb{R}^2 \to \mathbb{R}$ has additional properties. More exactly, he proved that (9.14) cannot be sharpened by replacing its right-hand side by $|e^{-tH_{B_0}}(x, y)|$, which has faster Gaussian decay as $|x - y| \to \infty$ [35].

EXAMPLE 9.15. Define $B(x) = (1 + x_1^2/\mu^2)B_0$ with two constants $B_0, \lambda > 0$. Then there exist some $\lambda > 0$, some $x, y \in \mathbb{R}^2$ and some $B_0 > 0$, such that $B_0 \leqslant B(x)$ for all $x \in \mathbb{R}^2$, but $|e^{tH_B}(x, y)| > |e^{tH_{B_0}}(x, y)|$.

It is, however, possible to improve (9.14) at the price of permitting only a restricted class of nonconstant magnetic fields (including the one in Example 9.15). For a field which is constant in one direction Leschke, Ruder and Warzel obtained the following result [73], Theorem 2.

THEOREM 9.16. *Let B_1, B_2 and V_1, V_2 be two magnetic fields and two potentials of which neither depends on the second coordinate x_2. Then the pointwise inequalities $|B_1(x_1)| \leqslant |B_2(x_1)|$ and $V_1(x_1) \leqslant V_2(x_1)$ for all $x_1 \in \mathbb{R}$ imply the inequality*

$$\left|e^{-tH_{B_2,V_2}}(x, y)\right| \leqslant \left|e^{-tH_{B_1,V_1}}\big((x_1, 0), (y_1, 0)\big)\right|$$

for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$ and all $t > 0$.

[6]Here $q = (q_1, q_2)$ and $p = (p_1, p_2)$ denote the usual two-component vector operators of position and canonical momentum, respectively.

Pauli operators. For large temperature (short time in the heat kernel) the diamagnetic and paramagnetic effects can be compared and Erdös established the following general result in dimensions $d = 2, 3$. Again, this statement includes the semiclassical result by using $t := \beta\hbar^2$, $\mu = \hbar^{-1}$.

THEOREM 9.17. *Suppose $d = 3$ and let V be a semibounded continuously differentiable function. Let $\mathbf{B} \in C^3$ with $\|\mathbf{B}\|_{L^\infty} + \|\nabla\mathbf{B}\|_{L^\infty} < \infty$ and let $\mu \geqslant 1$ be a scaling parameter. For any $x \in \mathbb{R}^d$, one has that*

$$\begin{aligned}&\operatorname{tr}_{\mathbb{C}^2} \mathrm{e}^{-t\mathcal{P}_{\mu\mathbf{B},V}}(x,x)\\ &\quad = 2(4\pi t)^{-d/2}\big(1 + \mathrm{O}\big(t^{5/2}\mu^2 + t^4\mu^4\big)\big)\mathrm{e}^{-tV(x)}\frac{t\mu\cosh t\mu\lfloor\mathbf{B}(x)\rfloor}{\sinh t\mu\lfloor\mathbf{B}(x)\rfloor}\\ &\qquad + 2\big(\mathrm{e}^{-tH_{\mathbf{B}=0,V}}(x,x) - \mathrm{e}^{-tV(x)}\mathrm{e}^{-tH_{\mathbf{B}=0,V}}(x,x)\big)\end{aligned} \tag{9.15}$$

as $t \to 0$ and $t\mu \to 0$. The error depends on $\max_{y:\,|x-y|\leqslant 1}[|\nabla^2\mathbf{B}(y)| + |\nabla^3\mathbf{B}(y)|]$, *the C^1-norm of $\mathbf{B}$, on $V_0 := \min_{y\in\mathbb{R}^d} V(y)$ and on $\max_{y:\,|y-x|\leqslant 1}|\nabla V(y)|$. The same holds for $d = 2$; here $\mathbf{B}(x)$, $\lfloor\mathbf{B}(x)\rfloor$ is replaced by $B(x)$ everywhere above.*

The latter implies Conjecture 9.8(ii) in the sense of (D.3) and (D.4) [35], Corollary III.2.

COROLLARY 9.18. *Suppose $d = 3$, $\mathbf{B}_1, \mathbf{B}_2 \in C^3$ with finite C^1-norms, and that $V \in C^1$ is semibounded. For any x satisfying $\lfloor\mathbf{B}_1(x)\rfloor \leqslant \lfloor\mathbf{B}_2(x)\rfloor$, one has that*

$$\begin{aligned}&\operatorname{tr}_{\mathbb{C}^2} \mathrm{e}^{-t\mathcal{P}_{\mu\lfloor\mathbf{B}_2(x)\rfloor,V}}(x,x) - \operatorname{tr}_{\mathbb{C}^2} \mathrm{e}^{-t\mathcal{P}_{\mu\lfloor\mathbf{B}_1(x)\rfloor,V}}(x,x)\\ &\quad \geqslant C^*(x)(4\pi t)^{-d/2}\mathrm{e}^{-tV(x)}t^2\mu^2\big(\lfloor\mathbf{B}_2(x)\rfloor^2 - \lfloor\mathbf{B}_1(x)\rfloor^2\big) > 0\end{aligned} \tag{9.16}$$

for small enough $t\mu \leqslant \varepsilon_0$ and some $C^*(x)$, where $\varepsilon_0, C^*(x) > 0$ depend on the global lower bound of V, on $\max_{y:\,|y-x|\leqslant 1}[|\nabla V(y)| + |\nabla^2\mathbf{B}(y)| + |\nabla^3\mathbf{B}(y)|]$, *and on the global C^1-norms of $\mathbf{B}_1$ and $\mathbf{B}_2$. If, moreover, $\int \exp(-tV(x))\,\mathrm{d}x < \infty$ for all t, and the C^1-norm of V and the C^3-norms of $\mathbf{B}_1, \mathbf{B}_2$ are bounded, then domination in the sense of* (D.4) *is valid for t small enough. The same holds for $d = 2$; here $\mathbf{B}_i(x)$, $\lfloor\mathbf{B}_i(x)\rfloor$ is replaced by $B_i(x)$ everywhere above.*

An interesting open problem is to investigate paramagnetism in $d = 3$ for a general magnetic field away from the short time regime.

9.5. *CLR estimate for generators of positively dominated semigroups*

The CLR estimate for Schrödinger operators with magnetic fields takes the following form.

THEOREM 9.19 (CLR estimate). *For $\mathcal{A} \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ and $V_+ \in L^{d/2}(\mathbb{R}^d)$,*

$$N(0; -\Delta_{\mathcal{A}} - V) \leqslant C_d \int V_+^{d/2}\,\mathrm{d}x \tag{9.17}$$

with constant C_d depending only on the dimension.

The first proof of Theorem 9.19, following Lieb's method [77], was based upon the path integral technique [9,138]. Later Melgaard and Rozenblum [99] obtained an analytic proof by means of a "customized" diamagnetic inequality expressed in terms of certain semigroups combined with Rozenblum's original approach. Among the applications of the "magnetic" CLR estimate are asymptotics of eigenvalues and stability of matter in magnetic fields (even in the more realistic Pauli setting, where one takes into account spin of the particles).

Let S generate a positively dominated semigroup. Like in the positive case (see Section 7.5), suppose that multiplication by a given $V \geqslant 0$ is form-bounded with respect to S, with a bound smaller than 1. (Note that this is always the case, provided the above condition is satisfied for the dominator T; this is an easy consequence of (9.11).) Then we can define the self-adjoint operator $S - V$ and, as above, consider the quantity $N(0; S - V)$.

Rozenblum and Solomyak obtained the following result [121].

THEOREM 9.20. *Let $T \in \mathcal{P}$ and $S \in \mathcal{PD}(T)$. Suppose that $M_T(t)$ satisfies* (7.30) *and $M_T(t) = \mathrm{O}(t^{-\alpha})$ at zero, with some $\alpha > 0$. Let G be a function as in Theorem* 7.11 *and let g_0 be the constant therein. Then*

$$N(0; S - V) \leqslant \frac{1}{g_0} \int_0^\infty \frac{\mathrm{d}t}{t} \int_\Omega M_T(t) G\big(tV(x)\big)\, \mathrm{d}x, \tag{9.18}$$

as long as the expression on the right-hand side is finite.

Comparing Theorems 7.11 and 9.20, we see that $N(0; S - V)$ and $N(0; T - V)$ have the same estimate. However, the natural conjecture $N(0; S - V) \leqslant N(0; T - V)$ is, generally, wrong as noted in [9].

Evidently, Theorem 7.11 is a particular case of Theorem 9.20, for $S = T$. A more general version of Theorem 9.20 exists which does not require (7.30) [118].

PROOF OF THE MAGNETIC CLR ESTIMATE. For a given magnetic vector potential $\mathcal{A}(x) = (A_j(x))_{j=1}^d \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$, consider the magnetic Laplacian $S = -\Delta_{\mathcal{A}} = -(\nabla - \mathrm{i}\mathcal{A})^2$. According to Theorem 9.2, $-\Delta_{\mathcal{A}} \in \mathcal{PD}(-\Delta)$ and thus Theorem 9.20 gives

$$N(0; -\Delta_{\mathcal{A}} - V) \leqslant C(G) \int_{\mathbb{R}^d} V(x)^{d/2}\, \mathrm{d}x, \quad d \geqslant 3, \tag{9.19}$$

with the same constant as in (7.35). □

The proof of (9.19), outlined in [138], page 168, uses the stochastic Itô integral. A more elementary proof in [99], avoiding path integration, gives a somewhat worse constant than in the nonmagnetic case.

The same scheme enables one to derive CLR-type estimates for several other operators of Mathematical Physics with magnetic fields [118].

The proof of the diamagnetic inequality for the *one-vortex* Aharonov–Bohm Hamiltonian (see Theorem 9.6) can be carried over to the case of finitely, and even infinitely many vortices (e.g., a lattice). Equipped with the latter diamagnetic inequality, Melgaard, Ouhabaz and Rozenblum used Theorem 9.20 to derive the Lieb–Thirring inequality, as well as CLR-type estimates, for the (multivortex) Aharonov–Bohm Hamiltonian. Moreover, the presence of special Hardy-type inequalities gives much better eigenvalue estimates than without magnetic fields [98].

9.6. *Comparison theorem for semigroup generators*

We start with a general eigenvalue comparison theorem for semigroup generators. In the Hilbert space $L^2(\Omega)$ we consider two positive self-adjoint operators S and T. These operators generate contraction semigroups $P(t) = \exp(-tS)$ and $Q(t) = \exp(-tT)$, $t \geqslant 0$, in $L^2(\Omega)$. We suppose that the semigroup $Q(t)$ is positivity preserving and $P(t) \preccurlyeq Q(t)$, $t \geqslant 0$. For a function $\phi(\lambda)$, $\lambda > 0$, growing subexponentially, we denote by $\hat{\phi}(t)$ the Laplace transform of ϕ. Such function ϕ is called *regular* if

$$t\hat{\phi}(t) \leqslant C_\phi \phi\bigl(t^{-1}\bigr), \quad t > 0, \tag{9.20}$$

for a certain constant C_ϕ. There are plenty of regular functions, which, obviously, form a convex cone in the space of functions on the semiaxis. In particular, power functions, these times logarithm to some power, etc. are regular.

Rozenblum showed the following result [118].

THEOREM 9.21. *Suppose that for the operator T the estimate*

$$N(\lambda, T) \leqslant \phi(\lambda) \tag{9.21}$$

holds for all $\lambda > 0$, with a certain nondecreasing function $\phi(\lambda)$ of subexponential growth. If the semigroup $P(t) = \exp(-tS)$ is dominated by $Q(t) = \exp(-tT)$, $t \geqslant 0$, then

(i) *the eigenvalue distribution function of S obeys*

$$N(\lambda, S) \leqslant \mathrm{e}\lambda^{-1}\hat{\phi}\bigl(\lambda^{-1}\bigr), \quad 0 < \lambda < \infty, \tag{9.22}$$

in particular, for a regular *function ϕ,*

$$N(\lambda, S) \leqslant \mathrm{e}C_\phi\phi(\lambda), \quad 0 < \lambda < \infty; \tag{9.23}$$

(ii) *if the operator T^{-1} belongs to the class $\mathfrak{S}^p_w$, $0 < p < \infty$, then $S^{-1} \in \mathfrak{S}^p_w$ and $\|S^{-1}\|_{\mathfrak{S}^p_w} \leqslant C(p)\|T^{-1}\|_{\mathfrak{S}^p_w}$;*

(iii) *if the operator T^{-1} belongs to the class $\mathfrak{S}^p$, $0 < p < \infty$, then $S^{-1} \in \mathfrak{S}^p$ and $\|S^{-1}\|_{\mathfrak{S}^p} \leqslant \|T^{-1}\|_{\mathfrak{S}^p}$.*

PROOF. The trace of the positive operator $Q(t)$ can be expressed as

$$t\int_0^\infty N(\lambda, T)\exp(-t\lambda)\,\mathrm{d}\lambda \leqslant t\hat{\phi}(t), \quad t>0.$$

Since the function ϕ grows subexponentially, this integral converges, and thus $Q(t)$ belongs to the trace class $\mathfrak{S}^1$ and, therefore, to the Hilbert–Schmidt class $\mathfrak{S}^2$ for any positive t. This implies that the operators $P(t)$ dominated by $Q(t)$ also belong to the Hilbert–Schmidt class for all positive t, with domination of Hilbert–Schmidt norms. Thus

$$\operatorname{tr} P(t) = \|P(t)\|_{\mathfrak{S}^1} = \left\|P\left(\frac{t}{2}\right)\right\|^2_{\mathfrak{S}^2} \leqslant \left\|Q\left(\frac{t}{2}\right)\right\|^2_{\mathfrak{S}^2} = \operatorname{tr} Q(t) \leqslant t\hat{\phi}(t).$$

Being expressed in terms of eigenvalues λ_j of the operator S, this gives

$$\operatorname{tr} P(t) = \sum_{j=1}^\infty \mathrm{e}^{-t\lambda_j} \leqslant t\hat{\phi}(t). \tag{9.24}$$

Now, for a given positive λ, we set $t=\lambda^{-1}$ in (9.24). Since the first $n(\lambda, S)$ terms in the sum in (9.24) are greater than e^{-1}, we arrive at (9.22). The regularity condition (9.20) gives (9.23). Since the power function is regular, we obtain part (ii) of the theorem, with $C(p) = \mathrm{e}\Gamma(1+\frac{1}{p})$. Finally, $\|S^{-1}\|^p_{\mathfrak{S}^p} = \Gamma(\frac{1}{p})^{-1}\int_0^\infty t^{1/p-1}\operatorname{tr}\exp(-tS)\,\mathrm{d}t$, and similarly, for the operator T^{-1}; in this way (iii) follows from the inequality $\operatorname{tr}\exp(-tS) \leqslant \operatorname{tr}\exp(-tT)$. □

9.7. *"Meta" CLR estimates for dominated pairs of semigroups*

We start with the most regular case; here it is possible to avoid troublesome technicalities present in the general situation. Let S and T be positive definite operators in $L^2(\Omega)$, with the semigroup $\exp(-tT)$ positivity preserving and $\exp(-tS)$ dominated by $\exp(-tT)$, $t>0$. For a nonnegative measurable function $V(x)$, with V and V^{-1} both bounded, we define the operators $T-qV$, $S-qV$, $q\geqslant 0$.

Rozenblum obtained the following result [118].

THEOREM 9.22. *Suppose that*

$$N(0; B-qV) \leqslant \phi(q) \tag{9.25}$$

for all $q>0$, for some nondecreasing function $\phi(q)$ of subexponential growth. Then

$$N(0; A-qV) \leqslant \mathrm{e}q^{-1}\hat{\phi}\big(q^{-1}\big), \quad q>0, \tag{9.26}$$

and

$$\int_0^\infty N(0; A - qV)q^{p-1}\,\mathrm{d}q \leqslant \int_0^\infty N(0; B - qV)q^{p-1}\,\mathrm{d}q, \quad 0 < p < \infty, \tag{9.27}$$

as long as the integral in the right-hand side is finite; *and if the function* ϕ *in* (9.25) *is regular, then*

$$N(0; A - qV) \leqslant \mathrm{e}C_\phi \phi(q). \tag{9.28}$$

PROOF. We will construct semigroup generators related to our Schrödinger-type operators, and apply Theorem 9.21. According to the Birman–Schwinger principle, see Theorem 6.7,

$$N(0; B - qV) = n(\tau, K_B), \qquad N(0; A - qV) = n(\tau, K_A), \quad \tau = q^{-1/2},$$

where

$$K_A = V^{1/2}A^{-1/2}, \qquad K_B = V^{1/2}B^{-1/2},$$

and $n(\tau, K)$ denotes the distribution function for s-numbers $s_j(K)$ of K (see Section A.8). Under our conditions, these operators are bounded, they are products of bounded operators, and K_B is compact. Moreover, they have trivial kernels and therefore the self-adjoint compact operators $K_B K_B^* = V^{1/2}B^{-1}V^{1/2}$ and $K_A K_A^* = V^{1/2}A^{-1}V^{1/2}$ have self-adjoint positive-definite inverses T_B and T_A, respectively. The estimate (9.25) can therefore be written as $N(\lambda, T_B) \leqslant \phi(\lambda)$, $\lambda > 0$. So, Theorem 9.21, with $T = T_B$ and $S = T_A$ will give us the required estimate as soon as we show that the semigroup $\exp(-tT_B)$ dominates $\exp(-tT_A)$. This is equivalent to the resolvent domination

$$(T_A + t)^{-1} \preccurlyeq (T_B + t)^{-1}, \quad t > 0.$$

We can write $(T_A + t)^{-1} = (V^{-1/2}AV^{-1/2} + t)^{-1} = V^{1/2}(A + tV)^{-1}V^{1/2}$, and a similar identity for T_B. Now it remains to notice that the Trotter formula implies $(A + tV)^{-1} \preccurlyeq (B + tV)^{-1}$ and to use positivity of V. □

REMARK 9.23. The relation (9.27) can be expressed also as

$$\|K_A\|_{\mathfrak{S}^{2p}} \leqslant \|K_B\|_{\mathfrak{S}^{2p}}, \quad 0 \leqslant p \leqslant \infty,$$

at the same time, for the regular function $\phi(q) = q^p$, (9.28) implies

$$\|K_A\|_{\mathfrak{S}^p_w} \leqslant C(p)\|K_B\|_{\mathfrak{S}^p_w}, \quad 0 \leqslant p \leqslant \infty.$$

Applied to Schrödinger operators, with $T = -\Delta$, $S = -\Delta_{\mathcal{A}}$, the conditions of boundedness of V and V^{-1} are too restrictive. In a more advanced version of Theorem 9.22, these restrictions are removed and, moreover, V can be a measure, like in Sections 5.2 and 5.3, the main idea of the proof being the same [118].

10. Zero modes for Pauli operator

One of the important features in the spectral properties of Pauli operators is the presence of zero modes, eigenfunctions corresponding to a zero eigenvalue. This phenomenon, rather unusual for nonmagnetic operators, became an interesting field of research lately, both for physicists and mathematicians. Zero modes induce certain pathological behavior of the operator under perturbations by an electric potential, since the potential energy cannot be controlled by the kinetic one. Therefore, in particular, even a very weak electric field may produce an infinite negative discrete spectrum. Analysis of zero modes becomes thus a crucial step in the study of stability of quantum systems (see [80,87]). At the same time, they play an important role in quantum electrodynamics, since they produce singularities in fermionic determinants; see, e.g., [45–47]. In this section we discuss different situations where zero modes arise. We start with the most classical example.

10.1. *Constant magnetic field*

We recall that the Pauli operator with constant magnetic field in dimension $d = 2$ is

$$\mathcal{P}_{\mathcal{A}} = \big(\sigma \cdot (\nabla + \mathrm{i}\mathcal{A})\big)^2 = \begin{pmatrix} Q_- Q_+ & 0 \\ 0 & Q_+ Q_- \end{pmatrix}, \tag{10.1}$$

$Q_\pm = (\partial_1 + \mathrm{i}A_1) \pm \mathrm{i}(\partial_2 + \mathrm{i}A_2)$ (cf. (2.8)) with the magnetic potential $\mathcal{A}(x)$ having the form $(Bx_2/2, -Bx_1/2)$. Here the field B is supposed to be positive.

Since the zero modes $\psi(x)$ are solutions of the equation $\mathcal{P}_{\mathcal{A}}\psi = 0$ and $\mathcal{P}_{\mathcal{A}}$ is the square of the Dirac operator, $\mathcal{P}_{\mathcal{A}} = \mathbf{D}^2$, they must be solutions of the first-order system $\mathbf{D}\psi = 0$. We write this equation for the components ψ_+, ψ_- of the vector function ψ, arriving at equations

$$\begin{aligned} &(\partial_1 + \mathrm{i}\partial_2 - \mathrm{i}A_1 + A_2)\psi_+ = 0, \\ &(\partial_1 - \mathrm{i}\partial_2 - \mathrm{i}A_1 - A_2)\psi_- = 0, \quad \partial_j = \partial_{x_j}. \end{aligned} \tag{10.2}$$

Taking into account the explicit expression for $\mathcal{A}$, we transform (10.2) to

$$(\partial_1 + \mathrm{i}\partial_2)\mathrm{e}^{-B/4(x_1^2+x_2^2)}\psi_+ = 0, \tag{10.3}$$

$$(\partial_1 - \mathrm{i}\partial_2)\mathrm{e}^{B/4(x_1^2+x_2^2)}\psi_- = 0. \tag{10.4}$$

These equations mean that the functions $f_\pm = \mathrm{e}^{\mp B/4(x_1^2+x_2^2)}\psi_\pm$ are analytical in the variables $\bar{z} = x_1 - \mathrm{i}x_2$ and $z = x_1 + \mathrm{i}x_2$, respectively. Consider (10.3). Supposing that it has a solution $\psi_+ \in L^2$, we obtain an analytical function $f_+ \in L^2$; there are, however, no such analytical functions. This means that (10.3) has no (nontrivial) solutions in L^2. On the other hand, (10.4) has quite a lot of L^2 solutions. In fact, if f_- is any entire function of the variable $\bar{z}$ growing at infinity slower than $\exp(|\bar{z}|^{2-\varepsilon})$, then the corresponding function $\psi_- = \mathrm{e}^{-B/4(x_1^2+x_2^2)} f_-$ solves (10.4).

The factorization (2.8) enables one to describe completely the spectrum of $\mathcal{P}_{\mathcal{A}}$. This description is based on the commutation relation

$$Q_+Q_- - Q_-Q_+ = -2BI, \tag{10.5}$$

and the well-known fact that the operators ST and TS have the same nonzero spectrum. Notice, moreover, that $Q_- = Q_+^*$. So, Q_- and, therefore, Q_+Q_- have zero eigenvalue. Since $Q_+Q_- \geqslant 0$ and $Q_-Q_+ = 2BI + Q_+Q_-$, the operator Q_-Q_+ has no spectrum below $2B$, and therefore, Q_+Q_- has no *nonzero* spectrum in $[0, 2B)$. This means that Q_-Q_+ has no spectrum in $(2B, 4B)$ and so on. Repeating this reasoning, one can see that the only possible points of spectrum for Q_+Q_- and Q_-Q_+ are the points of the arithmetical progression $2\nu B$, $\nu = 0, 1, 2, \dots$ for Q_+Q_- and $\nu = 1, 2, \dots$ for Q_-Q_+. Next, it follows again from (10.5) that Q_- transforms isomorphically the space of zero modes for Q_+Q_- onto the spectral subspace of Q_-Q_+ with eigenvalue $2B$. This means, moreover, that Q_+Q_- also has an infinitely degenerate eigenvalue at $2B$. We iterate this reasoning to conclude that all points $2\nu B$ are infinitely degenerate eigenvalues of $\mathcal{P}_{\mathcal{A}}$. They are called *Landau levels*. In a number of questions the explicit description of eigenspaces of $\mathcal{P}_{\mathcal{A}}$ is needed. We return to the study of these eigenspaces in Section 11.1.

Infinite degeneracy of the zero eigenvalue for a constant magnetic field was known quite long ago. It was only in 1979 that the existence of zero modes was discovered for a non-constant field and, moreover, the multiplicity was calculated.

10.2. *Aharonov–Casher theorem and zero modes for $d = 2$*

We consider the Pauli operator $\mathcal{P}_{\mathcal{A}}$ in $\mathbb{R}^2$ with a magnetic potential $\mathcal{A}$ which, for a start, we suppose to be Lipschitz, such that the field $B(x) = \partial A_1/\partial x_2 - \partial A_2/\partial x_1$ decays sufficiently fast at infinity (we may even suppose that it has compact support). This does not, however, mean that the magnetic potential $\mathcal{A}$ decays at infinity, so the operator is not a weak perturbation of the nonmagnetic operator.

To see that $\mathcal{A}$ cannot decay at infinity too fast, consider the circulation of the vector field $\mathcal{A}$ along a circle $\mathcal{C}_R$ with large radius R. According to the Green formula,

$$\int_{\mathcal{C}_R} \mathcal{A}(x)\,\mathrm{d}x = \int_{|x|\leqslant R} B(x)\,\mathrm{d}x, \tag{10.6}$$

so, if the total *flux* of the magnetic field,

$$\Phi = (2\pi)^{-1}\int_{\mathbb{R}^2} B(x)\,\mathrm{d}x, \tag{10.7}$$

is nonzero, $\mathcal{A}$ cannot decay at infinity faster than $C|x|^{-1}$, at least in some directions.

It was found by Aharonov and Casher that it is this flux which determines the number of zero modes [6].

THEOREM 10.1. *For a real number q, let $\{q\}$ denote the largest integer* strictly less *than q. Let the magnetic field B have compact support. Then, the dimension of the space of zero modes for the Pauli operator equals $\{|\Phi|\}$, if $|\Phi| > 0$ and 0 if $|\Phi| = 0$.*

PROOF. We introduce the *scalar potential*

$$\varphi(x) = (2\pi)^{-1} \int_{\mathbb{R}^2} \log\bigl(|x-y|\bigr) B(y)\,\mathrm{d}y, \tag{10.8}$$

which behaves like

$$\varphi(x) = \Phi \log|x| + \mathrm{O}\bigl(|x|^{-1}\bigr) \quad \text{as } |x| \to \infty. \tag{10.9}$$

Since $-(2\pi)^{-1}\log(|x|)$ is the Green function for the Laplacian, φ satisfies the equation $-\Delta\varphi = B$, and the vector field

$$\mathcal{A}(x) = (\varphi_y, -\varphi_x) \tag{10.10}$$

can serve as a magnetic potential for the field B. In addition, $\mathcal{A}(x)$ satisfies the Coulomb gauge $\operatorname{div}\mathcal{A} = 0$.

Now the system (10.2) determining the zero modes can be transformed to

$$(\partial_1 + \mathrm{i}\partial_2)\mathrm{e}^{-\varphi}\psi_+ = 0, \tag{10.11}$$

$$(\partial_1 - \mathrm{i}\partial_2)\mathrm{e}^{\varphi}\psi_+ = 0, \tag{10.12}$$

taking into account the expression of $\mathcal{A}$ via φ. The reasoning follows now the pattern of the case of a constant magnetic field. Equations (10.11)–(10.12), being simply the Cauchy–Riemann equations, mean that the function $f_+ = \mathrm{e}^{-\varphi}\psi_+$ is analytical in the variable $\bar{z} = x_1 - \mathrm{i}x_2$, and $f_- = \mathrm{e}^{\varphi}\psi_-$ is analytical in $z = x_1 + \mathrm{i}x_2$. Suppose that $\Phi \geqslant 0$. Then (10.9) implies that $\mathrm{e}^{-\varphi}$ is a bounded function in $\mathbb{R}^2$, and if ψ_+ belongs to L^2, this would mean that the analytical function f_+ belongs to L^2. However, there are no (nontrivial) analytical functions in L^2, and this means that (10.11) has no solutions. For $\psi_- = \mathrm{e}^{-\varphi} f_-$ to be in L^2, the function f_- may increase at infinity no faster than a polynomial, and being analytical, it must be a polynomial in z and, furthermore, it must be of degree $n < \Phi - 1$, as seen from the asymptotics (10.9). The space of such polynomials has dimension exactly $\{\Phi\}$, and this proves our theorem. Note, moreover, that all zero modes have zero components ψ_+, so they are "spin down", as physicists express this. If $\Phi < 0$, the same reasoning works, but now it is ψ_- which is forced to vanish, and all zero modes are "spin up". □

As we see, the two starting points in the study of zero modes in dimension $d = 2$ are the constant magnetic field, with infinitely many zero modes, and a compactly supported one, with finitely many. A considerable work was done in order to find more general situations, where zero modes can be studied.

10.3. *Zero modes for singular fields*

The conditions in the Aharonov–Casher theorem are far from being optimal. Even a superficial analysis of the proof shows that both smoothness and decay conditions can be considerably relaxed. Along this direction, we describe here the strongest result up to now, established by Erdös and Vugalter [41]. But first we must return to the problem of how to *define* a self-adjoint Pauli operator with a very singular magnetic field. This is achieved, again, by introducing the scalar potential and using the ideas presented in Section 3.2.

Let B be a signed locally-finite measure on $\mathbb{R}^2$. A direct definition of the operator by means of the formulas (2.6) or (2.7), using some magnetic potential, would immediately require some regularity conditions on B, encountering moreover nontrivial essential self-adjointness questions. Using quadratic forms, as it is explained in Section 3.2, proves to be more efficient.

First, supposing that B is nice and is defined by means of the magnetic potential $\mathcal{A}$, we write the quadratic form of the Pauli operator using the factorization (10.1) and the fact that $Q_-^* = Q_+$,

$$\mathfrak{p}_{\mathcal{A}}[\psi] \equiv \mathfrak{p}_{\mathcal{A}}\big[(\psi_+, \psi_-)^\top\big] = \int \big(|Q_+\psi_+|^2 + |Q_-\psi_-|^2\big)\,\mathrm{d}x. \tag{10.13}$$

This form is well defined for $\psi_\pm \in \mathfrak{D}(Q_\pm)$, provided $\mathcal{A} \in L^2_{\mathrm{loc}}$, which gives more freedom, but is still a too restrictive condition. So, suppose that there exists a scalar potential φ, as in (10.10). Then, using the complex derivatives, as we already did twice, we can rewrite the quadratic form (10.13) as

$$\mathfrak{p}_{\mathcal{A}}[\psi_+, \psi_-] = 4\int \big(\mathrm{e}^{2\varphi}\big|\partial_z\big(\mathrm{e}^{-\varphi}\psi_+\big)\big|^2 + \mathrm{e}^{-2\varphi}\big|\partial_{\bar z}\big(\mathrm{e}^{\varphi}\psi_-\big)\big|^2\big)\,\mathrm{d}x. \tag{10.14}$$

Note that the magnetic vector potential is not present in (10.14); the scalar potential φ can be defined directly, as a solution of the equation

$$-\Delta\varphi = B. \tag{10.15}$$

Now, suppose that B is a signed Borel measure with locally-finite total variation. We denote the set of such measures by $\overline{\mathfrak{M}}$. If, moreover, for any point $x \in \mathbb{R}^2$, $B(\{x\}) \in (-\pi, \pi]$, the measure B is said to belong to $\overline{\mathfrak{M}}^*$. Note that points with nonzero measure correspond to Aharonov–Bohm solenoids placed in these points.

The sets of measures in $\overline{\mathfrak{M}}$ and $\overline{\mathfrak{M}}^*$, resp., with finite global variation are denoted by $\mathfrak{M}$ and $\mathfrak{M}^*$, resp. Two measures B, B' are called equivalent if for any point x the difference $B(x) - B(x)^*$ is an integer multiple of 2π. For any measure $B \in \overline{\mathfrak{M}}$ there exists a unique $B' \in \overline{\mathfrak{M}}^*$ equivalent to B, the same holds also for $\mathfrak{M}$ and $\mathfrak{M}^*$. Now, supposing that for a given $B \in \overline{\mathfrak{M}}$ there exists a potential $\phi \in L^2_{\mathrm{loc}}$ satisfying (10.15) in the sense of distributions, one can define the Pauli operator $\mathcal{P}_B$ as the one corresponding to the quadratic form (10.14) with domain consisting of those pairs of functions $\psi_+, \psi_- \in L^2$ for which the expression (10.14) is finite. It is shown that, in fact, such a potential φ always exists and, moreover, it is a rather nice function: its derivatives locally belong to L^p for any $p < 2$, and furthermore, $\exp(\pm 2\varphi)$ belongs locally to $L^{1+\varepsilon}$ for some $\varepsilon > 0$. This regularity is used to prove that the form (10.14), thus defined, is closed and therefore determines a self-adjoint operator.

The Aharonov–Casher theorem can be extended to Pauli operators defined in this way [41].

THEOREM 10.2. *Suppose that the field B is a measure in $\mathfrak{M}^*$ with finite total variation. Define the total flux of B as $\langle B\rangle = (2\pi)^{-1}B(\mathbb{R}^2)$. Then the dimension of the space of zero modes associated with the Pauli operator $\mathcal{P}_B$ equals $\{\langle B\rangle\}$, if $\langle B\rangle$ is noninteger or $\langle B\rangle = 0$. If $\langle B\rangle$ is an integer, this dimension can be $\{\langle B\rangle\}$ or $\{\langle B\rangle\} - 1$.*

We note that for an integer flux, both cases may occur, and the difference depends on the behavior of B at infinity. Moreover, an example constructed in [41] shows that for a measure with an infinite local variation the statement may be wrong.

10.4. *Infinite number of zero modes*

Now we discuss more examples when there is an infinite number of zero modes, as this is the case for the constant magnetic field. The first idea here would be to consider some perturbations of the constant field. As the most simple example, we consider a weak perturbation of the magnetic potential, $\mathcal{A} = \mathcal{A}_B + \mathcal{A}'$, where $\mathcal{A}_B$ is the potential of the constant magnetic field, $\mathcal{A}_B = (Bx_2/2, -Bx_1/2)$ and $\mathcal{A}' \to 0$ at infinity. In this case the Pauli operator $\mathcal{P}_{\mathcal{A}}$ can be (using methods from Section 5) shown to be a form-compact perturbation of $\mathcal{P}_{\mathcal{A}_B}$. According to Weyl's theorem on the invariance of the essential spectrum under relatively compact perturbations (see, e.g., [155] or [114]), zero will still be a point in the essential spectrum of the perturbed operator. In fact, 0 is in the essential spectrum for Q_+Q_-; since 0 is not in the essential spectrum of Q_-Q_+, it is an isolated point of the spectrum, so it has to be an infinitely degenerate eigenvalue again. A more delicate reasoning is needed if it is the magnetic field B that is weakly perturbed but the corresponding perturbation of the magnetic potential $\mathcal{A}$ is not weak. For this case the infiniteness of zero modes was proved by Iwatsuka [59] using a perturbation of the commutator relation (10.5). The situation was finally cleared by Shigekawa [127] who showed, using again commutation relations, that there are infinitely many zero modes also in the case when B is continuous and tends to infinity at infinity; a sharp contrast to the magnetic Schrödinger operator, where in this case the essential spectrum is empty; see Section 6.4.

A recent development is related to the study of zero modes for infinite systems of Aharonov–Bohm solenoids. Let ω_1, ω_2 be two linearly independent vectors in $\mathbb{R}^2$ and in each point of the lattice, $z_{n_1,n_2} = n_1\omega_1 + n_2\omega_2$, an Aharonov–Bohm solenoid with a fixed flux α is placed. In the language of Section 10.3, the magnetic field here is a measure consisting of point loads with weight α placed in the points of the lattice. Since the total variation is infinite, this case is not covered in [41]. In [48] Geyler and Grishanov constructed explicitly the vector potential $\mathcal{A}$ for this singular field using methods of elliptic functions, and an infinite number of zero modes were shown to exist. This is the case even if such a lattice of Aharonov–Bohm solenoids is placed on the background of a constant magnetic field.

Just recently the problem of infinitely many zero modes was settled by Shirokov and Rozenblum (*http://front.math.ucdavis.edu/math-ph/0501059*). If the field B is *nonnegative* regular Borel measure as in [41], such that the total flux $\langle B\rangle$ of the field is infinite, then the space of zero modes is infinite-dimensional. This is still the case if the nonnegative condition is replaced by the requirement that for some R, $B(D_R) > 0$ for any disk D_R with radius R. If, moreover, $B(D_R) \geqslant a > 0$ then zero is an isolated point in the spectrum of the Pauli operator.

10.5. *Zero modes in dimension $d = 3$*

While the presence of zero modes for the Pauli operator in the plane seems to be rather common, for a long time no examples of zero modes were known in the three-dimensional case. Only in 1986, Loss and Yau [87] constructed a series of examples, where zero modes can occur. In particular, the operator with the magnetic field

$$\mathcal{B}(x) = \frac{12}{(1+|x|^2)^{-3}}\bigl(2x_1x_3 - 2x_2, 2x_2x_3 + 2x_1, 1 - x_1^2 - x_2^2 + x_3^2\bigr)$$

possesses at least one zero mode. It was established recently by Erdös and Solovej that the dimension of the space of zero modes can be arbitrarily large [38], and Elton found that for any open set $\Omega \subset \mathbb{R}^3$ and for any m, there exists a potential $\mathcal{A}$ with support in Ω possessing at least m zero modes [33]. For a decaying magnetic field one can estimate from above the number of zero modes. In fact, from the representation (2.6), it follows that this number can be majorized by the number of negative eigenvalues of the magnetic Schrödinger operator with electric potential $-|\mathcal{B}(x)|$. Applying the magnetic CLR estimate, we obtain that the number of zero modes is not greater than $c\int|\mathcal{B}(x)|^{3/2}\,\mathrm{d}x$. The study of this number and also of the *zero modes density* $n(x) = \sum|\psi_j(x)|^2$, where $\{\psi_j\}$ is the system of orthogonal zero modes, is an important problem related, again, to stability of different quantum systems and asymptotical behavior of spectral characteristics. Not much is known yet. It is found, however, that in dimension three zero modes are not a common feature. In particular, Balinsky and Evans showed that in $L^{3/2}(\mathbb{R}^3)$ magnetic fields without zero modes form a dense open set and, moreover, for any magnetic field $\mathcal{B} \in L^{3/2}$, the set of $t \in (0,\infty)$ such that the Pauli operator with magnetic field $t\mathcal{B}$ has zero modes, is locally finite [10]. In the language of potentials, Elton established that in the space of continuous potentials $\mathcal{A}(x)$ decaying as $\mathrm{o}(|x|^{-1})$, potentials possessing at least m zero modes form a smooth submanifold of co-dimension m^2, when $m = 1, 2$ and is contained in a smooth submanifold of co-dimension $2m - 1$ for bigger m [33]. With all this experimental evidence, the search is in the process for some quantization conditions which are expected to determine the presence of zero modes. This might have great importance for quantum theory. The reader is referred to [1,2] and references therein for further discussion on zero modes and their relation to physical problems.

11. Perturbed Pauli operators and Lieb–Thirring inequalities

There are serious reasons to consider the Pauli operator in atomic physics when one takes into account the interaction of the magnetic moments of the electrons with the magnetic field via the Zeeman term $\boldsymbol{\sigma} \cdot \mathcal{B}$. This interaction becomes most essential when the magnetic fields in question are strong, for example, in astrophysics, and also quantum dots (e.g., artificial atoms). However the changes in the spectral properties of the operator are felt even for weak fields.

As mentioned in Sections 8 and 9.5, the optimal form of CLR and Lieb–Thirring inequalities for Schrödinger operators is well established, even in the case of very singular magnetic fields, and the remaining challenge consists in obtaining the optimal constants in these inequalities.

When one replaces the Schrödinger operator by the Pauli operator, the situation changes completely. The main observation here is that generally, a perturbation by a negative electric potential, even a very small one, may produce an infinite number of negative eigenvalues (as it was first noticed in [9]), and it is the presence of zero modes which is the primary source for that. Hence, one cannot hope to find an analogy to the CLR estimate under general conditions. Thus in order to describe the distribution of these eigenvalues, one has to concentrate on the Lieb–Thirring-type inequalities, or look for some other characteristics of eigenvalues.

The first result about Lieb–Thirring inequalities for the Pauli operator appeared only in 1994 [81] and concerned the *constant magnetic field*. Even here the situation turned out to be far from being simple.

11.1. *Constant magnetic field: The unperturbed operator*

We have described the spectrum of the Pauli operator in dimension $d = 2$ with constant magnetic field in Section 10.1. In order to perturb this operator and its three-dimensional counterpart by an electric potential, we need a more detailed information about the corresponding spectral subspaces, and it is more convenient to start by discussing the spectral properties of the magnetic Laplacian $-\Delta_{\mathcal{A}}$ with constant magnetic field in $\mathbb{R}^2$ and $\mathbb{R}^3$ (see, e.g., [29,138,149]).

The potential $\mathcal{A}(x)$ has the form $(Bx_2/2, -Bx_1/2, 0)$, where the strength of the field B is supposed to be positive. In dimension $d = 2$ the spectrum of the operator $-\Delta_{\mathcal{A}}^{(2)} = -(\nabla + \mathrm{i}\mathcal{A})^2$ (the superscript refers to the dimension) consists of eigenvalues with *infinite multiplicity*,

$$\Lambda_\nu = (2\nu + 1)B, \quad \nu = 0, 1, \ldots, \tag{11.1}$$

which are called *Landau levels*. This description of the spectrum follows from the one for the Pauli operator; see Section 10.1 and the formula (2.6) relating the operators we are studying now. The spectral projections Ψ_ν onto the corresponding spectral subspaces $\mathcal{H}_\nu^{(2)}$ are integral operators with smooth kernels $\Psi_\nu(x', y')$ (we denote by $x' = (x_1, x_2)$ the two-dimensional variable in order to distinguish it from the three-dimensional one, x). All kernels can be expressed explicitly via the Laguerre functions (see, especially, [150] where many other useful formulas related to this operator are given as well as its relation to the Heisenberg group). We will need here only the explicit expression for $\Psi_0(x', y')$,

$$\Psi_0\big(x', y'\big) = \frac{B}{2\pi} \exp\left[\mathrm{i}(x_1 y_2 - x_2 y_1)\frac{B}{2} - \big(x' - y'\big)^2 \frac{B}{4} \right] \delta(x_3 - y_3). \qquad (11.2)$$

Consider now the magnetic Laplacian $-\Delta_{\mathcal{A}}^{(3)} = -\Delta_{\mathcal{A}}^{(2)} + p_3^2$ in three dimensions, where $p_3 = -\mathrm{i}\partial/\partial x_3$. Define the subspaces $\mathcal{H}_\nu^{(3)} = \mathcal{H}_\nu^{(2)} \otimes L^2(\mathbb{R}^1)$ (for readers not used to the tensor product, this simply means that these subspaces consist of L^2-functions of three variables (x', x_3), that belong to $\mathcal{H}_\nu^{(2)}$ for almost all values of x_3). Separation of variables shows that each subspace $\mathcal{H}_\nu^{(3)}$ is invariant with respect to $-\Delta_{\mathcal{A}}^{(3)}$ and the spectrum of $-\Delta_{\mathcal{A}}^{(3)}$ restricted to $\mathcal{H}_\nu^{(3)}$ is the semiaxis $\Lambda_\nu + \mathbb{R}_+$; the latter are called *Landau bands* and the corresponding subspaces are *band subspaces*. The spectrum of the operator $-\Delta_{\mathcal{A}}^{(3)}$ thus coincides with the union of Landau bands, i.e., the semiaxis $[B, \infty)$. The projections Π_ν onto subspaces $\mathcal{H}_\nu^{(3)}$ have the form $\Pi_\nu = \Psi_\nu \otimes I$ (which means that they act as Ψ_ν in x' and as identity operator in x_3). One can formally (but rather conveniently) write Π_0 as an integral operator with kernel

$$\Pi_0(x, y) = \frac{B}{2\pi} \exp\left[\mathrm{i}(x_1 y_2 - x_2 y_1)\frac{B}{2} - \big(x' - y'\big)^2 \frac{B}{4} \right] \delta(x_3 - y_3). \qquad (11.3)$$

According to (2.6), the three-dimensional Pauli operator $\mathcal{P}_{\mathcal{A}}$ with a constant magnetic field B is the matrix operator acting on two-component vector functions and being the direct sum of operators $P_+ = -\Delta_{\mathcal{A}} + B$ and $P_- = -\Delta_{\mathcal{A}} - B$. The spectral structure of P_+, resp. P_-, is obtained by shifting the spectrum of $-\Delta_{\mathcal{A}}$ by B, resp. by $-B$. So, in particular, the spectrum of P_+ equals the semiaxis $[2B, \infty)$, and the one of P_- equals $\mathbb{R}_+ = [0, \infty)$. Each Landau band shifts accordingly but the band subspaces do not change. On the band subspace associated with the zero band, the Pauli operator $\mathcal{P}_{\mathcal{A}}$ acts simply as p_3^2, which can be written as $\Pi_0 \mathcal{P}_{\mathcal{A}} \Pi_0 = \Pi_0 p_3^2 \Pi_0$. The same simplification can be made for the resolvents

$$\Pi_0 (\mathcal{P}_{\mathcal{A}} + \mu)^{-1} \Pi_0 = \Pi_0 \big(p_3^2 + \mu\big)^{-1} \Pi_0, \quad \mu > 0. \qquad (11.4)$$

It is the zero band (reaching zero), which is the cause of many interesting effects for the Pauli operator.

11.2. *Lieb–Thirring inequality for a constant magnetic field*

Now we perturb the three-dimensional Pauli operator by an electric potential V, viz.

$$\mathcal{P}_{\mathcal{A},V} = \mathcal{P}_{\mathcal{A}} - V(x). \qquad (11.5)$$

We suppose that $V_+ \in L^{3/2}(\mathbb{R}^3) \cap L^{5/2}(\mathbb{R}^3)$.

As we already mentioned, simple considerations based on the Glazman lemma (see Section A.9 and [9] for details) show that $\mathcal{P}_{\mathcal{A},V}$ may have infinitely many eigenvalues even when V is a nice function with compact support (and it surely has if V is positive). However, the following Lieb–Thirring inequality obtained by Lieb, Solovej and Yngvason shows that these infinitely many eigenvalues are summable [81].

THEOREM 11.1. *Let $V_+ \in L^{5/2}(\mathbb{R}^3) \cap L^{3/2}(\mathbb{R}^3)$ and let $E_j(B,V) \leqslant 0$, $j = 1, 2, 3, \ldots,$ denote the negative eigenvalues of the operator $\mathcal{P}_{\mathcal{A}} - V(x)$ in $L^2(\mathbb{R}^3, \mathbb{C}^2)$. Then*

$$\sum_j \left|E_j(B,V)\right| \leqslant C_1 B \int_{\mathbb{R}^3} V_+(x)^{3/2}\,\mathrm{d}x + C_2 \int_{\mathbb{R}^3} V_+(x)^{5/2}\,\mathrm{d}x \tag{11.6}$$

with $C_1 = 4/3\pi$ and $C_2 = 8\sqrt{6}/5\pi$.

More generally, for each $0 < \delta < 1$, one can choose $C_1 = (2/3\pi)(1-\delta)^{-1}$ and $C_2 = (2\sqrt{6}/5\pi)\delta^{-2}$.

Inequality (11.6) does not match the classical phase-space expression which is given by

$$\sum_j \left|E_j(B,V)\right| \simeq \frac{B}{3\pi^2} \int_{\mathbb{R}^3} \left\{ V_+(x)^{3/2} + 2\sum_{\nu=1}^{\infty} \bigl(V(x) - 2\nu B\bigr)_+^{5/2} \right\} \mathrm{d}x. \tag{11.7}$$

However, the two terms in (11.6) correspond, resp., to the $B \to \infty$ (first term) and the $B \to 0$ (second term) asymptotics of (11.7), that was rigorously established later (see [101,109,144]).

The proof we present here is a somewhat simplified version of the one in [82]; we, however, do not struggle for the best constants C_1, C_2 in (11.6).

In order not to interrupt the proof, we make some preparations. First, consider the operator p_3^2 in $L^2(\mathbb{R}^1)$. The resolvent $(p_3^2 + \mu)^{-1}$ of this operator can be found easily by means of the Fourier transform. It is an integral operator with kernel

$$\bigl(p_3^2 + \mu\bigr)^{-1}(x_3, y_3) = (2\pi)^{-1} \int \exp\bigl(\mathrm{i}\xi_3(x_3 - y_3)\bigr)\bigl(\xi_3^2 + \mu\bigr)^{-1}\,\mathrm{d}\xi_3.$$

We are interested in the value of this kernel on the diagonal $x_3 = y_3$. A direct calculation gives

$$\bigl(p_3^2 + \mu\bigr)^{-1}(x_3, x_3) = (2\pi)^{-1} \int_{\mathbb{R}} \bigl(\xi_3^2 + \mu\bigr)^{-1}\,\mathrm{d}\xi_3 = (4\mu)^{-1/2}. \tag{11.8}$$

In the three-dimensional case we will need the expression for the integral of the square of the resolvent kernel. Since

$$(-\Delta + \mu)^{-1}(x,y) = (2\pi)^{-3} \int \exp\bigl(\mathrm{i}\xi(x-y)\bigr)\bigl(\xi^2 + \mu\bigr)^{-1}\,\mathrm{d}\xi,$$

Plancherel's equality yields

$$\int \bigl[(-\Delta+\mu)^{-1}(y,y)\bigr]^2\,\mathrm{d}y = (2\pi)^{-3}\int \bigl(\xi^2+\mu\bigr)^{-2}\,\mathrm{d}\xi$$
$$= (12\pi)^{-1}\mu^{-1/2}. \tag{11.9}$$

Next we give a simple inequality for functions. Let $\chi_l(\lambda)$ be the characteristic function of the semiaxis $\lambda \geqslant l$ and let μ be some positive number. Then

$$(\lambda+\mu)^{-1}\chi_l(\lambda) \leqslant 2(\lambda+l+\mu)^{-1}, \quad \lambda \geqslant 0. \tag{11.10}$$

The inequality follows immediately since $\lambda \geqslant (\lambda+l)/2$ for $\lambda \geqslant l$.

Finally, we establish a useful inequality for the distribution function. (For trace class operators, see Section A.8.)

LEMMA 11.2. *If S and T are positive semidefinite trace class operators, then the number N of eigenvalues greater than or equal to* 1 *of $S+T$ satisfies*

$$N \leqslant 2\,\mathrm{tr}\,S + 4\,\mathrm{tr}\,T^2. \tag{11.11}$$

PROOF. Let $\{\phi_j\}_{j=1}^N$ be the orthonormal eigenfunctions for $S+T$ with eigenvalues greater than or equal to 1. For each j we have either

$$\bigl\langle \phi_j, |S|\phi_j\bigr\rangle \geqslant \frac{1}{2} \quad \text{or} \quad \bigl\langle \phi_j, |T|\phi_j\bigr\rangle \geqslant \frac{1}{2}. \tag{11.12}$$

Since $\langle \phi_j, |T|\phi_j\rangle^2 \leqslant \langle \phi_j,\phi_j\rangle\langle \phi_j, |T^2|\phi_j\rangle = \langle \phi_j, |T^2|\phi_j\rangle$, we infer that $2\,\mathrm{tr}\,S + 4\,\mathrm{tr}\,T^2 \geqslant \sum_{j=1}^N 2\langle \phi_j, |T|\phi_j\rangle + 4\langle \phi_j, |T^2|\phi_j\rangle \geqslant N$. □

Now we can proceed with the proof of Theorem 11.1.

PROOF OF THEOREM 11.1. First of all, we note that the operator $\mathcal{P}_{\mathcal{A},V}$ is a direct sum of $P_+ - V$ and $P_- - V$. Since $P_+ - V \geqslant P_- - V$, it suffices to prove our estimate for $P_- - V$ only.

For $E > 0$ we define the Birman–Schwinger operator

$$K_E = V_E^{1/2}\left(P_- + \frac{E}{2}\right)^{-1} V_E^{1/2}, \tag{11.13}$$

where $V_E = (V - (E/2))_+$. We divide K_E into a part coming from the lowest Landau band and a part coming from the higher bands. For this purpose we define

$$K_E^0 = V_E^{1/2}\varPi_0\left(P_- + \frac{E}{2}\right)^{-1}\varPi_0 V_E^{1/2}$$
$$= V_E^{1/2}\varPi_0\left(p_3^2 + \frac{E}{2}\right)^{-1}\varPi_0 V_E^{1/2}, \tag{11.14}$$

where, in the last equality, we used (11.4). Moreover, we define

$$K_E^{>} = V_E^{1/2}(I - \Pi_0)\left(P_- + \frac{E}{2}\right)^{-1}(1 - \Pi_0)V_E^{1/2}. \tag{11.15}$$

Clearly, $K_E = K_E^0 + K_E^{>}$ since P_- commutes with Π_0.

To calculate the trace of K_E^0, we first cyclically rearrange factors in K_E^0 (this does not change trace; cf. Section A.8) and then use the explicit formulas (11.3) and (11.8) to integrate the kernel over the diagonal $x = y$

$$\begin{aligned}
\operatorname{tr}\bigl(K_E^0\bigr) &= \operatorname{tr}\left[V_E \Pi_0\left(p_3^2 + \frac{E}{2}\right)^{-1}\right] \\
&= \iint V_E(x)\bigl(\Psi_0\bigl(x', x'\bigr)\bigr)\left(p_3^2 + \frac{E}{2}\right)^{-1}(x_3, x_3)\,\mathrm{d}x'\,\mathrm{d}x_3 \\
&= \frac{B}{2\pi}(2E)^{-1/2}\int V_E(x)\,\mathrm{d}x.
\end{aligned} \tag{11.16}$$

The second part of the Birman–Schwinger operator will be evaluated in the following way. Denote by Q the spectral projection of P_- corresponding to $[B, \infty)$. We have

$$(1 - \Pi_0)Q = (1 - \Pi_0). \tag{11.17}$$

According to the spectral theorem (see Section A.5), the operator $Q(P_- + \frac{E}{2})^{-1}Q$ equals $g(P_-)$, where $g(\lambda) = (\lambda + \frac{E}{2})^{-1}\chi_B(\lambda)$. Using (11.10) and the functional calculus (Theorem A.2 in Section A.5), we obtain the operator inequality

$$Q\left(P_- + \frac{E}{2}\right)^{-1}Q \leqslant 2\left(P_- + B + \frac{E}{2}\right)^{-1} = 2\left(-\Delta_{\mathcal{A}} + \frac{E}{2}\right)^{-1}. \tag{11.18}$$

According to (11.17), this implies that

$$(I - \Pi_0)\left[P_- + \frac{E}{2}\right]^{-1}(I - \Pi_0) \leqslant 2\left(-\Delta_{\mathcal{A}} + \frac{E}{2}\right)^{-1}. \tag{11.19}$$

This operator inequality keeps being true after we multiply it from both sides by V_E,

$$V_E(I - \Pi_0)\left[P_- + \frac{E}{2}\right]^{-1}(I - \Pi_0)V_E \leqslant 2V_E\left(-\Delta_{\mathcal{A}} + \frac{E}{2}\right)^{-1}V_E. \tag{11.20}$$

The above inequalities are operator ones, understood in the Hilbert space sense. Now we pass to integral kernels and use the diamagnetic inequality for resolvents,

$$\bigl|(-\Delta_{\mathcal{A}} + \mu)^{-1}(x, y)\bigr| \leqslant (-\Delta + \mu)^{-1}(x - y).$$

Thus we get the following estimate for the trace of $(K_E^{>})^2$

$$\operatorname{tr}\big[\big(K_E^{>}\big)^2\big] \leqslant 4\operatorname{tr}\bigg[V_E\bigg(-\Delta_{\mathcal{A}}+\frac{E}{2}\bigg)^{-1}V_E\bigg]^2$$
$$\leqslant 4\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} V_E(x)\bigg\{\bigg[-\Delta+\frac{E}{2}\bigg]^{-1}(x-y)\bigg\}^2 V_E(y)\,\mathrm{d}x\,\mathrm{d}y.$$

We apply Schwarz' inequality to the latter integral, which gives us the estimate

$$\operatorname{tr}\big[\big(K_E^{>}\big)^2\big] \leqslant 4\int_{\mathbb{R}^3} V_E(x)^2\,\mathrm{d}x\int_{\mathbb{R}^3}\bigg\{\bigg[-\Delta+\frac{E}{2}\bigg]^{-1}(y)\bigg\}^2\,\mathrm{d}y.$$

The value of the latter integral we know, see (11.9), so we obtain

$$\operatorname{tr}\big[\big(K_E^{>}\big)^2\big] \leqslant C\int_{\mathbb{R}^3} V_E(x)^2\,\mathrm{d}x.$$

To complete the proof of Theorem 11.1, we use that, as a consequence of the Birman–Schwinger principle,

$$\sum_j \big|E_j(B,V)\big| \leqslant \int_0^\infty N_E\,\mathrm{d}E \leqslant \int_0^\infty \big\{2\operatorname{tr}\big(K_E^0\big)+4\operatorname{tr}\big[\big(K_E^{>}\big)^2\big]\big\}\,\mathrm{d}E. \tag{11.21}$$

We insert here the estimates for $\operatorname{tr}K_E^0$ and $\operatorname{tr}[(K_E^{>})^2]$ we have just derived, and after performing integration we arrive at (11.6). □

A similar result, with a somewhat simpler proof, is valid in the two-dimensional case [83].

THEOREM 11.3. *Let $V_+ \in L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$. Then the negative eigenvalues E_j of the Pauli operator $\mathcal{P}_{\mathcal{A}} - V$ with a constant magnetic field B satisfy the inequality*

$$\sum |E_j| \leqslant \frac{B}{\pi}\int_{\mathbb{R}^2} V_+(x)\,\mathrm{d}x + 3\int_{\mathbb{R}^2} V_+(x)^2\,\mathrm{d}x. \tag{11.22}$$

In [81–83] spectacular applications of these inequalities to the problem of stability of matter and analysis of ground states of large quantum systems are given.

11.3. *Distribution of eigenvalues*

Another way to describe the spectrum of the Pauli operator with a constant magnetic field is to study the distribution of eigenvalues near the Landau levels in dimension 2. If the electric potential V decays at infinity then it is a relatively compact perturbation of $\mathcal{P}_B$ and, according to Weyl's theorem, the essential spectrum of the perturbed operator $\mathcal{P}_B + V$ is the same as of $\mathcal{P}_B$. This means that some, finite or infinite, eigenvalue clusters may form around each Landau level; if the cluster is finite, the Landau level remains intact, if the cluster is infinite, the eigenvalues there must converge to the Landau level, the latter may disappear. To find out what really happens, we introduce the distribution functions. Let Λ be a Landau level, we fix some $s < \Lambda$, so that the interval (s, Λ) does not contain other Landau levels, and for $\lambda \in (s, \Lambda)$ denote by $n(s, \lambda; V)$ the number of eigenvalues of the operator $\mathcal{P} + V$ in (s, λ). The problem consists in studying the behavior of this quantity as $\lambda \to \Lambda$: if it is bounded then the number of eigenvalues which split *down* from Λ is finite; if $n(s, \lambda; V)$ is unbounded, one has to find estimates and, if possible, asymptotics as $\lambda \to \Lambda - 0$. In a similar way, fixing some s above Λ one considers the behavior of $n(\lambda, s)$ as $\lambda \to \Lambda + 0$, i.e., distribution of eigenvalues splitting *up* from Λ. In the three-dimensional case, similar problems can be stated only for the distribution of eigenvalues lying below the lowest Landau level, since the whole positive semiaxis is covered by the essential spectrum. These problems were studied in [108,139–141,146,147]. It was found there that in three dimensions, and in the case of not too fast decaying V, in two dimensions, there is an infinite set of eigenvalues splitting from each Landau level, up or down, if the electric potential has, respectively, nonzero positive or negative part, and additionally asymptotic formulas were proved, admitting explanation in terms of the phase-space volume. Only recently, Raikov and Warzel [111] and Melgaard and Rozenblum [100], considered the case of very fast decaying potentials V (including compactly supported ones), and a number of pathologies were discovered. In particular, if V has constant sign, an infinite number of eigenvalues split from each Landau level, however they converge there so fast that the asymptotic behavior of $n(s, \lambda; V)$ or $n(\lambda, s)$ does not fit into the phase volume pattern and even does not depend on V. For the case of variable magnetic field, when zero is an isolated eigenvalue of the unperturbed Pauli operator with infinite multiplicity, similar results on the distribution of eigenvalues of $\mathcal{P} \pm V$ near zero were obtained by Raykov [110].

11.4. *Arbitrary magnetic field*

Erdös initiated the study of Lieb–Thirring inequalities for Pauli operators with nonhomogeneous fields [34]. He observed that the direct extension of (11.6) to the case of the nonconstant fields, namely

$$\sum_j \left|E_j(\mathcal{B}, V)\right| \leqslant C_{1,\gamma} \int_{\mathbb{R}^3} \left|\mathcal{B}(x)\right| V_+(x)^{\gamma+1/2}\, \mathrm{d}x + C_{2,\gamma} \int_{\mathbb{R}^3} V_+(x)^{\gamma+3/2}\, \mathrm{d}x, \tag{11.23}$$

as well as its possible consequence (via Hölder's inequality)

$$\sum_j \left|E_j(\mathcal{B}, V)\right| \leqslant \widetilde{C}_{1,\gamma} \int_{\mathbb{R}^3} \left|\mathcal{B}(x)\right|^{3/2} V_+(x)^{\gamma}\, \mathrm{d}x + \widetilde{C}_{2,\gamma} \int_{\mathbb{R}^3} V_+(x)^{\gamma+3/2}\, \mathrm{d}x, \tag{11.24}$$

is false without substantial regularity conditions on $\mathcal{B}$. In addition, he conjectured in [34] that it would be necessary to replace $|\mathcal{B}|$ by some effective field strength, $B_{\mathrm{eff}}(x)$, obtained by averaging $|\mathcal{B}|$ locally on the magnetic lengthscale, $|\mathcal{B}|^{-1/2}$.

Finding proper Lieb–Thirring-type inequalities for the Pauli operator is crucial for the problem of stability of relativistic matter: in proving that the relativistic Hamiltonian for a large Coulomb system of particles in a magnetic field is semibounded from below as well as to find reasonable dependence of the lower bound on the number of particles and the energy of the field. One more source of inspiration for construction Lieb–Thirring inequalities lies in establishing asymptotic formulas, with respect to all parameters involved, for regular and some singular potentials [36,144]. Since the "natural" inequality is wrong and there is no intrinsic candidate for the "effective" field, the search in recent years went in the direction of finding estimates satisfying some natural scaling conditions, and then checking how good they serve the main problem. The scaling properties involve behavior for large and small fields as well as with respect to Planck's constant $\hbar$. We describe some key results here.

Motivated by Erdös' observation, Sobolev [142,143] obtained estimates in the forms of (11.23) and (11.24), but with $|\mathcal{B}|$ replaced by a effective (scalar) magnetic field $B_{\mathrm{eff}}(x)$,

$$\sum_j \bigl|E_j(\mathcal{B}, V)\bigr| \leqslant C_1 \int_{\mathbb{R}^3} B_{\mathrm{eff}}(x)^{3/2} V_+(x)\,\mathrm{d}x + C_2 \int_{\mathbb{R}^3} V_+(x)^{5/2}\,\mathrm{d}x. \tag{11.25}$$

Roughly speaking, $B_{\mathrm{eff}}(x)$ is a slow varying function which dominates $|\mathcal{B}(x)|$ pointwise. Subsequently, Bugliaro et al. [21] and Shen [125,126] established (11.25) with a $B_{\mathrm{eff}}(x)$ whose energy is comparable to that of $|\mathcal{B}|$, viz. $\|B_{\mathrm{eff}}\|_{L^2} \approx \|\mathcal{B}\|_{L^2}$ (Shen showed that the same holds for any L^p norm). The latter in particular implies the following estimate by Lieb, Loss and Solovej [80].

THEOREM 11.4. *Let $V_+ \in L^{5/2}(\mathbb{R}^3) \cap L^4(\mathbb{R}^3)$ and let $E_j(B, V) \leqslant 0$, $j = 1, 2, 3, \ldots$, denote the negative eigenvalues of the operator $\mathcal{P}_{\mathcal{A}} - V(x)$ in $L^2(\mathbb{R}^3, \mathbb{C}^2)$. Then*

$$\sum_j \bigl|E_j(\mathcal{B}, V)\bigr| \leqslant a_\gamma \int_{\mathbb{R}^3} V_+(x)^{5/2}\,\mathrm{d}x + b_\gamma \left(\int_{\mathbb{R}^3} \mathcal{B}(x)^2\,\mathrm{d}x\right)^{3/4} \left(\int_{\mathbb{R}^3} V_+(x)^4\,\mathrm{d}x\right)^{1/4} \tag{11.26}$$

for all $0 < 0 < \gamma$, where $a_\gamma = (2^{3/2}/5)(1-\gamma)^{-1} L_3$ and $b_\gamma = 3^{1/4} 2^{-9/4} \pi \gamma^{-3/8} (1-\gamma)^{-5/8} L_3$. One can take L_3, defined above, to be 0.1156.

The estimate (11.26) and the estimate by Bugliaro et al. [21] (and Shen [125,126]) have been very useful in the proof of stability of matter within the Pauli setting; in this case the magnetic energy, $\int |\mathcal{B}|^2$, is also part of the total energy to be minimized and thus even the second moment of the magnetic field is controlled [80].

Another reason why (11.23) cannot hold for an arbitrary magnetic field is due to the existence of the zero modes [87], see Section 10. Indeed, for certain magnetic fields with nonconstant direction the Dirac operator $\mathbf{D} := \boldsymbol{\sigma} \cdot (-\mathrm{i}\nabla + \mathcal{A})$ has a nontrivial L^2-kernel. If $n(x)$ denotes the density of zero modes, $n(x) = \sum_j |\psi_j(x)|^2$, where $\{\psi_j\}$ is an orthonormal basis in Ker $\mathbf{D}$, then under a small potential perturbation of $\mathbf{D}^2$ the sum $\sum_j |E_j(\mathcal{P}_{\mathcal{A},V})|$ behaves as $\int n(x)V_+(x)\,\mathrm{d}x$, i.e., it is linear in V_+. Hence an extra term linear in V_+ must be added to (11.23) and $n(x)$ has to be estimated; being an important characteristic by itself. In the works [21,34,80,125,126,142,143] the density $n(x)$ is estimated by a function that behaves quantitatively as $|\mathcal{B}(x)|^{3/2}$. In particular, in the strong field regime these estimates are not sufficient to prove semiclassical asymptotics for $\mathcal{P}_{\mathcal{A},V}(h,b)$ uniformly in b; they give the asymptotics only up to $b \leqslant \text{const} \cdot h^{-1}$ [144].

The key task is thus to understand the density of Loss–Yau zero modes. Since $n(x)$ scales like $(\textit{length})^{-3}$ and $|\mathcal{B}(x)|$ scales like $(\textit{length})^{-2}$ under the dilation of the space, a simple dimension counting shows that $n(x)$, therefore $\sum_j |E_j|$, cannot be estimated in general by $|\mathcal{B}(x)|$ or by its smoothed version. However, if an extra lengthscale is introduced, for example, certain derivatives of the field are allowed in the estimate, then it is possible to give a bound on the eigenvalue sum that grows slower than $|\mathcal{B}|^{3/2}$ in the large field regime.

The semiclassical formula for the sum of the negative eigenvalues, $\sum_j |E_j(\mathcal{P}_{\mathcal{A},V})|$, for a constant magnetic field behaves linearly in the field strength, $|\mathcal{B}|$, see (11.7). This fact suggests that $\sum_j |E_j(\mathcal{P}_{\mathcal{A},V})|$ may be bounded by an expression that grows with the first power of $|\mathcal{B}|$ even for nonconstant magnetic fields and away from the semiclassical asymptotic regime.

Two physical applications, especially, motivate the search for Lieb–Thirring inequalities which are optimal in the strong $\mathcal{B}$-field regime with respect to the field strength $|\mathcal{B}|$, namely strong magnetic fields in astrophysics, and quantum dots (in particular, artificial atoms).

Recently, Erdös and Solovej [39,40] obtained two qualitatively different Lieb–Thirring inequalities which fulfill these conditions with as weak assumptions on $\mathcal{B}$ as possible and no technical assumptions on V. The methods behind the proofs of these two estimates are very different and they are somewhat complementary.

While both Lieb–Thirring inequalities are sharp as far as the potential and the strength of the magnetic field are concerned, they require additional technical assumptions on the magnetic field. These are usually formulated in terms of *variation lengthscales*, and practically they are regularity assumptions on $\mathcal{B}$. This means that supremum norms of derivatives of the magnetic field appear in the final Lieb–Thirring inequality.

The difference between the two inequalities is that the more involved bound, see Theorem 11.7, contains only local supremum norms. Therefore it enjoys an important *locality property*: the estimate is insensitive to the behavior of the magnetic field far away from the support of V_+. The simpler inequality, Theorem 11.5, involves the global C^5-norm of the direction of the magnetic field, $\mathbf{n} := \mathcal{B}/|\mathcal{B}|$. In particular, irregular behavior of $\mathbf{n}$ far away from the support of V_+ renders our estimate large despite that it should not substantially influence the negative spectrum. As a compensation, less assumptions on the regularity of the field strength $|\mathcal{B}|$ are required and the proof is much shorter.

The simpler inequality. Consider the three-dimensional Pauli operator, $\mathcal{P}_{\mathcal{A},V} = \mathbf{D}^2 - V$, with a differentiable magnetic field $\mathcal{B} = \nabla \times \mathcal{A}$, $\mathbf{D} := \boldsymbol{\sigma} \cdot (-\mathrm{i}\nabla + \mathcal{A})$ being the Dirac operator. We make two global assumptions:

ASSUMPTION 1. $\mathcal{B}(x) \neq 0$ for all $x \in \mathbb{R}^3$, i.e., the unit vector field $\mathbf{n} := \mathcal{B}/|\mathcal{B}|$ is well defined.

ASSUMPTION 2. The vector field $\mathbf{n}$ satisfies the following global regularity condition

$$L_{\mathbf{n}}^{-1} := \sum_{\gamma=1}^{5} \left\| \nabla^\gamma \mathbf{n} \right\|_\infty^{1/\gamma} < \infty, \tag{11.27}$$

where $L_{\mathbf{n}}$ is called the *global variation lengthscale* of $\mathbf{n}$.

For any $L > 0$, $x \in \mathbb{R}^3$, we also define

$$B_L^*(x) := \sup\left\{ \left|\mathcal{B}(y)\right| : |y - x| \leqslant L \right\} + L \cdot \sup\left\{ \left|\nabla\mathcal{B}(y)\right| : |y - x| \leqslant L \right\}.$$

The simpler inequality takes the following form [40].

THEOREM 11.5 (Lieb–Thirring inequality without a locality property). *For any $0 < L \leqslant L_{\mathbf{n}}$, the sum of the absolute values of the negative eigenvalues, $E_1 \leqslant E_2 \leqslant \cdots \leqslant 0$, for $\mathcal{P}_{\mathcal{A},V}$ satisfies*

$$\sum_j \left| E_j(\mathcal{P}_{\mathcal{A},V}) \right| \leqslant c\left(L^{-1} \int_{\mathbb{R}^3} \left(B_L^*(x) + L^{-2} \right) V_+(x)\,\mathrm{d}x + \int_{\mathbb{R}^3} B_L^*(x) V_+(x)^{3/2}\,\mathrm{d}x + \int_{\mathbb{R}^3} V_+(x)^{5/2}\,\mathrm{d}x \right) \tag{11.28}$$

with a universal constant c.

The involved inequality. Next we give the basic set-up necessary to present the more involved Lieb–Thirring inequality.

We assume that $\mathcal{B} \in C^4(\mathbb{R}^3, \mathbb{R}^3)$, and define three basic lengthscales of $\mathcal{B}$. The Pauli operator will be localized on these lengthscales. Let $\mathbf{n} := \mathcal{B}/|\mathcal{B}|$ be the unit vector field in the direction of the magnetic field at all points where $\mathcal{B}$ does not vanish. For any $L > 0$ and $x \in \mathbb{R}^3$, we define

$$B_L(x) := \sup\left\{ \left|\mathcal{B}(y)\right| : |x - y| \leqslant L \right\} \tag{11.29}$$

and

$$b_L(x) := \inf\left\{ \left|\mathcal{B}(y)\right| : |x - y| \leqslant L \right\} \tag{11.30}$$

to be the supremum and the infimum of the magnetic field strength on the ball of radius L about x.

DEFINITION 11.6 (Lengthscales of a magnetic field). The *magnetic lengthscale* of $\mathcal{B}$ is defined as

$$L_m(x) := \sup\{L > 0\colon\ B_L(x) \leqslant L^{-2}\}.$$

The *variation lengthscale* of $\mathcal{B}$ at x is given by

$$L_v(x) := \min\{L_s(x), L_n(x)\},$$

where

$$\begin{aligned} L_s(x) &:= \sup\{L > 0\colon \\ &\qquad L^\gamma \sup\{|\nabla^\gamma|\mathcal{B}(y)||\colon\ |x-y| \leqslant L\} \leqslant b_L(x), \gamma = 1,2,3,4\}, \\ L_n(x) &:= \sup\{L > 0\colon \\ &\qquad L^\gamma \sup\{|\nabla^\gamma \mathbf{n}(y)|\colon\ |x-y| \leqslant L, \mathcal{B}(y) \neq 0\} \leqslant 1, \gamma = 1,2,3,4\} \end{aligned}$$

(with the convention that $\sup\emptyset = -\infty$). Finally we set

$$L_c(x) := \max\{L_m(x), L_v(x)\}. \tag{11.31}$$

A magnetic field $\mathcal{B}\colon \mathbb{R}^3 \to \mathbb{R}^3$ determines three local lengthscales. The magnetic lengthscale, L_m, is comparable with $|\mathcal{B}|^{-1/2}$. The lengthscale L_s determines the scale on which the strength of the field varies, i.e., it is the variation scale of $\log|\mathcal{B}|$. The field line structure, determined by $\mathbf{n}$, varies on the scale of L_n. The variation lengthscale L_v is the smaller of these last two scales, i.e., it is the scale of variation of the vector field $\mathcal{B}$.

For weak magnetic fields the magnetic effects can be neglected in the final eigenvalue estimate, so the variational lengthscale becomes irrelevant. This idea is reflected in the definition of L_c; there is no need to localize on scales shorter than the magnetic scale L_m.

With these preparations we are ready to formulate the main result in [39].

THEOREM 11.7 (Lieb–Thirring inequality with a locality property). *Assume that the magnetic field $\mathcal{B} = \nabla \times \mathcal{A}$ is in $C^4(\mathbb{R}^3, \mathbb{R}^3)$. Then the sum of the negative eigenvalues of $\mathcal{P}_{\mathcal{A},V} = [\boldsymbol{\sigma}\cdot(\mathbf{P}+\mathcal{A})]^2 - V$ satisfies*

$$\begin{aligned} \sum_j |E_j| &\leqslant c\int_{\mathbb{R}^3} V_+(x)^{5/2}\,\mathrm{d}x + c\int_{\mathbb{R}^3} |\mathcal{B}(x)| V_+(x)^{3/2}\,\mathrm{d}x \\ &\quad + c\int_{\mathbb{R}^3} \bigl(|\mathcal{B}(x)| + L_c(x)^{-2}\bigr) L_c(x)^{-1} V_+(x)\,\mathrm{d}x. \end{aligned} \tag{11.32}$$

For the proof we refer to the paper [39]. More estimates having different forms were obtained in recent years [22,37]. In general, however, there is still no conjecture for the form of Lieb–Thirring inequalities for arbitrary magnetic fields, optimal in all aspects.

Lately much effort has been put into the search for bounds on $\sum_j |E_j(\mathcal{P}_{\mathcal{A},V})|$ which grows like $|\mathcal{B}|$ even for nonconstant fields and away from the semiclassical asymptotical regime. Such bounds would allow one to prove the semiclassic asymptotics for $\sum_j |E_j(\mathcal{P}_{\mathcal{A},V})|$ uniformly in $\mathcal{B}$ as $\hbar \downarrow 0$.

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Appendix. Basic facts from abstract spectral theory

A.1. *Linear operators. Closed operators*

Let $\mathcal{H}$ be a complex Hilbert space, let $D \subset \mathcal{H}$ be a linear subset, usually dense, and let $T: D \to \mathcal{H}$ be a linear (not necessarily continuous) map. For brevity, T is said to be a *linear operator* in $\mathcal{H}$. The set D is denoted $\mathfrak{D}(T)$ and called the *domain* of the operator. If D_0 is a linear subset of D then $T_0 = T|_{D_0}$ is said to be a *restriction* of T. The operator T is then called an *extension* of T. We shall write $T_0 \subset T$.

On $\mathfrak{D}(T)$ one can define the *graph norm* or T *norm* $\|\cdot\|_T$ by

$$\|u\|_T^2 = \|Tu\|^2 + \|u\|^2, \quad u \in \mathfrak{D}(T). \tag{A.1}$$

T is said to be a *closed operator* if $\mathfrak{D}(T)$ is complete in the T norm. An equivalent definition is this: T is closed of its *graph* $\mathcal{G}(T) = \{\{u, v\} \in \mathcal{H} \oplus \mathcal{H}\colon u \in \mathfrak{D}(T), v = Tu\}$ is closed in $\mathcal{H} \oplus \mathcal{H}$. We say that T is a *closable operator* if the closure of the graph of T in $\mathcal{H} \oplus \mathcal{H}$ is also the graph of an operator. An equivalent condition is that if $\{u_n\}$, where $u_n \in \mathfrak{D}(T)$, is a Cauchy sequence in the T norm and $\|u_n\| \to 0$ then $\|u_n\|_T \to 0$. The latter property means that the topologies generated by the norm of $\mathcal{H}$ and by the T norm on $\mathfrak{D}(T)$ are compatible.

If T is closable then the operator $\overline{T}$ defined by $\mathcal{G}(\overline{T}) = \overline{\mathcal{G}(T)}$ is called the *closure* of T. If T is bounded then $\overline{T}$ coincides with the extension of T by continuity.

An operator T from $\mathcal{H}_1$ into $\mathcal{H}_2$ is said to be *bounded* if there exists a $C \geqslant 0$ such that $\|Tu\|_{\mathcal{H}_2} \leqslant C\|u\|_{\mathcal{H}_1}$ for all $u \in \mathfrak{D}(T)$. Any such C is called a *bound* of T.

An operator T from $\mathcal{H}_1$ into $\mathcal{H}_2$ is said to be *compact* if every bounded sequence $\{u_n\}$ from $\mathfrak{D}(T)$ contains a subsequence $\{u_{n_j}\}$ for which $\{Tu_{n_j}\}$ is convergent.

By $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $\mathcal{B}_\infty(\mathcal{H}_1, \mathcal{H}_2)$ we denote respectively the spaces of everywhere defined bounded and compact operators acting from $\mathcal{H}_1$ into $\mathcal{H}_2$; $\mathcal{B}(\mathcal{H}) := \mathcal{B}(\mathcal{H}, \mathcal{H})$, $\mathcal{B}_\infty(\mathcal{H}) := \mathcal{B}_\infty(\mathcal{H}, \mathcal{H})$. The *norm* of an operator T in any of these spaces is given by

$$\|T\| = \inf\bigl\{C \geqslant 0 \bigm| \|Tu\|_{\mathcal{H}_2} \leqslant C\|u\|_{\mathcal{H}_1} \text{ for all } u \in \mathfrak{D}(T)\bigr\}.$$

A.2. *The adjoint operator*

Let T be a *densely defined* operator, i.e., $\overline{\mathfrak{D}(T)} = \mathcal{H}$. Then the *adjoint operator* T^* can be constructed as follows. The domain of T^* is

$$\mathfrak{D}\big(T^*\big) := \big\{v \in \mathcal{H} \,\big|\, \exists h \in \mathcal{H}\colon \langle Tu, v\rangle = \langle u, h\rangle \ \forall u \in \mathfrak{D}(T)\big\}. \tag{A.2}$$

The vector h is uniquely determined by v, and we set $h = T^*v$. Thus

$$\langle Tu, v\rangle = \big\langle u, T^*v\big\rangle \quad \forall u \in \mathfrak{D}(T), \forall v \in \mathfrak{D}\big(T^*\big). \tag{A.3}$$

As opposed to the case of $T \in \mathcal{B}(\mathcal{H})$, this equality is used not only to describe the "action" of T^* but, as we shall see, also to describe the domain of T^*.

The operator T^* is always a closed operator, and $\overline{\mathfrak{D}(T^*)} = \mathcal{H}$ if and only if T is closable. If the latter holds then $(T^*)^* = \overline{T}$. If $T_0 \subset T$ and $\overline{\mathfrak{D}(T_0)} = \mathcal{H}$ then $T_0^* \supset T^*$.

A.3. *Self-adjoint operators*

An operator T which fulfills $T^* = T$ is said to be *self-adjoint*.

An operator T, such that $\overline{\mathfrak{D}(T)} = \mathcal{H}$ and

$$\langle Tu, v\rangle = \langle u, Tv\rangle \quad \forall u, v \in \mathfrak{D}(T), \tag{A.4}$$

is called *symmetric*. These two notions are equivalent for $T \in \mathcal{B}(\mathcal{H})$. If $T^* = \overline{T}$ then T is said to be *essentially self-adjoint*. If T is symmetric and $\overline{T} \neq T^*$ then T^* is seen not to be symmetric.

The self-adjointness of an operator can often be established by means of perturbation theory, i.e., from the fact that the operator is close to another operator known in advance to be self-adjoint.

A.4. *Spectrum of an operator*

Let T be a closed operator. By definition, the *resolvent set* $\rho(T)$ consists of points $\lambda \in \mathbb{C}$ such that there exists $(T - \lambda I)^{-1} \in \mathcal{B}(\mathcal{H})$ (I being the identity operator in $\mathcal{H}$). The complement $\sigma(T) = \mathbb{C}\backslash\rho(T)$ of the resolvent set is called the *spectrum* of T. The set $\rho(T)$ is open and $\sigma(T)$ is closed. It is possible that $\sigma(T) = \mathbb{C}$ or $\sigma(T) = \emptyset$. (For $T \in \mathcal{B}(\mathcal{H})$ neither of these possibilities can be realized.)

If $T = T^*$ then the spectrum of T is nonempty and lies on the real axis. The spectrum $\sigma(T)$ of a self-adjoint operator can be represented as the union of the *point spectrum* $\sigma_{\mathrm{p}}(T)$, i.e., the set of all eigenvalues, and the *continuous spectrum*

$$\sigma_{\mathrm{c}}(T) = \big\{\lambda \in \mathbb{R} \,\big|\, \operatorname{Im}(T - \lambda I) \text{ is a nonclosed set}\big\}. \tag{A.5}$$

The spectra $\sigma_{\mathrm{p}}(T)$ and $\sigma_{\mathrm{c}}(T)$ can have a nonempty intersection. If $\sigma_{\mathrm{p}}(T) = \emptyset$ then T has a *purely continuous spectrum*. If the linear hull of the eigenspaces $\operatorname{Ker}(T - \lambda I)$, where $\lambda \in \sigma_{\mathrm{p}}(T)$, is dense in $\mathcal{H}$ then T has a *purely point spectrum*. In this case the continuous spectrum coincides with the set of limit points of the point spectrum and, generally speaking, is nonempty.

The union of the continuous spectrum and the set of eigenvalues of infinite multiplicity is called the *essential spectrum* of a self-adjoint operator T, denoted $\sigma_{\mathrm{ess}}(T)$. If $\sigma_{\mathrm{ess}}(T) = \emptyset$ then T has *discrete spectrum*. An equivalent condition for T to have discrete spectrum is that $(T - \lambda I)^{-1}$ be a compact operator for some $\lambda \in \rho(T)$ (and then for all such λ).

A.5. *The spectral theorem*

Suppose that associated with every Borel set $\Omega \subset \mathbb{R}$ is an orthogonal projection $E(\Omega)$ in $\mathcal{H}$. Let $E(\mathbb{R}) = I$ and let the following condition of countable additivity be fulfilled: if $\{\Omega_n\}$, $n = 1, 2, \ldots$, are pairwise disjoint Borel sets, then $\sum_n E(\Omega_n) = E(\bigcup_n \Omega_n)$. (The series on the left-hand side converges in the strong operator topology.) Any such map $E : \Omega \mapsto E(\Omega)$ is called a *spectral measure* in $\mathcal{H}$ (defined on the Borel subsets of the real axis).

If E is a spectral measure then, for any $u \in \mathcal{H}$, $E(\cdot)u$ is a vector-valued measure and $\mu_u(\cdot) = \langle E(\cdot)u, u\rangle$ is a scalar-valued Boreal measure normalized by $\mu_u(\cdot) = \|u\|^2$.

For any $u, v \in \mathcal{H}$, $\mu_{u,v}(\cdot) = \langle E(\cdot)u, v\rangle$ is a complex-valued Borel measure.

As in the case of scalar measures, the *support of a spectral measure* ($\operatorname{supp} E$) can be defined as the smallest closed subset $F \subset \mathbb{R}$ such that $E(F) = I$. The phrase “almost everywhere with respect to E” (E-a.e.) has the standard meaning.

Let E be a spectral measure and let f be a Borel measurable scalar function defined E-a.e. on $\mathbb{R}$. Then one can define the integral

$$J_f = \int f \, \mathrm{d}E \quad \left(= \int f(s) \, \mathrm{d}E(s) \right), \tag{A.6}$$

which is a closed operator in $\mathcal{H}$ with dense domain

$$\mathfrak{D}(J_f) = \mathfrak{D}_f = \left\{ u \in \mathcal{H} \,\middle|\, \int |f|^2 \, \mathrm{d}\mu_u < \infty \right\}. \tag{A.7}$$

The integral (A.6) can be understood, for example, in the “weak sense”, that is, $\langle J_f u, v\rangle = \int f \, \mathrm{d}\mu_{u,v}$ for $u \in \mathfrak{D}_\varphi$ and $v \in \mathcal{H}$. The operator J_f is self-adjoint if and only if f is an E-a.e. real-valued function. J_f is bounded if and only if f is E-a.e. bounded.

The following *spectral theorem* plays a central role in the spectral analysis of self-adjoint operators.

THEOREM A.1. *To every self-adjoint operator T there corresponds a unique spectral measure E_T such that*

$$T = \int s \, \mathrm{d}E_T(s).$$

One has that $\operatorname{supp} E_T = \sigma(T)$.

Having a spectral measure, we can associate to any bounded Borel function f on $\sigma(T)$ the operator J_f. This mapping $f \mapsto J_f \equiv \varphi(f)$ gives the following functional calculus form of the spectral theorem which is very useful.

THEOREM A.2. *Let T be a self-adjoint operator on $\mathcal{H}$. Then there exists a unique map φ from the bounded Borel functions on $\sigma(T)$ into $\mathcal{B}(\mathcal{H})$ having the following properties*:

(i) *φ is an algebraic $*$-homeomorphism, i.e.,*

$$\varphi(fg) = \varphi(f)\varphi(g), \qquad \varphi(\lambda f) = \lambda\varphi(f),$$
$$\varphi(1) = I, \qquad \varphi(\bar{f}) = \varphi(f)^*.$$

(ii) *φ is norm continuous, i.e.,* $\|\varphi(f)\|_{\mathcal{B}(\mathcal{H})} \leqslant \|f\|_{L^\infty}$.

(iii) *Let $\{f_n\}$ be a sequence of bounded Borel functions obeying $f_n(x) \to x$ for each x (as $n \to \infty$) and $|f_n(x)| \leqslant |x|$ for all x and n. Then, for any $\psi \in \mathfrak{D}(T)$,* $\lim_{n\to\infty} \varphi(f_n)\psi = T\psi$.

(iv) *If $f_n(x) \to f(x)$ pointwise and if the sequence $\{\|f_n\|_{L^\infty}\}$ is bounded, then $\varphi(f_n) \to \varphi(f)$ strongly.*

(v) *If $T\psi = \lambda\psi$, then $\varphi(f)\psi = f(\lambda)\psi$.*

(vi) *If $f \geqslant 0$, then $\varphi(f) \geqslant 0$.*

A.6. *Various spectra*

Let T be a self-adjoint operator in $\mathcal{H}$ and let E_T be the spectral measure of T. One can distinguish the following subspaces: $\mathcal{H}_{\mathrm{p}}$ defined as the closure of the linear hull of all eigenspaces of T, $\mathcal{H}_{\mathrm{ac}}$ defined as the set of all $u \in \mathcal{H}$ such that the measure $\mu_u^T(\cdot) = \langle E_T(\cdot)u, u\rangle$ is absolute continuous with respect to the Lebesgue measure, and $\mathcal{H}_{\mathrm{sc}}$, the orthogonal complement of $\mathcal{H}_{\mathrm{p}} \oplus \mathcal{H}_{\mathrm{ac}}$ in $\mathcal{H}$. If $u \in \mathcal{H}_{\mathrm{sc}}$ then μ_u^T is a continuous measure singular relative to the Lebesgue measure.

The subspaces $\mathcal{H}_{\mathrm{p}}$, $\mathcal{H}_{\mathrm{ac}}$ and $\mathcal{H}_{\mathrm{sc}}$ are orthogonal to each other and invariant with respect to T. The parts T_{p}, T_{ac} and T_{sc} of T in these subspaces (for example, $T_{\mathrm{ac}} = T|_{\mathfrak{D}(T)\cap\mathcal{H}_{\mathrm{ac}}}$) are self-adjoint as operators in $\mathcal{H}_{\mathrm{p}}$, $\mathcal{H}_{\mathrm{ac}}$ and $\mathcal{H}_{\mathrm{sc}}$, respectively. They are called the *pure point*, *absolute continuous*, and *singular continuous* components of T. One can also speak about the *absolute continuous components of the spectrum*, etc. If $T = T_{\mathrm{ac}}$ then the spectrum of T is said to be *absolute continuous*.

A.7. *Distribution function*

The *distribution function*

$$N(I; T) = \dim \operatorname{Ran} E_T(I) \tag{A.8}$$

($I \subset \mathbb{R}$ being an arbitrary interval) serves as an important characteristic of the location of the spectrum of a self-adjoint operator T. The case when $N(I;T) < \infty$ is the most interesting one. In this case, the spectrum of T on I consists of finitely many eigenvalues of finite multiplicity and (A.8) is equal to the sum of the multiplicities. If $N(I;T) = \infty$ and I is a bounded interval, then the closure of I contains at least one point belonging to $\sigma_{\mathrm{ess}}(T)$. If $I \cap \sigma_{\mathrm{ess}}(T) = \emptyset$ then the spectrum of T is said to be *discrete on* I.

For a lower semibounded self-adjoint operator, we set

$$N(\lambda; T) = N\big((-\infty, \lambda); T\big) \quad \forall \lambda \in \mathbb{R}. \tag{A.9}$$

A.8. *Classes of compact operators*

The following information on compact operators can be found in, e.g., [17], Chapter 11.

If T is a compact operator on some Hilbert space $\mathcal{H}$ and $\mu_j = \mu_j(T)$, $j \in \mathbb{N}$, denote the eigenvalues of T enumerated counting algebraic multiplicity in the order of nonincreasing moduli $|\mu_j|$, then the nonzero eigenvalues $s_j = s_j(T) = \sqrt{\mu_j(T^*T)}$ of $|T|$ are called the *singular numbers* or *singular values* or s*-numbers* of T. If $\sum_j s_j(T) < \infty$, then T is called a *trace class* operator. For trace class operators, the series

$$\operatorname{tr} T = \sum_j \mu_j(T)$$

is absolutely convergent. The set of all trace class operators is denoted by $\mathfrak{S}^1$ and it is an $*$-ideal in $\mathcal{B}(\mathcal{H})$, viz.

1. $\mathfrak{S}^1$ is a vector space.
2. If $T_1 \in \mathfrak{S}^1$ and $T_2 \in \mathcal{B}(\mathcal{H})$, then $T_1T_2 \in \mathfrak{S}^1$ and $T_2T_1 \in \mathfrak{S}^1$.
3. If $T_1 \in \mathfrak{S}^1$, then $T_1^* \in \mathfrak{S}^1$.

If $T \in \mathfrak{S}^1$ and $\{\varphi_n\}$ is any orthonormal basis, then $\operatorname{tr} T$ equals the sum $\sum_n \langle \varphi_n, T\varphi_n \rangle$ which also converges absolutely and the limit is independent of the choice of the basis. The map $\operatorname{tr} : \mathfrak{S}^1 \to \mathbb{C}$, defined either way, obeys:

1. $\operatorname{tr}(\cdot)$ is a linear map.
2. $\operatorname{tr} T^* = \overline{\operatorname{tr} T}$.
3. $\operatorname{tr} T_1T_2 = \operatorname{tr} T_2T_1$ if $T_1 \in \mathfrak{S}^1$ and $T_2 \in \mathcal{B}(\mathcal{H})$.

Let $n(s,T) = \#\{s_j(T) > s\}$, $s > 0$, be the *counting function*. By the definition of s_j, one has that

$$n\big(s^2, T^*T\big) = n(s,T), \tag{A.10}$$

and for any self-adjoint nonnegative operator, one has $s_j(T) = \mu_j(T)$. The counting function satisfies the *Weyl inequality*,

$$n(s_1 + s_2, T_1 + T_2) \leqslant n(s_1, T_1) + n(s_2, T_2). \tag{A.11}$$

For $0 < p < \infty$ the trace class (or Neumann–Schatten class) $\mathfrak{S}^p$ is a set of $T \in \mathcal{B}_\infty(\mathcal{H}_1, \mathcal{H}_2)$ for which the following functional is finite

$$\|T\|^p_{\mathfrak{S}^p(\mathcal{H}_1,\mathcal{H}_2} := \sum_j \bigl[s_j(T)\bigr]^p = p\int_0^\infty s^{p-1} n(s,T)\,\mathrm{d}s. \tag{A.12}$$

If $p \geqslant 1$ then (A.12) defines a norm on $\mathfrak{S}^p$ and a quasinorm for $p < 1$. Evidently, $T \in \mathfrak{S}^q(\mathcal{H}_1, \mathcal{H}_2)$ if and only if

$$\operatorname{tr}\bigl(|T|^q\bigr) = \operatorname{tr}\bigl((T^*T)^{q/2}\bigr) < \infty.$$

For $0 < p < \infty$ the class $\mathfrak{S}^p_w(\mathcal{H}_1, \mathcal{H}_2) \subset \mathcal{B}_\infty(\mathcal{H}_1, \mathcal{H}_2)$ is the set of all compact operators T such that the following functional is finite:

$$\|T\|_{\mathfrak{S}^p_w} := \Bigl(\sup_{s>0} s^p n(s,T)\Bigr)^{1/p}.$$

The functional $\|\cdot\|_{\mathfrak{S}^p_w}$ is a quasinorm. The classes $\mathfrak{S}^p_w(\mathcal{H}_1, \mathcal{H}_2)$ are not separable (if $\dim \mathcal{H}_1 = \dim \mathcal{H}_2 = \infty$); a separable subspace $\mathfrak{S}^p_0 \subset \mathfrak{S}^p_w$ is defined by

$$\mathfrak{S}^p_0 := \Bigl\{T \in \mathfrak{s}^p \Bigm| \lim_{s\to+0} s^p n(s,T) = 0\Bigr\}.$$

Note that $\mathfrak{S}^p \subset \mathfrak{S}^p_0$.

For a compact self-adjoint operator T we set

$$n_\pm(s,T) = N\bigl((\lambda,\infty); \pm T\bigr). \tag{A.13}$$

For operators $T = T^* \in \mathcal{B}_\infty$ the following functionals are introduced:

$$\Delta^{(\pm)}_p(T) := \limsup_{s\to\infty} s^p n_\pm(s,T), \tag{A.14}$$

$$\delta^{(\pm)}_p(T) := \liminf_{s\to\infty} s^p n_\pm(s,T), \tag{A.15}$$

so that $0 \leqslant \delta^{(\pm)}_p(T) \leqslant \Delta^{(\pm)}_p(T) \leqslant \infty$. The functionals $\Delta^{(\pm)}_p$, $\delta^{(\pm)}_p$ are continuous in $\mathfrak{s}_p$ and do not change if their argument changes by an operator of the class $\mathfrak{S}^p_0$.

A.9. *Glazman's lemma*

The spectral theorem leads to the variational principle for the distribution function. There are many different formulations of this principle, the one most convenient for our purposes is called the *Glazman lemma*, which plays an important role in both qualitative and quantitative spectral analysis.

For an operator with quadratic form $\mathfrak{t}[u]$, the number of points of the spectrum, finite or infinite on some interval, is described by the dimensions of subspaces in $\mathfrak{D}(\mathfrak{t})$ where the ratio $\mathfrak{t}[u]/\|u\|_{\mathcal{H}} = \chi[u]$ is controlled. In particular, $N(\lambda; T)$, for the operator T defined by the quadratic form $\mathfrak{t}$, equals the maximum of the dimensions of subspaces where $\chi[u] < \lambda$, or minimum of the co-dimensions of subspaces where $\chi[u] \geqslant \lambda$. Construction of subspaces satisfying these inequalities immediately gives estimates for $N(\lambda; T)$ from below, resp. from above. Taken together, they, in particular, prove various versions of the Birman–Schwinger principle; see Section 6.2.

LEMMA A.3. *Let T be a lower (upper) semibounded self-adjoint operator and $\mathcal{M} \subset \mathfrak{D}(\mathfrak{t})$ be a linear subset, dense in the T-norm. Then*

$$N_{\pm}(\lambda, T) = \max \dim\bigl\{\mathcal{L} \subset \mathcal{M}\colon \pm\bigl(\mathfrak{t}[u] - \lambda\|u\|^2\bigr) < 0, u \in \mathcal{L} \setminus \{0\}\bigr\}, \tag{A.16}$$

$$N_{\pm}(\lambda, T) = \min \operatorname{codim}\bigl\{\mathcal{L} \subset \mathfrak{D}(\mathfrak{t})\colon \pm\bigl(\mathfrak{t}[u] - \lambda\|u\|^2\bigr) \geqslant 0, u \in \mathcal{L}\bigr\}. \tag{A.17}$$

Here codim $\mathcal{L}$ *for a subspace $\mathcal{L} \subset \mathfrak{D}(\mathfrak{t})$ denotes the minimal number of orthogonality conditions which determine $\mathcal{L}$. Minus and plus signs correspond to upper, resp. lower, semibounded operators.*

An important consequence of the Glazman lemma is that the distribution function depends on the quadratic form in a monotone way. Let, for example, two operators T_1 and T_2 correspond to quadratic forms $\mathfrak{t}_1[u], \mathfrak{t}_2[u]$, so that $\mathfrak{D}(\mathfrak{t}_1) \subset \mathfrak{D}(\mathfrak{t}_2)$ and $\mathfrak{t}_1[u] \geqslant \mathfrak{t}_2[u]$, $u \in \mathfrak{D}(\mathfrak{t}_1)$. Then in (A.16), the set of subspaces over which we maximize is larger for T_2 than for T_1, and therefore, $N(\lambda, T_2) \geqslant N(\lambda, T_1)$ for any λ.

We collect some properties of the distribution function for the Schrödinger operator $H_V = -\Delta - V$. The following lemma follows immediately from the Glazman lemma.

LEMMA A.4. (i) *If $V_1 \leqslant V_2$ pointwise, then $N(-E; H_{V_2}) \leqslant N(-E; H_{V_1})$ for all $E \geqslant 0$. In particular, $N(-E; H_V) \leqslant N(-E; H_{V_+})$.*

(ii) *For all $\alpha \in [0, 1]$ and $E \geqslant 0$,*

$$N(-E; H_V) \leqslant N(-\alpha E; H_{(V(x)-(1-\alpha)E)_+}), \tag{A.18}$$

where $(V(x) - (1-\alpha)E)_+$ denotes the positive part of the potential $V(x) - (1-\alpha)E$.

Looking at the graph of $N(-E; H_V)$ one sees that integrating $N(-E; H_V)$ with respect to E yields (minus) the sum of the negative eigenvalues. More generally, since

$$\frac{\partial}{\partial E} N(-E; H_V) = -\sum_j \delta(E - E_j),$$

where $\{E_j\}_j$ denote the eigenvalues of H_V, one has the following result [84,85].

LEMMA A.5. *Let* $\gamma > 0$ *and* $S_{\gamma,d}(V) := \sum_{E_j<0} |E_j|^\gamma$. *Then*

$$S_{\gamma,d}(V) = \gamma \int_0^\infty E^{\gamma-1} N(-E; H_V)\, \mathrm{d}E. \tag{A.19}$$

A.10. *Birman–Schwinger principle*

A variant of the method of forms based on the notion of a variational triple is useful in a number of cases, in particular in the study of spectral problems of the form

$$G\psi = \lambda H_0 \psi. \tag{A.20}$$

A *variational triple* $\{\mathcal{H}; \mathfrak{h}_0, \mathfrak{g}\}$ consists of a Hilbert space $\mathcal{H}$ with a metric form $\mathfrak{h}_0[u]$ and a bounded sesquilinear Hermitian form $\mathfrak{g}[u, v]$ in $\mathcal{H}$. The relation

$$\mathfrak{g}[u, v] = \mathfrak{h}_0[Tu, v] \quad \forall u, v \in \mathcal{H}$$

assigns a unique operator $T = T(\mathcal{H}; \mathfrak{h}_0, \mathfrak{g})$ to $\{\mathcal{H}; \mathfrak{h}_0, \mathfrak{g}\}$. The operator T is bounded and self-adjoint in $\mathcal{H}$.

In particular, let $\mathfrak{h}_0$ and $\mathfrak{g}$ be the quadratic forms of operators H_0 and G acting in a Hilbert space $\mathcal{H}$. More precisely, let H_0 be a positive definite self-adjoint operator and let $\mathfrak{h}_0$ be its associated quadratic form. We assume that G is a symmetric operator defined on a dense set $\mathcal{D} \subset \mathfrak{Q}(\mathfrak{h}_0)$ (in the simplest case $\mathcal{D} = \mathfrak{D}(H_0)$) and the quadratic form $\langle Gu, v\rangle$ is bounded in $\mathfrak{Q}(\mathfrak{h}_0)$. Extending the form by continuity, we can obtain a bounded form $\mathfrak{g}[u]$ on $\mathfrak{Q}(\mathfrak{h}_0)$. Hence we have constructed a variational triple $\{\mathcal{H}; \mathfrak{h}_0, \mathfrak{g}\}$. The operator determined by the triple coincides with $H_0^{-1}G : \mathfrak{Q}(\mathfrak{h}_0) \to \mathfrak{Q}(\mathfrak{h}_0)$ on $\mathcal{D}$. It is therefore natural to associate the spectrum of this operator with (A.20).

The study of the negative spectrum of the Schrödinger operator with a decreasing potential can be reduced to the investigation of the spectrum of an equation of the form

$$p\psi = \lambda(-\Delta\psi + \varepsilon\psi), \quad \varepsilon \geqslant 0.$$

The abstract scheme of such a reduction was developed by Birman [14]. We shall state only the simplest result from [14].

THEOREM A.6 (Birman–Schwinger principle). *Let* H *and* H_0 *be semibounded self-adjoint operators in a Hilbert space* $\mathcal{H}$ *and let* $H_0 > 0$. *Let* $\mathfrak{h}$ *and* $\mathfrak{h}_0$ *be the corresponding quadratic forms, and let* $\mathfrak{Q}(\mathfrak{h}) = \mathfrak{Q}(\mathfrak{h}_0) = \mathcal{K}$. *Let* T_ε, *where* $\varepsilon > 0$, *be the operator determined by the variational triple* $\{\mathcal{K}; \mathfrak{h}_0[u] + \varepsilon\|u\|^2_{\mathcal{K}}, \mathfrak{g} = \mathfrak{h}_0 - \mathfrak{h}\}$. *Then*

$$N(-\varepsilon; H) = n_+(1, T_\varepsilon). \tag{A.21}$$

If H_0 *is positive definite, then* (A.21) *is also valid for* $\varepsilon = 0$.

The operator T_ε is traditionally called the *Birman–Schwinger operator*. The proof of Theorem A.6 can be reduced to comparing the formulae

$$N(-\varepsilon, H) = \max\dim\bigl\{\mathcal{L} \subset \mathcal{K}\colon\ \mathfrak{h}[u] + \varepsilon\|u\|^2 < 0\bigr\},$$
$$n_+(1, T_\varepsilon) = \max\dim\bigl\{\mathcal{L} \subset \mathcal{K}\colon\ \mathfrak{g}[u] > \mathfrak{h}_0[u] + \varepsilon\|u\|^2\bigr\},$$

which follow directly from Lemma A.3. The assertion concerning the case $\varepsilon = 0$ can be extended to any $H_0 \geqslant 0$, but the formulation becomes more involved (see [14]).

A.11. *Asymptotic perturbation lemma*

Generally, one should expect that if one perturbs an operator with a weaker one, the main properties must not change. The lemma we give here (established first in [16]) assigns concrete meaning to this vague statement, as it concerns asymptotics of the spectrum.

LEMMA A.7. *Let K be a compact self-adjoint operator, and for some $q > 0$ and any $\varepsilon > 0$, K may be represented a sum, $K = K_\varepsilon + K'_e$, where*

$$\lim_{t\to+0} n_\pm(t, K_\varepsilon)t^q = c_\pm(\varepsilon), \qquad \limsup_{t\to+0} n_\pm\bigl(t, K'_\varepsilon\bigr)t^q \leqslant \varepsilon.$$

Then there exist limits $\lim_{\varepsilon\to 0} c_\pm(\varepsilon) = c_\pm$ *and* $\lim_{t\to+0} n_\pm(t, K)t^q = c_\pm$.

PROOF. Fix some $\delta > 0$. The Weyl inequality (A.11) gives $n_+(t, K) \leqslant n_+(t(1-\delta), K_\varepsilon) + n_+(t\delta, K'_\varepsilon)$. Passing to lim sup, we obtain

$$c_+^{(+)} = \limsup_{t\to 0} n_+(t, K)t^q \leqslant c_+(\varepsilon)(1-\delta)^q + \delta^{-q}\varepsilon.$$

On the other hand, applying Weyl inequality to $K_\varepsilon = K + (-K'_\varepsilon)$, we obtain $n_+(t, K) \geqslant n_+(t(1+\delta), K_\varepsilon) - n_-(t\delta, K'_\varepsilon)$. Passing here to lim inf, we get for $c_+^{(-)} = \liminf_{t\to 0} N_+(t, K)t^{-q}$

$$c_+^{(-)} \geqslant \lim_{t\to 0} n_+\bigl(t(1+\delta), K_\varepsilon\bigr)t^q - \limsup_{t\to 0} n_-\bigl(t\delta, K'_\varepsilon\bigr)t^q$$
$$\geqslant c_+(\varepsilon)(1+\delta)^{-q} - \delta^{-q}\varepsilon.$$

Thus

$$c_+(\varepsilon)(1+\delta)^{-q} - \delta^{-q}\varepsilon \leqslant c_+^{(-)} \leqslant c_+^{(+)} \leqslant c_+(\varepsilon)(1-\delta)^q + \delta^{-q}\varepsilon. \tag{A.22}$$

We set here $\delta = \varepsilon^{1/(q+1)}$ so that $\delta^{-q}\varepsilon \to 0$. Then (A.22) gives $c_+^{(-)} = c_+^{(-)} = \lim c_+(\varepsilon)$. □

The lemma plays a crucial role whenever one wants to establish an asymptotic formula. Here the general scheme is to prove the asymptotic formula first for some regular case (say, smooth and compactly supported potentials if one considers Schrödinger operators) and then use some uniform estimate (say, the CLR estimate) in combination with Lemma A.7 to extend the asymptotics to more singular cases.

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CHAPTER 7

Multiplicity Techniques for Problems without Compactness

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Contents

HANDBOOK OF DIFFERENTIAL EQUATIONS
Stationary Partial Differential Equations, volume 2
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Introduction

The aim of this survey relies on focusing some recent multiplicity results for nonlinear problems with a lack of compactness which could probably find new and different applications. The corresponding existence results are well known long before but a full understanding of the multiplicity problem has required more specific and recent techniques. Here we shall show those techniques by stressing on the geometric ideas underlying them. More precisely, two main problems will be addressed:

1. *Elliptic problems at critical growth on a bounded domain*.
2. *Elliptic problems at subcritical growth on the whole domain*.

For both the above problems there is a lack of compactness which is due to the existence of extremely concentrated solutions in case 1 and to the existence of solutions whose centers of mass escape to infinity in case 2. In both cases the existence results are available thanks to suitable hypotheses on the linear term which make the compactness degeneracy increase the functional which is going to be minimized. Thus deviation from compactness is possible in principle but it is not advantageous, under different hypotheses on the lower-order terms one would easily show nonexistence results. This survey includes the results in [13,19,20,43], some parts of which are borrowed with minor modifications, and it is in large part devoted to a careful analysis of the two above problems following mainly [19] and [13]. For the common features shared by these problems, they well offer the opportunity to clarify and to focus the ideas and the techniques employed in those works. Moreover, we shall make the exposition self-contained as far as possible and organized in a heuristic way. Nevertheless, a good knowledge of the use of the variational and topological methods in nonlinear analysis and some familiarity with the study of nonlinear elliptic equations is required to the reader.

The exposition begins with an introductory part in which we recall the known facts concerning the two already mentioned problems and, after introducing suitable concentration–compactness tools, briefly sketches the main ideas which lead to prove the existence of a nontrivial solution, recovering compactness thanks to suitable estimates on the energy levels. In Section 2 we shall prove the compactness theorems which show how, substituting the Palais–Smale sequences with the sequences of solutions of approximating problems, the concentration–compactness tools introduced in the first section lead to complete compactness results. Section 3 is devoted to the proof of some decay estimates, inspired by the analysis of some particular cases, employed in the previous section. Section 4 concerns the proof of the multiplicity theorems, which will be given after a preliminary part in which the notion of genus, together with a recent variant, is introduced and employed to construct the suitable min–max approach. Finally, in Section 5 some concluding remarks are stated and a result which falls out of the main hypotheses assumed here is shown. Further introductory indications are given at the beginning of each section.

Notation. Throughout the paper we make use of the following notation:

- Ω denotes a bounded subset of $\mathbb{R}^N$.
- $L^p(\Omega)$, $1 \leqslant p \leqslant +\infty$, $\Omega \subseteq \mathbb{R}^N$, denotes a Lebesgue space, the norm in $L^p(\Omega)$ is denoted by $\|\cdot\|_p$.

- $H_0^1(\Omega)$, $\Omega \subset \mathbb{R}^N$, denote the Sobolev space obtained as closure of $\mathcal{C}_0^\infty(\Omega)$, with respect to the norm

$$\|u\| = \left[\int_\Omega |\nabla u|^2 \,\mathrm{d}x\right]^{1/2}.$$

- $H^{-1}(\Omega)$, $\Omega \subset \mathbb{R}^N$, denotes the dual spaces of $H_0^1(\Omega)$.
- If $u \in H_0^1(\Omega)$, $\Omega \subset \mathbb{R}^N$, and if there is no risk of ambiguity, we denote also by u its extension to $\mathbb{R}^N$ made by setting $u \equiv 0$ on $\mathbb{R}^N \setminus \Omega$.
- S denotes the Sobolev constant, i.e., $S = \inf\{\|\nabla u\|_2^2/\|u\|_{2^*}^2 \mid u \in H_0^1(\Omega), u \neq 0\}$.
- We denote by $\lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots \leqslant \lambda_n \leqslant \cdots$ the sequence of the eigenvalues of the Laplacian operator $-\Delta$ on $H_0^1(\Omega)$ and by $\varphi_1, \varphi_2, \ldots, \varphi_n, \ldots$ the corresponding sequence of orthonormal eigenvectors, we set $E_n = \operatorname{Span}\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$.

For every real number c we shall say that a sequence $(u_n)_{n\in\mathbb{N}}$ is a Palais–Smale (briefly PS) sequence for the functional $I : H^1 \to \mathbb{R}$ at level c if the following two conditions hold:

(1) $I(u_n) \to c$;

(2) $\mathrm{d}I(u_n) \to 0$ in H^{-1}, where $\mathrm{d}I$ is the Fréchet derivative of I.

We shall briefly say that $(u_n)_{n\in\mathbb{N}}$ is a PS sequence if there exists a level $c \in \mathbb{R}$ such that $(u_n)_{n\in\mathbb{N}}$ is a PS sequence at level c.

1. Statement of the problems and sketch of the existence results

1.1. *Elliptic problems at critical growth on a bounded domain*

Let us consider the critical growth problem

$$\begin{cases} -\Delta u = |u|^{2^*-2}u + \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{CP}$$

where Ω is an open regular subset (without any shape condition) of $\mathbb{R}^N$ ($N \geqslant 3$), $2^* = 2N/(N-2)$ is the critical Sobolev exponent for the embedding of $H_0^1(\Omega)$ into $L^p(\Omega)$, and $\lambda > 0$. Several people have got involved with this problem (see [2,3]) and here, for the reader's convenience, we summarize the main known results in the field.

1. If $\lambda \leqslant 0$, the Pohozaev identity (see [37]) allows us to say that problem (CP) has, in general (for a star-shaped Ω), no nontrivial solution.
2. There exists a constant $\lambda^* \in [0, \lambda_1[$ such that (CP) has a positive solution if $\lambda \in]\lambda^*, \lambda_1[$, where λ_1 is the first eigenvalue of $-\Delta$ defined on $H_0^1(\Omega)$. When $N \geqslant 4$ then $\lambda^* = 0$ (see [11]). The existence of a nontrivial solution also for $\lambda \geqslant \lambda_1$ has been subsequently proved in [12]. In the three-dimensional case and when Ω is a ball then $\lambda^* = \lambda_1/4$. Moreover, by using also in this case a suitable version of Pohozaev identity we know that, for $\lambda \in]0, \lambda^*[$, (CP) has no radial solution (see [11,12]) but it is still unknown if there exist nonradial solutions (changing sign) to (CP).

3. If $N \geqslant 4$ and Ω is a ball, then for any $\lambda > 0$, (CP) has infinitely many changing sign solutions (which sometimes cannot be radial, as shown in [2,3]) which are built by using the particular symmetry of the domain Ω (see [24]).
4. If $N \geqslant 7$ and Ω is a ball, then for each $\lambda > 0$, (CP) has infinitely many changing sign-radial solutions, see [39] and a previous paper by Cerami, Solimini and Struwe [14], where it is also shown that for $N \geqslant 6$, (CP) has at least two (pairs of) solutions on any smooth bounded domain.
5. When $4 \leqslant N \leqslant 6$ and Ω is a ball there exists a constant $\lambda^* > 0$ such that (CP) has no changing sign-radial solution if $\lambda \in]0, \lambda^*[$. So the bound $N \geqslant 7$ in the previous result cannot be removed (see [2,3]).
6. In [19] the question about existence of infinitely many solutions to problem (CP), for any bounded smooth domain $\Omega \subset \mathbb{R}^N$ in the case $N \geqslant 7$, is affirmatively answered. Furthermore, by the above mentioned result in [2,3], the compactness arguments, which can be also employed in the radial case, cannot be extended to lower values of N.
7. Finally, in [20] it is shown that, for $\lambda \in]0, \lambda_1[$ and $N \geqslant 4$, problem (CP) has at least $\frac{N}{2} + 1$ (pairs of) solutions ($N + 1$ for λ close enough to 0) improving thus the result in [14]. Such result has been extended in [15] to the case $\lambda \geqslant \lambda_1$.

The main difficulty in dealing with problem (CP) is the existence of noncompact Palais–Smale sequences (PS sequences) of the corresponding functional

$$I_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_\Omega |u|^2 - \frac{1}{2^*}\int_\Omega |u|^{2^*}, \tag{1.1}$$

defined on the Hilbert space $H_0^1(\Omega)$. The behavior of noncompact PS sequences has been studied in [45] which, roughly speaking, assures the existence of a subsequence approximated by its weak limit plus terms which tend to concentrate around a finite number of points (see Theorem 1.6). This result allows a precise description of the behavior of noncompact PS sequences of the functional I_λ and an estimate of their possible levels, suggesting the idea to look for *good levels* in order to get compactness. We shall employ this analysis in conjunction with other suitable compactness techniques to deal with the multiplicity problem. Then, following [19], we shall show as, in dimension $N \geqslant 7$, every min–max admissible class produces precompact PS sequences. This will follow as a consequence of a uniform bound theorem stated for bounded sets U of solutions to

$$\begin{cases} -\Delta u = |u|^{p-2}u + \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{SP}$$

with $p \in [2, 2^*]$. This result will require suitable a priori estimates on some norms of the functions in U. Such estimates will be employed to the aim of finding a suitable control on the functions and on their derivatives and, finally, a local Pohozaev identity will allow the proof of the following uniform bound theorem.

THEOREM 1.1 (Uniform bound through concentration estimates). *Let $N \geqslant 7$ and U be a bounded set in $H_0^1(\Omega)$ whose elements are solutions, for a fixed $\lambda > 0$, to problems* (SP), *for p varying in* $[2, 2^*]$. *Then U is uniformly bounded, i.e., there exists a constant $C > 0$ such that*

$$\sup_{u \in U} \sup_{x \in \Omega} \left| u(x) \right| \leqslant C.$$

The above result is equivalent to a compactness property in $H^1(\Omega)$ (see [10]) which allows uniform L^∞ estimates in the case of a precompact set of solutions. Though there is a lack of compactness for PS sequences, we have compactness for the bounded sets of solutions. Thus the key idea, in the case of a critical growth, relies in using the variational methods to solve slightly subcritical problems, where the usual arguments based on PS sequences produce solutions, and then to pass to the limit on such a set of solutions. In the light of these considerations, it becomes evident that Theorem 1.1 plays a crucial role in this program of work, indeed its proof has involved the major difficulties. Furthermore, we shall show how this technique allows to apply classical min–max arguments to problem (CP) and to prove, in this way, the existence of infinitely many solutions, as it is stated in the following theorem.

THEOREM 1.2 (Infinitely many solutions to (CP) in large dimension). *If $N \geqslant 7$, then problem* (CP) *admits infinitely many solutions*.

As we have just observed, analogous multiplicity results, like the existence of infinitely many radial solutions to (CP) when Ω is a ball, can be obtained from Theorem 1.1 in the same way as Theorem 1.2 and so the uniqueness result in [3], Theorem A, leads to the following remark.

REMARK 1.1. The restriction $N \geqslant 7$ in Theorem 1.1 cannot be removed. Indeed, the theorem is false for $N \leqslant 6$.

Theorem 1.2 does not give any answer to the existence of infinitely many solutions to (CP) when $N \leqslant 6$. In such a case it is only known the negative answer for radial solutions and the affirmative one for symmetric domains. Here we shall show through different techniques that when $N \geqslant 4$, for $\lambda \in]0, \lambda_1[$, problem (CP) has at least $\frac{N}{2} + 1$ (pairs of) solutions (see [20]).

1.2. *Elliptic problems at subcritical growth on the whole domain*

Let us consider the problem

$$\begin{cases} -\Delta u + a(x)u = |u|^{p-2}u \quad \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{P}$$

where $N \geqslant 2$, $p > 2$ and $p < 2N/(N-2)$ when $N > 2$, and the potential $a(x)$ is a continuous function, positive in $\mathbb{R}^N$, except at most a bounded set, satisfying suitable decay assumptions. We do not impose any symmetry property to $a(x)$.

Problems like (P) naturally arise in various branches of Mathematical Physics, indeed the solutions to (P) can be seen as solitary waves (stationary states) in nonlinear equations of the Klein–Gordon or Schrödinger type, moreover, they present specific mathematical difficulties that make them challenging to the researchers.

The solutions to problem (P) can be searched as critical points of the energy functional $I : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + a(x)u^2\right) \mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p \, \mathrm{d}x. \tag{1.2}$$

The usual variational methods, that allow to prove the existence of infinitely many solutions to (P) in a bounded domain, cannot be straightly applied to I. Indeed, the embedding $j : H^1(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ is continuous but not compact, therefore the basic Palais–Smale condition is not satisfied by I at all the energy levels. This difficulty can be avoided when $a(x)$ enjoys some symmetry. Indeed, the first known results have been obtained considering $a(x) = a(|x|)$ or even $a(x) = a_\infty \in \mathbb{R}^+ \setminus \{0\}$ (see [8,9,16,17,35,42]). In this case, the restriction of I to $H^1_r(\mathbb{R}^N)$, the subspace of $H^1(\mathbb{R}^N)$ consisting of spherically symmetric functions, restores compactness, because the embedding of $H^1_r(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$ is compact. So, the existence of a positive solution to (P) can be proved either by using mountain pass theorem or by minimization on a natural constraint, while the existence of infinitely many solutions follows by standard minimax arguments. Moreover, it is worth recalling that, still under the assumption $a(x) = a(|x|)$, one can also find the existence of infinitely many nonradial changing sign solutions, breaking the radial symmetry of the equation (see [6] and reference therein).

When $a(x)$ does not enjoy any symmetry property, the problem becomes more difficult and even proving the existence of one positive solution is not a trivial matter. This situation requires a deeper understanding of the nature of the obstructions to the compactness and the use of more subtle tools. Most of the researches have been concerned with the case

$$\lim_{|x|\to+\infty} a(x) = a_\infty > 0 \tag{1.3}$$

so that (P) can be related to the “problem at infinity”,

$$-\Delta u + a_\infty u = |u|^{p-2}u \quad \text{in } \mathbb{R}^N. \tag{P$_\infty$}$$

A first answer to the existence question has been given proving that, in some cases, being true some inequalities relating (P) and (P$_\infty$), the concentration–compactness principle can be applied and (P) can be solved by minimization [28]. This is the case, for instance, when $a(x)$ is a continuous function that, besides (1.3) and some decay assumptions, satisfies

$$0 < \delta_1 \leqslant a(x) \leqslant a_\infty \quad \forall x \in \mathbb{R}^N. \tag{1.4}$$

Subsequently, a careful analysis of the behavior of the Palais–Smale sequences (see [4,7]) has allowed to state that the compactness can be lost (in the sense that a PS sequence does not converge to a critical point) if and only if such a sequence breaks into a finite number of solutions to (P_∞) which are *centered* at points which go to infinity. As a consequence, it has been possible to give an estimate of the energy levels in which the PS condition fails in terms of the energy of such masses and to face better some existence and multiplicity questions for (P). Indeed, the existence of a positive solution to (P) has been proved (see [4]) even when a ground state solution cannot exist, that is, for instance, when, besides (1.3) and suitably decay assumptions, the potential satisfies the condition $a(x) > a_\infty\ \forall x \in \mathbb{R}^N$; moreover, under conditions (1.3), (1.4) and a suitable decay at infinity, it has been shown the existence of a changing sign solution in addition to the positive one (see [33]).

To conclude this brief review of known results, let us mention that there is some other work involving the use of variational methods to deal with standing waves of nonlinear Schrödinger equations. Some of these papers mainly deal with the existence of solutions to (P) using mountain pass and comparison arguments. See, e.g., [21,38] as well as the references therein. In particular, we point out that in [38] the existence of a positive and a negative solution is proved, provided

$$\text{(i)}\quad \inf_{\mathbb{R}^N} a(x) > 0, \qquad \text{(ii)}\quad \lim_{|x|\to+\infty} a(x) = +\infty, \tag{1.5}$$

while in [5] the existence of a third changing sign solution is shown.

Some other papers study cases in which the potential $a(x)$ possesses nondegenerate critical points and depends on a parameter, i.e., it appears like $a_h(x) = a(hx)$, and contain results of multiplicity of positive solutions under restrictions on the size of h (see [1,23,36] and for $a(x)$ of a special form [34]). Finally we remind that, under assumptions of periodicity on a, (P) has been shown to posses infinitely many solutions (see [18]).

Following [13], we shall assume the function a to satisfy the following conditions.

(a_1) $a \in \mathcal{C}^1(\mathbb{R}^N, \mathbb{R})$;
(a_2) $\liminf_{|x|\to+\infty} a(x) = a_\infty > 0$;
(a_3) $\frac{\partial a}{\partial \vec{x}}(x) e^{\alpha|x|} \underset{|x|\to+\infty}{\longrightarrow} +\infty\ \forall \alpha > 0$, where $\forall x \in \mathbb{R}^N \setminus \{0\}$, $\vec{x} = x/|x|$;
(a_4) there exists a constant $\bar{c} > 1$ such that

$$\left|\nabla_{\tau_x} a(x)\right| \leqslant \bar{c}\frac{\partial a}{\partial \vec{x}}(x) \quad \forall x \in \mathbb{R}^N\colon\ |x| > \bar{c},$$

where $\nabla_{\tau_x} a(x)$ denotes the component of $\nabla a(x)$ lying in the hyperplane orthogonal to $\vec{x}$ and containing x.

Therefore, in such a setting, we shall prove the next theorems. The ingredients of the proof recall the techniques and the estimates employed in the previous case for the problem at critical growth on a bounded domain. The analogous of the subcritical problem (SP) is, in this case, the same problem (P) on a bounded domain or, more specifically, on a ball B_r centered in the origin, with homogeneous Dirichlet boundary conditions. Given $r > 0$, we shall therefore consider the approximating problem

$$\begin{cases} -\Delta u + a(x)u = |u|^{p-2}u & \text{in } B_r(0), \\ u = 0 & \text{on } \partial B_r(0). \end{cases} \tag{AP_r}$$

The first theorem states a compactness property of the bounded sets of solutions to the approximating problems.

THEOREM 1.3. *Assume that $a(x)$ satisfies* (a_1)–(a_4). *Let $U \subset H^1(\mathbb{R}^N)$ be a bounded set consisting of solutions to* (AP_r) *for some $r > 0$. Then U is a precompact subset of $H^1(\mathbb{R}^N)$.*

From such compactness property we shall deduce the infinite multiplicity result.

THEOREM 1.4. *If $a(x)$ satisfies the assumptions* (a_1)–(a_4), *then* (P) *has infinitely many solutions.*

Let us consider a sequence of balls in $\mathbb{R}^N$, $B_{r_n}(0) = \{x \in \mathbb{R}^N\colon |x| < r_n\}$, $r_n \underset{n\to+\infty}{\longrightarrow} +\infty$, and the related problems $(P_n) = (AP_{r_n})$ approaching (P).

Since it is possible to prove that, for every n, (P_n) possesses infinitely many solutions, obtained by constructing infinitely many critical levels for the related functionals as minimax on suitable classes of functions, it is a natural idea considering sequences $\{u_n\}$ of solutions to (P_n), corresponding to minimax classes of the same type, and then trying to pass to the limit.

Clearly, once again, we need to prove that such sequences are precompact. Hence some additional tool is needed to control the situation. This is again a local Pohozaev-type inequality that, combined with some uniform decay estimates and integral bounds on any bounded sequence of solutions to (P_n), allows to conclude that, under our assumptions, the lack of compactness due to translations cannot occur for such sequences because it is possible, in principle, but it is not convenient in order to minimize the functional.

REMARK 1.2. It is worth pointing out that, if in (P) we replace $\mathbb{R}^N$ by $\mathbb{R}^N \setminus \overline{\Omega}$, where Ω is any bounded smooth open set in $\mathbb{R}^N$, Theorem 1.4 is still true, because the arguments we shall develop still hold after very simple modification.

1.3. *Concentration–compactness tools*

The presence of the critical exponent in the Sobolev embedding and the unbounded measure of the domain does not allow using the classical compactness techniques based on Rellich theorem but requires more fine tools as the ones studied by Lions in [28,29]. Compactness theorems are due to Struwe [45,46] for the problem at critical growth and to Benci and Cerami [7] for the problem on the whole domain. We shall follow the approach pursued in [42], where a sufficiently general statement including the two previous results is given. To this aim let us begin by introducing some terminology.

We shall call *scaling* of *center* x_0 and *modulus* σ the mapping $\rho : x \mapsto x_0 + \sigma(x - x_0)$. In order to have always the possibility to compose two scalings, we shall include among them also the translations, which are the product of two scalings with inverse moduli and different centers. In such a case there is no center, the modulus is of course 1 and the function is determined by the translation vector. If $(\rho_n)_{n\in\mathbb{N}}$ is a sequence of scalings we shall say that it is *diverging by concentration* if the corresponding sequence of the moduli diverges to $+\infty$. We shall say that it is *diverging by vanishing* if the corresponding sequence of the moduli converges to zero and that it is *diverging by translation* if the corresponding sequence of the moduli is bounded and bounded away from zero and the corresponding sequence of the centers or of the translation vectors is diverging. We shall say that $(\rho_n)_{n\in\mathbb{N}}$ is *diverging* if every subsequence admits a subsequence which is diverging by concentration or by vanishing or by translation. Two sequences of scalings $(\rho_n)_{n\in\mathbb{N}}$ and $(\rho'_n)_{n\in\mathbb{N}}$ are said to be *mutually diverging* if the sequence $((\rho_n)^{-1}\circ\rho'_n)_{n\in\mathbb{N}}$ is diverging. If ρ is a scaling with modulus σ and u is a function defined on $\mathbb{R}^N$, for $\alpha\in\mathbb{R}$ fixed, we shall refer to the function $\lambda^\alpha u\circ\rho$ as the scaled function u by ρ and we shall denote it by the symbol $\rho(u)$. Fixed $1\leqslant p<+\infty$, we shall take $\alpha=N/p$ in order to keep invariant the L^p norm and $\alpha=N/p^*$, for $p<N$, to keep invariant the $H^{1,p}$ norm. We shall transfer to the scalings of the functions the same terminology which we have introduced for the scalings of the variable.

It turns out that a sequence of scalings $(\rho_n)_{n\in\mathbb{N}}$ is diverging if for every u the sequence $(\rho_n(u))_{n\in\mathbb{N}}$ weakly converges to zero or, equivalently, if there exists at least one function $u\neq 0$ such that $(\rho_n(u))_{n\in\mathbb{N}}$ weakly converges to zero. In such a case, we can pass to a subsequence which is diverging by concentration or vanishing or translation.

DEFINITION 1.1. Let $U\subset H^{1,p}(\mathbb{R}^{\mathbb{N}})$ be a bounded subset, we shall say that U has a *bounded scale* if, for every diverging sequence of scalings $(\rho_n)_{n\in\mathbb{N}}$ and for every sequence $(u_n)_{n\in\mathbb{N}}\subset U$, the sequence of scaled functions $(\rho_n(u_n))_{n\in\mathbb{N}}$ weakly converges to zero in $L^{p^*}(\mathbb{R}^N)$.

The following theorem has been proved in [42].

THEOREM 1.5. *Let $(u_n)_{n\in\mathbb{N}}$ be a given bounded sequence of functions in $H^{1,p}(\mathbb{R}^{\mathbb{N}})$, with index p satisfying $1<p<N$. Then, replacing $(u_n)_{n\in\mathbb{N}}$ with a suitable subsequence, we can find a sequence of functions $(\varphi_i)_{i\in\mathbb{N}}$ belonging to $H^{1,p}(\mathbb{R}^{\mathbb{N}})$ and, in correspondence of any index i, we can find a sequence of scalings $(\rho_n^i)_{i\in\mathbb{N}}$ in such a way that the sequence $(\rho_n^i(\varphi)_i)_{i\in\mathbb{N}}$ is summable in $H^{1,p}(\mathbb{R}^{\mathbb{N}})$, uniformly with respect to n, and that the sequence $(u_n-\sum_{i\in\mathbb{N}}\rho_n^i(\varphi)_i)_{n\in\mathbb{N}}$ converges to zero in L^{p^*}. Moreover, we have that, for any pair of indexes i and j, the two corresponding sequences of scalings $(\rho_n^i)_{i\in\mathbb{N}}$ and $(\rho_n^j)_{i\in\mathbb{N}}$ are mutually diverging, that*

$$\sum_{i=0}^{+\infty}\|\varphi_i\|_{1,p}^p\leqslant M, \tag{1.6}$$

where M is the limit of $(\|u_n\|_{1,p}^p)_{n\in\mathbb{N}}$, and that the sequence $(u_n-\sum_{i\in\mathbb{N}}\rho_n^i(\varphi)_i)_{n\in\mathbb{N}}$ converges to zero in $H^{1,p}(\mathbb{R}^{\mathbb{N}})$ if and only if (1.6) *is an equality.*

REMARK 1.3. We notice that the above theorem still holds in the more general context of Lorentz spaces $L(p^*,q)$ for $q>p$, it does not hold in the case $q=p$. Moreover, it is equivalent to state the compact embedding of bounded subsets of $H^{1,p}(\mathbb{R}^{\mathbb{N}})$ with a bounded scale into $L(p^*,q)$ for $q>p$ (see [42]).

REMARK 1.4. In the above theorem all the limits φ_i are the weak limits of $(\rho_n^i)^{-1}(u_n)$. Since for any two indexes i, j the corresponding sequences of scalings are mutually diverging, there exists at most one index i such that ρ_n^i admits the limit scaling. We shall denote by 0 such an index. Thus, it is not restrictive to assume $\rho_n^0=\mathrm{id}$ and so that φ_0 is the weak limit of the sequence. We always reserve the index 0 to this aim, by taking φ_0 as the weak limit of $(u_n)_{n\in\mathbb{N}}$ even when $\varphi_0=0$ and it does not need to be taken into account in Theorem 1.5. For $i\geqslant 1$ we can suppose, by passing to a subsequence, that every sequence $(\rho_n^i)_{n\in\mathbb{N}}$ is diverging by concentration or by vanishing or by translation.

It is quite clear from the above theorem that the deviation from compactness for a PS sequence of the problem at critical growth on a bounded domain can be controlled by sequences of scalings only diverging by concentration. Whereas, for the subcritical problem on the whole domain the analogous phenomenon can be controlled in L^p by scalings only diverging by translation.

DEFINITION 1.2. Let $(u_n)_{n\in\mathbb{N}}\subset H^{1,p}(\mathbb{R}^N)$ be a given sequence. We shall say that $(u_n)_{n\in\mathbb{N}}$ is a fragmented sequence in $H^{1,p}$ or, respectively, in L^q if there exists a finite number $k\geqslant 1$ of functions $\varphi_0,\varphi_1,\dots,\varphi_k$ belonging to $H^{1,p}(\mathbb{R}^N)$ and in correspondence of any index $i\geqslant 1$, there exists a sequence of mutually diverging scalings $(\rho_n^i)_{i\in\mathbb{N}}$ such that the sequence $(u_n-\varphi_0-\sum_i\rho_n^i(\varphi)_i)_{n\in\mathbb{N}}$ converges to zero in $H^{1,p}(\mathbb{R}^N)$ or, respectively, in L^q.

DEFINITION 1.3. Let $(u_n)_{n\in\mathbb{N}}$ be a fragmented sequence. In the case all of the ρ_n^i for $i\geqslant 1$, are diverging by concentration, we shall say that the sequence is concentrating. If all of the ρ_i for $i\geqslant 1$, are diverging by translation we shall say that the sequence is broken.

Now we are in a position to apply the above results to the analysis of the two elliptic problems under consideration. Let us begin with problem (CP).

1.3.1. *Concentrating sequences.* Given $\sigma>0$ and $\bar{x}\in\mathbb{R}^N$, let us consider the following scaled function

$$\rho(u)=u_\sigma : x\mapsto \sigma^{N/2^*}u\big(\bar{x}+\sigma(x-\bar{x})\big),$$

where the choice of the exponent $\alpha=N/2^*$ makes the scaling operation ρ keep constant the norms $\|\nabla u_\sigma\|_2$ and $\|u_\sigma\|_{2^*}$.

In order to produce estimates on the values of solutions u to (SP), we observe that $v=|u|$ (extended by zero out of Ω) solves

$$\begin{cases} -\Delta v\leqslant bv^{2^*-1}+A,\\ v\in H^1\big(\mathbb{R}^N\big), \quad v\geqslant 0, \end{cases} \tag{EI}$$

in the sense of distributions, where b is any coefficient greater than one and $A = -\inf(bs^{2^*-1} - s^{p-1} - \lambda s)$ (taken for $1 \leqslant p \leqslant 2^*$, $s > 0$) is a constant which does not depend on u and p. Since b can be trivially normalized, we shall always take $b = 1$ in (EI). So most of the estimates employed for the solutions to (SP) will be derived for solutions to (EI) in $H^1(\mathbb{R}^N)$ and this will let us free from caring about the sign of u or taking into account the domain Ω.

DEFINITION 1.4. Let $(u_n)_{n\in\mathbb{N}}$ be a given sequence. We shall say that $(u_n)_{n\in\mathbb{N}}$ is

- a *controlled sequence* if each u_n is a solution to (EI),
- a *balanced sequence* if each u_n solves (SP) for some $p \in [2, 2^*]$.

REMARK 1.5. As we have already pointed out, the absolute value of every solution to (SP) (under an extension by zero out of Ω and multiplied by a constant) is also solution to (EI). Therefore any sequence consisting, term by term, of the absolute value of a balanced sequence is a controlled sequence. On the other side, when we shall deal with controlled sequences, we shall know that each term is positive and that $\Omega = \mathbb{R}^N$.

REMARK 1.6. Let $(u_n)_{n\in\mathbb{N}}$ be any PS sequence for I_λ. Then, if a sequence of functions $(\varphi_i)_{i\in\mathbb{N}}$ is as in Theorem 1.5, for every $i \geqslant 1$, φ_i solves (CP) in $H^1(\mathbb{R}^N)$ with $\lambda = 0$ and, in particular, (EI) with $A = 0$.

Since every solution to (EI) with $A = 0$ has the H_0^1 norm greater or equal to a positive constant (see Remark 1.10), then from the above results we get the existence of a constant $C > 0$ such that $\|\varphi_i\|_{H_0^1} > C$ for every $i \geqslant 1$. Since the sequence of such norms is summable, one can conclude that there are only finitely many φ_i. Therefore we are lead to recall the main result due to Struwe in [45] as a corollary of Theorem 1.5 and so we have the following statement which is appropriate to deal with the present situation.

THEOREM 1.6. *Let $(u_n)_{n\in\mathbb{N}}$ be a noncompact PS sequence. Then, by replacing $(u_n)_{n\in\mathbb{N}}$ with a suitable subsequence, there exists a finite number k, depending on a bound M on $\|u_n\|_{H_0^1}$ (namely $k \leqslant MS^{-N/2}$, where S is the Sobolev constant), of global solutions φ_i to* (CP) *in $H^1(\mathbb{R}^N)$ with $\lambda = 0$ with corresponding k sequences of mutually diverging scalings $(\rho_n^i)_{n\in\mathbb{N}}$ with respective concentration points x_n^i and diverging moduli σ_n^i (i.e., $\lim_{n\to+\infty} \sigma_n^i = +\infty$) such that*

$$u_n - \sum_{i=1}^{k} \rho_n^i(\varphi_i) \to \varphi_0 \quad \text{in } H_0^1(\Omega), \tag{1.7}$$

where φ_0 is the weak limit of the sequence and solves (CP).

PROOF. Let $(u_n)_{n\in\mathbb{N}}$ be a PS sequence and let $v_n \in H_0^1$ minimize the distance from $\sum_{i=0}^{k} \rho_n^i(\varphi_i)$ in $H^1(\mathbb{R}^N)$ for every $n \in \mathbb{N}$. Then we have $v_n - \sum_{i=0}^{k} \rho_n^i(\varphi_i) \to 0$

in $H^1(\mathbb{R}^N)$. We notice that $(v_n)_{n\in\mathbb{N}}$ is a PS sequence, since all of the φ_i are solutions to (CP) in $H^1(\mathbb{R}^N)$. Then, by an integration by parts, by denoting by ε_n a small term in the H^{-1} norm, we get

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left|\nabla(u_n - v_n)\right|^2 \mathrm{d}x \\
&\quad= \int_{\Omega} \left|\nabla(u_n - v_n)\right|^2 \mathrm{d}x \\
&\quad= -\int_{\Omega} \Delta(u_n - v_n)(u_n - v_n)\,\mathrm{d}x \\
&\quad= \int_{\Omega} \bigl(|u_n|^{2^*-2}u_n - |v_n|^{2^*-2}v_n + \lambda(u_n - v_n) + \varepsilon_n\bigr)(u_n - v_n)\,\mathrm{d}x \\
&\quad= \int_{\Omega} \bigl(|u_n|^{2^*-2}u_n - |v_n|^{2^*-2}v_n + \lambda(u_n - v_n)\bigr)(u_n - v_n)\,\mathrm{d}x \\
&\qquad+ \int_{\Omega} \varepsilon_n(u_n - v_n)\,\mathrm{d}x.
\end{aligned}$$

Since u_n and v_n are bounded in L^{2^*}, $|u_n|^{2^*-2}u_n$ and $|v_n|^{2^*-2}v_n$ are bounded in $L^{2^{*\prime}}$. Moreover, by Theorem 1.5 we know that $u_n - v_n \to 0$ in L^{2^*}, so we have the first term on the right-hand side converging to zero by duality. Furthermore, since u_n and v_n are bounded in $H^1(\mathbb{R}^N)$ and $\varepsilon_n \to 0$ in H^{-1} we have that also the second term on the right-hand side converges to zero in H^1 and so the thesis is proved. □

Theorem 1.6 says, in other terms, that from any noncompact PS sequence we can extract a concentrating sequence in H^1. Given any concentrating sequence, we shall also consider the scalings ρ_n^i and the limit functions φ_i (which are not uniquely determined by Theorem 1.6) as also given.

The next statement, which can be seen as a variant of Theorem 1.6, allows us to say that also from a noncompact balanced sequence $(u_n)_{n\in\mathbb{N}}$ we can always extract a concentrating sequence, even if we do not know if $(u_n)_{n\in\mathbb{N}}$ is a PS sequence.

LEMMA 1.1. *Let $(u_n)_{n\in\mathbb{N}}$ be a noncompact bounded balanced sequence in $H_0^1(\Omega)$. Then from $(u_n)_{n\in\mathbb{N}}$ we can extract a concentrating subsequence in H_0^1.*

PROOF. Under a null extension of u_n to the whole of $\mathbb{R}^N$, we can use the structure theorem for bounded sequences Theorem 1.5. Then we shall show that for every $i \geqslant 1$, $|\varphi_i|$ solves (EI) with $A = 0$. Indeed, $(\rho_n^i)^{-1}|u_n| \to |\varphi_i|$ and $|u_n|$ solves (EI). If we denote by ν_n the modulus of $(\rho_n^i)^{-1}$ we have $\nu_n \to 0$. By scaling $|u_n|$, we have

$$-\Delta\bigl(\rho_n^i\bigr)^{-1}|u_n| \leqslant \bigl(\rho_n^i\bigr)^{-1}|u_n|^{2^*-1} + \nu_n^{(N+2)/2}A, \tag{1.8}$$

which, passing to the weak limit, gives (EI) for $v = |\varphi_i|$ and $A = 0$. This implies, in particular, that $\|\varphi_i\| \geqslant S^{N/4}$, where S is the Sobolev constant, see Remark 1.10, so we have an estimate on the number of the limits φ_i, $i \geqslant 1$. Finally, we can show that (1.7) holds as in the proof of Theorem 1.6. □

REMARK 1.7. A more detailed argument, see [19], Lemma 6.2, shows that, while φ_0 solves (CP) on Ω, every φ_i for $i \geqslant 1$, is a solution to (CP) on the whole domain for $\lambda = 0$ multiplied by the number

$$\mu = \lim_n \nu_n^{-N(N-2)/4(1-p_n/2^*)} \geqslant 1.$$

From now on we shall denote by σ_n^i the modulus of the scaling ρ_n^i, so that for every given i we have $\sigma_n^i \to +\infty$ as $n \to +\infty$.

For every i, j, through a selection argument, we can suppose that, for every $n \in \mathbb{N}$, $\sigma_n^i \leqslant \sigma_n^j$ (or vice versa), then we can order the indexes in such a way that, for every $n \in \mathbb{N}$, it results $\sigma_n^1 \leqslant \sigma_n^2 \leqslant \cdots \leqslant \sigma_n^k$. Therefore, with such an ordering, we have that ρ_n^1 corresponds to a function concentrating in x_n^1 in the slowest way. For every $n \in \mathbb{N}$, we set $\sigma_n = \sigma_n^1$ and $x_n = x_n^1$.

To the aim of establishing some local uniform estimates around the concentration points, we perform the following construction. In view of making estimates at a distance of the order of $\sigma_n^{-1/2}$ from the concentration point x_n, we need to exclude that, for a suitable constant c, someone of the functions φ_i, for $i \geqslant 1$, could have a concentration point x_n^i closer to $\partial B_{c\sigma_n^{-1/2}}(x_n)$ than $\sigma_n^{-1/2}$ (let us recall that x_n corresponds to the function concentrating in the slowest way), thus we argue as follows. For any $n \in \mathbb{N}$, let us consider k concentric annuli of width $7\sigma_n^{-1/2}$ and centered in x_n. Since, by Theorem 1.6, the total number of global solutions φ_i is k, we are sure that among those annuli there is at least one without any concentration point x_n^i. Let $\mathcal{A}_n^0$ be that annulus. Since $k \leqslant MS^{-N/2}$ does not depend by n, this procedure allows, passing to a subsequence, to choose a constant $\overline{C}$, which does not depend on n, such that $1 \leqslant \overline{C} \leqslant 7k + 1 \leqslant 7MS^{-N/2} + 1$, and such that $\mathcal{A}_n^0 = B_{(\overline{C}+7)\sigma_n^{-1/2}}(x_n) \setminus \overline{B}_{\overline{C}\sigma_n^{-1/2}}(x_n)$. Then we set $\mathcal{A}_n^1 = B_{(\overline{C}+6)\sigma_n^{-1/2}}(x_n) \setminus \overline{B}_{(\overline{C}+1)\sigma_n^{-1/2}}(x_n)$, $\mathcal{A}_n^2 = B_{(\overline{C}+5)\sigma_n^{-1/2}}(x_n) \setminus \overline{B}_{(\overline{C}+2)\sigma_n^{-1/2}}(x_n)$, $\mathcal{A}_n^3 = B_{(\overline{C}+4)\sigma_n^{-1/2}}(x_n) \setminus \overline{B}_{(\overline{C}+3)\sigma_n^{-1/2}}(x_n)$, getting in this way four sequences of annuli, of width $7\sigma_n^{-1/2}$, $5\sigma_n^{-1/2}$, $3\sigma_n^{-1/2}$ and $\sigma_n^{-1/2}$, respectively, such that, for $i = 1, 2, 3$, $\mathcal{A}_n^{i-1}$ is the $\sigma_n^{-1/2}$-neighborhood of $\mathcal{A}_n^i$. So when i increases $\mathcal{A}_n^i$ gets thinner and we are going to establish finer estimates on it. When $i = 0$ we only know that $\mathcal{A}_n^0$ does not contain concentration points x_n^i, we shall see in Section 2 how this rough estimate improves, in the case of a balanced sequence, for $i = 1, 2, 3$. When we shall deal with a balanced sequence $(u_n)_{n\in\mathbb{N}}$, we shall assume to have fixed a constant $\overline{C}$ as above and so we shall consider the four sequences $\mathcal{A}_n^i$ as also given and we shall call such sets *safe regions* of $(u_n)_{n\in\mathbb{N}}$.

1.3.2. *Broken sequences.* We pass now to the analogous analysis for problem (P). Let us begin by considering some inequalities, related to (P), that will be useful in producing estimates on the solutions to the approximating problems:

$$\begin{cases} -\Delta u + a(x)u \leqslant u^{p-1} & \text{in } \mathbb{R}^N, \\ u \geqslant 0 & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{EI$_1$}$$

$$\begin{cases} -\Delta u + a_\infty u \leqslant u^{p-1} & \text{in } \mathbb{R}^N, \\ u \geqslant 0 & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N). \end{cases} \tag{EI$_\infty$}$$

We remark that if u is weak solution of

$$\begin{cases} -\Delta u + a(x)u = |u|^{p-2}u & \text{in } \Omega, \\ u \in H^1_0(\Omega), \end{cases} \tag{P$_\Omega$}$$

$\Omega \subseteq \mathbb{R}^N$, then $|u|$, extended by 0 out of Ω, is a weak solution of (EI$_1$).

According to the previous definitions, we shall say that $(u_n)_{n\in\mathbb{N}}$ is a balanced sequence if, for every n, u_n is a nontrivial weak solution to (AP$_{r_n}$), where $(r_n)_{n\in\mathbb{N}}$, $r_n \in \mathbb{R}^+$, is any sequence so that $r_n > 0$ and that it is a controlled sequence if, for each n, u_n is a nontrivial weak solution to (EI$_1$). Once again we remark that to any balanced sequence $(u_n)_{n\in\mathbb{N}}$ there corresponds a controlled sequence $(v_n)_{n\in\mathbb{N}}$, where $v_n = |u_n|$ in $B_{r_n}(0)$ and $v_n = 0$ in $\mathbb{R}^N \setminus B_{r_n}(0)$.

REMARK 1.8. We see that, given any balanced sequence $(u_n)_{n\in\mathbb{N}}$ and a sequence of translation vectors $(t_n)_{n\in\mathbb{N}}$, $t_n \in \mathbb{R}^N$, $|t_n| \underset{n\to+\infty}{\longrightarrow} +\infty$, if

$$u(\cdot) = \lim_{n\to+\infty} u_n(\cdot - t_n) \quad \text{a.e. in } \mathbb{R}^N,$$

then $|u|$ is a weak solution of (EI$_\infty$).

A basic tool to face problems in unbounded domains has been the analysis of the PS sequences behavior and the information that in the framework of Theorem 1.5, when (1.3) is satisfied, a noncompact PS sequence differs from its weak limit by one or more sequences that, after suitable translations, go to a solution of (P$_\infty$) (see [7] and [4]).

Here, since our aim relies in finding solutions to (P) that are limit of balanced sequences, we need to know how a noncompact bounded balanced sequence can look like. Moreover, since we want to analyze a rather general case by working with (a$_2$) instead than (1.3), we cannot state the existence of a limit equation corresponding to (P). Nevertheless, by virtue of the previous considerations about the behavior of the balanced sequences and taking into account Remark 1.8, it is easy to realize that in our case the role of the limit problem can be played by (EI$_\infty$). The following lemma gives the necessary information leading to the conclusion that the set of norms of the solutions to (EI_∞) is bounded from below by a positive constant.

LEMMA 1.2. *There exists a positive constant* $C_0 > 0$ *such that for any nontrivial solution* φ *to* (EI_∞)

$$\|\varphi\|_p \geqslant C_0 \tag{1.9}$$

holds.

PROOF. Let φ be a nontrivial solution to (EI_∞), then φ satisfies

$$\|\nabla\varphi\|_2^2 + a_\infty\|\varphi\|_2^2 \leqslant \|\varphi\|_p^p.$$

By using Sobolev embedding theorem and by interpolating the L^p norm (taking into account that $2 < p < 2^*$) we have

$$S\|\varphi\|_{2^*}^2 + a_\infty|\varphi|_2^2 \leqslant \|\varphi\|_p^p \leqslant \left(\|\varphi\|_{2^*}^\alpha\|\varphi\|_2^{1-\alpha}\right)^p,$$

where S denotes the best Sobolev constant and $\alpha \in (0,1)$ is such that $\alpha/2^* + (1-\alpha)/2 = 1/p$. By applying Young inequality, we get

$$\begin{aligned}
\|\varphi\|_{2^*}^\alpha\|\varphi\|_2^{1-\alpha} &\leqslant \alpha\|\varphi\|_{2^*} + (1-\alpha)\|\varphi\|_2 \\
&\leqslant k_1\left[\left(\sqrt{S}\|\varphi\|_{2^*} + \sqrt{a_\infty}\|\varphi\|_2\right)^2\right]^{1/2} \\
&\leqslant k_1 2^{1/2}\left(S\|\varphi\|_{2^*}^2 + a_\infty\|\varphi\|_2^2\right)^{1/2},
\end{aligned}$$

where k_1 is chosen so that $k_1 \geqslant \max(\alpha/\sqrt{S}, (1-\alpha)/\sqrt{a_\infty})$. Hence,

$$\left(S\|\varphi\|_{2^*}^2 + a_\infty\|\varphi\|_2^2\right)^{p/2-1} \geqslant \frac{1}{k_1 2^{p/2}}$$

and so we deduce, as desired, $\|\varphi\|_p \geqslant C_0 > 0$, where C_0 is a constant not depending on φ. □

Taking advantage of Lemma 1.2 and by using the previous arguments, we can state the following assertions as corollaries of Theorem 1.5. Such results provide the desired picture of the controlled and therefore of the balanced sequences behavior.

PROPOSITION 1.1. *Let* $a(x)$ *satisfy* (a_1) *and* (a_2). *Let* $(u_n)_{n\in\mathbb{N}}$ *be a noncompact controlled sequence bounded in* $H^1(\mathbb{R}^N)$. *Then, there exists a subsequence* (*still denoted by* u_n) *for which the following holds*: *there exist an integer* $k > 0$, *nontrivial solutions to* (EI_∞) φ_i,

$1 \leqslant i \leqslant k$, *sequences* $(t_n^i)_{n\in\mathbb{N}}$, $1 \leqslant i \leqslant k$, *such that*

$$\begin{aligned} &u_n - \sum_{i=1}^{k} \varphi_i\left(\cdot - t_n^i\right) \to \varphi_0 \quad \textit{in } L^p\left(\mathbb{R}^N\right), \\ &\left|t_n^i\right| \underset{n\to+\infty}{\longrightarrow} +\infty, \qquad \left|t_n^i - t_n^j\right| \underset{n\to+\infty}{\longrightarrow} +\infty, \quad 1 \leqslant i \neq j \leqslant k. \end{aligned} \tag{1.10}$$

Now we are going to apply the same idea, used in dealing with the concentrating sequences, to the present case of diverging sequences. Specifically, we shall proceed in ordering the sequence of translation vectors and in constructing suitable regions on which we shall establish local uniform estimates. Thus, given any broken sequence $(u_n)_{n\in\mathbb{N}}$, we assume as given also the functions φ_i and the translation vectors t_n^i (even if they are not uniquely determined) that appear in (1.10). Through a selection argument, we can suppose that, for every $n \in \mathbb{N}$, $|t_n^i| \leqslant |t_n^j|$ (or vice versa), then we can order the indexes in such a way that, for every $n \in \mathbb{N}$, it results $|t_n^1| \leqslant |t_n^2| \leqslant \cdots \leqslant |t_n^k|$. For every $n \in \mathbb{N}$, we set $t_n = t_n^1$ the basic sequence of translations. In view of constructing the safe regions of the space to associate to any broken sequence, we introduce the following terminology.

DEFINITION 1.5. Let $A \subset \mathbb{R}^N$ be a subset of $\mathbb{R}^N$ and $v \in \mathbb{R}^N$ a point $v \notin A$. We call *cone of vertex* v *generated by* A the smallest set containing A and positively homogeneous with respect to the vertex v, i.e., the set

$$\left\{w \in \mathbb{R}^N \mid w = v + \lambda(x - v), x \in A, \lambda \in \mathbb{R}^+\right\}.$$

Let $(u_n)_{n\in\mathbb{N}}$ be a broken sequence and let $(t_n)_{n\in\mathbb{N}}$ be the above defined sequence. In view of making estimates involving diverging sequences, we shall work in the *safe regions* connected with the basic sequence of translations in which we can deduce some a priori estimates which are not affected by the presence of other masses which are escaping to infinity. This time we shall not be concerned with annuli (so set differences of concentric balls) centered in x_n but with set differences of coaxial cones with the axis parallel to t_n. In order to avoid the other masses, we have to perform a similar argument to the one used for (CP). To this aim we proceed in constructing the following sequences of subsets of $\mathbb{R}^N$ related to $(t_n)_{n\in\mathbb{N}}$. For any $n \in \mathbb{N}$, let us consider the cone $\mathcal{C}_n$ with vertex $t_n/2$ and generated by a ball $B_{R_n}(t_n)$. We begin by taking the cone $\mathcal{C}_{1,n}$ generated by the ball $B_{1,n} = B_{r_n}(t_n)$, where

$$r_n = \frac{\hat{\gamma}}{k}\frac{|t_n|}{2} \quad \text{with } 0 < \hat{\gamma} < \min\left(\frac{1}{5}, \frac{1}{4(\bar{c}+1)}\right), \tag{1.11}$$

$\bar{c}$ being the constant appearing in (a_4). If $\partial\mathcal{C}_{1,n} \cap B_{r_n/2}(t_n^i) = \emptyset$ for all $t_n^i \neq t_n$, $1 \leqslant i \leqslant k$, we set $\mathcal{C}_n = \mathcal{C}_{1,n}$ and $R_n = r_n$, otherwise we consider the larger cone $\mathcal{C}_{2,n}$ having vertex $t_n/2$ and generated by $B_{2r_n}(t_n)$. Since $|t_n| \leqslant |t_n^i|$, $1 \leqslant i \leqslant k$, for any index i for which $\partial\mathcal{C}_{1,n} \cap B_{r_n/2}(t_n^i) \neq \emptyset$, we have $B_{r_n/2}(t_n^i) \subset \mathcal{C}_{2,n}$, and we set $\mathcal{C}_n = \mathcal{C}_{2,n}$ if $\partial\mathcal{C}_{2,n}$ does not touch any of the other balls $B_{r_n/2}(t_n^i)$, $t_n^i \neq t_n$. Otherwise we pass to the cone $\mathcal{C}_{3,n}$, having vertex $t_n/2$, generated by $B_{3r_n}(t_n)$ that surely contains the balls, of radius $r_n/2$ centered at the points t_n^i, touching $\partial\mathcal{C}_{2,n}$.

We iterate this procedure and, since the number of the functions φ_i is k, we are sure that after at most k steps, we can associate to t_n a cone $\mathcal{C}_n$, having vertex $t_n/2$, generated by a ball $B_{R_n}(t_n)$, with $\frac{\hat{\gamma}}{k}\frac{|t_n|}{2} = r_n \leqslant R_n \leqslant kr_n = \hat{\gamma}\frac{|t_n|}{2}$, having the property that $\partial\mathcal{C}_n \cap B_{r_n/2}(t_n^i) = \emptyset$, for any index i, $1 \leqslant i \leqslant k$, such that $t_n^i \neq t_n$.

REMARK 1.9. Let θ_n denote the width angle of the cone $\mathcal{C}_n$. We emphasize that, since $R_n = \frac{|t_n|}{2}\tan\theta_n$, then

$$0 < \frac{\hat{\gamma}}{2k} \leqslant \tan\theta_n \leqslant \hat{\gamma} < \min\left(\frac{1}{5}, \frac{1}{4(\bar{c}+1)}\right).$$

Now we introduce some tools which will be useful in dealing with problem (P). Let $s \in \mathbb{R}$ and $n \in \mathbb{N}$, we consider the cones

$$\mathcal{C}_{s,n} = \mathcal{C}_n - s\vec{t}_n \tag{1.12}$$

and the regions around the boundary of $\mathcal{C}_n$

$$\mathcal{S}_{2s,n} = \mathcal{C}_{s,n} \setminus \mathcal{C}_{-s,n}. \tag{1.13}$$

Lastly we set

$$\mathcal{S}_n = \mathbb{R}^N \Big/ \bigcup_{i=0}^{k} B_{r_n/2}\big(t_n^i\big). \tag{1.14}$$

1.4. *Natural constraint*

The existence and the multiplicity results related to problems (CP) and (P) will be achieved by working with variational methods on the so-called natural constraint manifold, which is a subset $\mathcal{V}$ of H_0^1 which contains all the critical points of a functional I and such that every constrained critical point of I on $\mathcal{V}$ is a critical point with respect to the whole space. We shall introduce this concept for the case of (CP), the other case is analogous.

Let

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_\Omega |u|^2 - \frac{1}{p}\int_\Omega |u|^p \tag{1.15}$$

be the functional corresponding to (SP). We introduce the *natural constraint* $\mathcal{V}$ for the functional I as the manifold defined by

$$\mathcal{V} = \big\{u \mid u \neq 0, \nabla I(u)\cdot u = 0\big\}. \tag{1.16}$$

Then the defining equation for the natural constraint is

$$\Phi(u) = \int_\Omega |\nabla u|^2 - \lambda \int_\Omega |u|^2 - \int_\Omega |u|^p = 0. \tag{1.17}$$

PROPOSITION 1.2. *For every $u \in \mathcal{V}$, the following properties hold true*:

(i) $I(u) > 0$;

(ii) $I(u) > c > 0$, *if* $\lambda < \lambda_1$;

(iii) $I(u) \geqslant \frac{1}{N} S^{N/2}$, *if* $\lambda = 0$ *and* $p = 2^*$, *where* S *is the Sobolev constant*.

PROOF. By (1.17), a simple computation gives that, for every $u \in \mathcal{V}$,

$$I(u) = \left(\frac{1}{2} - \frac{1}{p}\right)\left(\int_\Omega |\nabla u|^2 - \lambda \int_\Omega |u|^2\right) = \left(\frac{1}{2} - \frac{1}{p}\right) \int_\Omega |u|^p > 0. \tag{1.18}$$

So by (1.18), (i) follows. If $\lambda < \lambda_1$, then

$$\int_\Omega |\nabla u|^2 - \lambda \int_\Omega |u|^2 \geqslant c\|u\|^2_{H^1(\Omega)} \geqslant c\|u\|_p^2$$

and hence, by (1.17),

$$\|u\|_p^2 \leqslant c\|u\|_p^p, \tag{1.19}$$

from which we have by the Sobolev embedding $\|u\|_p \geqslant c$. Finally, by combining this last inequality with (1.18) we have

$$I(u) = \left(\frac{1}{2} - \frac{1}{p}\right) \int_\Omega |u|^p \geqslant c$$

and so we get (ii). Moreover, when $\lambda = 0$ and $p = 2^*$, $u \in \mathcal{V}$ implies $\|\nabla u\|_2^2 = \|u\|_{2^*}^{2^*}$ therefore, by the Sobolev inequality, we have $S \leqslant \|u\|_{2^*}^{2^*-2}$ and so $\|u\|_{2^*}^{2^*} \geqslant S^{2^*/(2^*-2)} = S^{N/2}$. Then, by (1.18), we finally get

$$I(u) = \left(\frac{1}{2} - \frac{1}{2^*}\right)\|u\|_{2^*}^{2^*} \geqslant \frac{1}{N} S^{N/2}$$

from which (iii) follows. □

REMARK 1.10. The proof of (iii) also applies to positive solutions to (EI) when $A \leqslant 0$, since we only use the inequality $\|\nabla u\|_2^{2^*} \leqslant \|u\|_2^{2^*}$, trivially implied by (EI).

The main property of $\mathcal{V}$ is stated in the following proposition.

PROPOSITION 1.3. *Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{V}$ be a constrained PS sequence for I_λ. Then $(u_n)_{n\in\mathbb{N}}$ is a PS sequence for I_λ.*

PROOF. Passing to a subsequence we can assume $I(u_n) \to c \geqslant 0$. If $c = 0$, by (1.18) we get $\|u\|_p \to 0$, so $\|u\|_2 \to 0$ by Hölder inequality and finally $\|u\|_{H^1} = (\|u\|_p^p + \lambda\|u\|_2^2) \to 0$ by the constraint equation (1.17). So $u_n \to 0$ and therefore $(u_n)_{n\in\mathbb{N}}$ is a PS sequence. Thus let us assume $c > 0$. For every $n \in \mathbb{N}$, we have

$$\nabla I(u_n) = \mu_n \nabla \Phi(u_n) + r_n,$$

where r_n is an infinitesimal term in $H^{-1}(\Omega)$ and μ_n is the Lagrange multiplier. Then, for every $n \in \mathbb{N}$,

$$\nabla I(u_n) = \mu_n\big(\nabla^2 I(u_n) \cdot u_n + \nabla I(u_n)\big) + r_n. \tag{1.20}$$

After an easy computation, multiplying both the sides of (1.20) by u_n and integrating, we get from (1.20)

$$\begin{aligned}(1-\mu_n)&\left(\int_\Omega |\nabla u_n|^2 - \lambda\int_\Omega |u_n|^2 - \int_\Omega |u_n|^p\right)\\ &= \mu_n\left(\int_\Omega |\nabla u_n|^2 - \lambda\int_\Omega |u_n|^2 - (p-1)\int_\Omega |u_n|^p\right) + \int_\Omega r_n \cdot u_n,\end{aligned}$$

and, by taking into account the constraint equation (1.17), we have

$$(p-2)\mu_n \int_\Omega |u_n|^p = \int_\Omega r_n \cdot u_n.$$

Now, by estimating the term on the right-hand side of the previous equation and by (1.17), we obtain

$$\begin{aligned}\int_\Omega r_n \cdot u_n &\leqslant \|r_n\|_{H^{-1}(\Omega)}\|u_n\|_{H^1(\Omega)}\\ &\leqslant \|r_n\|_{H^{-1}(\Omega)}\big(\|u_n\|_p^p + \lambda\|u_n\|_2^2\big)^{1/2},\end{aligned}$$

where c is a positive constant. So by Hölder inequality,

$$(p-2)\mu_n \leqslant c\|r_n\|_{H^{-1}(\Omega)}\big(\|u_n\|_p^{-p/2} + \|u_n\|_p^{1-p}\big).$$

Since $\|r_n\|_{H^{-1}(\Omega)} \to 0$ and $\|u_n\|_p^p \to \frac{2p}{p-2}c > 0$, by virtue of (1.18) and the condition $c > 0$, we get $\mu_n \to 0$ as $n \to \infty$. □

COROLLARY 1.7. *Let $u \in \mathcal{V}$ be a constrained critical point for I_λ. Then u is critical point for I_λ.*

1.5. *Existence of a nontrivial solution for the problem at critical growth*

In this subsection we shall only sketch the main arguments involved in the existence theory in the case $N \geqslant 4$ and $\lambda < \lambda_1$, we refer to [46] and the references therein for a more detailed treatment of the subject. On the other hand, the existence result is trivially implied by the multiplicity theorem which we are going to prove in Section 4 and which does not make use of this existence result. The only reason for which we are giving this sketch is to show the difference between a compactness argument based on a level estimate, which is enough to give the existence of a nontrivial solution and the compactness techniques required for the multiplicity results.

We recall that for $\lambda < \lambda_1$, since zero is an isolated point for $\mathcal{V}$, we also have $\inf_{\mathcal{V}} I_\lambda > 0$. To prove the existence of a nontrivial solution to the minimization problem for I_λ we shall assume to work with any minimizing sequence on $\mathcal{V}$ and then, by applying the results of Section 1.3 and after some estimates, we shall conclude that along any of such a sequence the functional cannot reach a level corresponding to a noncompact PS sequence and so, by recovering compactness, the standard variational methods allow to state the existence of a solution.

COROLLARY 1.8. *Let $(u_n)_{n\in\mathbb{N}}$ be a noncompact PS sequence for I_λ. Then $I_\lambda(u_n) \to c \geqslant N^{-1}S^{N/2}$ as $n \to \infty$.*

PROOF. Let $(u_n)_{n\in\mathbb{N}}$ be a noncompact PS sequence, by Theorem 1.6 we know that $u_n \to \varphi_0 + \sum_{i=1}^k \varphi_i$, where φ_0 solves the differential equation in (CP) on Ω and the functions φ_i, $i = 1, \ldots, k$, are concentrated solutions on $\mathbb{R}^N$ with $\lambda = 0$. Then

$$I_\lambda(u_n) \to I_\lambda(\varphi_0) + \sum_{i=1}^{k} I_0(\varphi_i)$$

as $n \to \infty$ and can easily estimate the terms in which the functional is split in the limit. Indeed, by virtue of Proposition 1.2(iii), we have that for every $i \geqslant 1$,

$$I_0(\varphi_i) \geqslant N^{-1}S^{N/2}.$$

So we can conclude that $\sum_{i=1}^k I_0(\varphi_i) \geqslant N^{-1}S^{N/2}$. Since $I_\lambda(\varphi_0) \geqslant 0$, we have the thesis. □

At this stage to conclude the argument regarding the existence of a nontrivial solution to (CP) we only have to show that really $\inf_{\mathcal{V}} I_\lambda < N^{-1}S^{N/2}$. To this aim we set $N \geqslant 4$ and we consider the family of *Talenti functions* u_σ, that is, $u_\sigma(x) = \sigma^{(N-2)/2}u(\sigma x)$, where $u(x) = (N(N-2))^{(N-2)/4}(1+|x|^2)^{(2-N)/2}$. We may assume that $0 \in \Omega$ and we can choose $\eta \in C_0^\infty(\Omega)$ be a fixed cut-off function such that $\eta = 1$ in a neighborhood $B_R(0)$ of 0. We set $u_\sigma^* = \eta u_\sigma$. After some computations based on the fact that u_σ optimize the Sobolev embedding and so $I_0(u_\sigma) = N^{-1}S^{N/2}$, we get that, for $\sigma > 0$ large enough, $\sup_{\alpha\in\mathbb{R}} I_\lambda(\alpha u_\sigma^*) < N^{-1}S^{N/2}$. Indeed, the negative contribution due to the subcritical term $-\lambda\int_\Omega |u|^2$ turns out to be less infinitesimal than the positive variation of $I_0(\eta u_\sigma)$

with respect to $I_0(u_\sigma) = N^{-1}S^{N/2}$, due to the presence of the cut-off function η, when σ is sufficiently large. So we can find $\alpha \in \mathbb{R}$ such that $\alpha u_\sigma^* \in \mathcal{V}$ and therefore we have $\inf_{\mathcal{V}} I_\lambda < N^{-1}S^{N/2}$ and $\inf_{\mathcal{V}} I_\lambda > 0$ if $\lambda < \lambda_1$. Consequently we get the existence of a nontrivial solution to (CP).

1.6. *Existence of a nontrivial solution for the problem on the whole domain*

Also in this case we shall only give a brief sketch of the proof in a particular case, with the same motivations as in the previous part.

Let

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + a(x)u^2\right) \mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p \,\mathrm{d}x$$

be the functional associated to the problem (P). Let us assume that $\lim_{|x|\to+\infty} a(x) = a_\infty > 0$, $a(x) < a_\infty$ for every $x \in \mathbb{R}^N$ and let us consider the corresponding functional at infinity

$$I_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + a_\infty u^2\right) \mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p \,\mathrm{d}x$$

which is associated to problem

$$-\Delta u + a_\infty u = |u|^{p-2}u \quad \text{in } \mathbb{R}^N. \tag{P$_\infty$}$$

We introduce the natural constraints $\mathcal{V}$ and $\mathcal{V}_\infty$ related to I and I_∞, respectively. By arguing as in the previous case, since $u = 0$ is an isolated point for $\mathcal{V}$, we begin by observing that

$$I(u) > 0 \quad \forall u \in \mathcal{V}.$$

We set $c_\infty = \inf_{u\in\mathcal{V}_\infty} I_\infty(u)$. As we have previously observed, in this case the failure of compactness for a PS sequence can be controlled by scalings diverging by translation and, in order to get the existence result, we have to exclude this case for a constrained minimizing sequence on $\mathcal{V}$. Thus, let $(u_n)_{n\in\mathbb{N}}$ be a noncompact PS sequence for I; by Theorem 1.6, we know that $I(u_n) \to I(\varphi_0) + \sum_{i=1}^k I_\infty(\varphi_i)$, where φ_0 solves the differential equation in (P), and the functions φ_i, $i = 1, \dots, k$, are solutions to (P$_\infty$) and the sequences of scalings $\rho_n^i(\varphi_i)$ are diverging by translation. So, for such a sequence we have $I(u_n) \to c \geqslant c_\infty$ and this shows that compactness is achieved for energy levels strictly lower than c_∞. Therefore, to get the existence of a nontrivial solution to (P), it remains to show that $\inf_{\mathcal{V}} I(u) < c_\infty$. To this aim let u_∞ be a ground state solution to (P$_\infty$), fix $\alpha \in \mathbb{R}$ such that $\alpha u_\infty \in \mathcal{V}$. Then we have $I_\infty(u_\infty) = \max_{\alpha\in\mathbb{R}} I_\infty(\alpha u_\infty)$ and so

$$I(\alpha u_\infty) < I_\infty(\alpha u_\infty) \leqslant I_\infty(u_\infty) = c_\infty,$$

from which the condition $\inf_{\mathcal{V}} I(u) < c_\infty$ follows.

2. Approximating problems and compactness of balanced sequences

This section is devoted to the proof of the two compactness theorems (Theorems 1.1 and 1.3), which are the most relevant step for proving the multiplicity of solutions (the variational approach through approximating problems discussed in Section 4 presents some difficulties but uses more standard ideas). We shall make use here of some estimates on the solutions which we shall justify roughly, by explaining why we can expect them, but which will be rigorously proved in the next section, which must be therefore considered as an essential part of the present one and which will make the proofs complete. Nevertheless, we prefer to develop the estimates in a separate section at the end, letting the reader already know how they should be and should be used. Let us begin by pointing out that no similar compactness result holds for PS sequences, as stated in the following easy remark.

REMARK 2.1. *Problems* (CP) *and* (P) *admit noncompact* PS *sequences.*

PROOF. For (CP) we just have to consider the cut-off Talenti function ηu_σ considered in the end of Section 1.5 by letting $\sigma = \sigma_n \to +\infty$. For (P), when a has limit a_∞ at infinity, we just have to fix a solution $\bar{u}$ to (P_∞) and take $u_n(x) = \bar{u}(x + t_n)$ with $|t_n| \to +\infty$. □

The impossibility of proving complete compactness results for PS sequences is the reason for which we are working with balanced sequences. The elements of a PS sequence are close to be solutions of the problem while the elements of a balanced sequence are *real* solutions of a close problem and this makes a big difference: when we deal with the terms of a balanced sequence, we know that we cannot have even a small improvement of some functional of the same type under any local modification of the same order.

On the other hand, when some masses are concentrating or escaping to infinity, we shall be able to produce a local modification which improves the value of the functional by respectively perturbing the concentration parameter or the translation vector. From this contradiction we shall be able to deduce the compactness theorems. The variation of the functional under such a local modification will be evaluated by a local Pohozaev inequality and the a priori decay estimates on the terms of a balanced sequence which will be found in the next subsection, carried in such inequalities, will formally produce the contradiction.

2.1. *Avoiding concentration*

In this subsection, we shall test the presence of concentrations which would prevent us to find solutions to (CP) as limits of a balanced concentrating sequence. To this aim, we shall evaluate the infinitesimal variation of the functional corresponding to (SP) under a scaling of a concentrated part of u_n. Such a variation must be null because we are dealing with a balanced sequence.

2.1.1. *Local Pohozaev identity.* The property that the variation of the functional under a scaling operation is null or reduced to a boundary term is equivalent to the well-known Pohozaev identity [37] which we must establish in a local form (namely without using boundary conditions) since it shall be tested on a small concentrated part of the functions u_n. We fix a general open smooth set B in $\mathbb{R}^N$ and shall consider, more in general, a semilinear elliptic equation of the form

$$-\Delta u = g(u). \tag{2.1}$$

LEMMA 2.1. *Let u be a smooth solution to* (2.1) *on a smooth domain B and let $G(u)$ be a primitive of the function $g(u)$. Then the following equation holds true*

$$N\int_B\left(G(u)+\frac{1}{2^*}g(u)u\right) = -\int_{\partial B}\left(\frac{1}{2}|\nabla u|^2(x\cdot\vec{n}) - G(u)(x\cdot\vec{n})\right) + \int_{\partial B}(\nabla u\cdot x)(\nabla u\cdot\vec{n}) + \frac{N}{2^*}\int_{\partial B}(\nabla u\cdot\vec{n})u. \tag{2.2}$$

PROOF. Multiplying by u and integrating by parts, we get

$$\int_B |\nabla u|^2 = \int_B g(u)u + \int_{\partial B}(\nabla u\cdot\vec{n})u, \tag{2.3}$$

where $\vec{n}$ is the outward normal to ∂B. Multiplying (2.1) for $\nabla u\cdot x$, since

$$\nabla\cdot\big((\nabla u\cdot x)\nabla u\big) = \Delta u(\nabla u\cdot x) + \big(\nabla(\nabla u\cdot x)\big)\cdot\nabla u,$$

using the divergence theorem and by integrating by parts, we get

$$\begin{aligned}
&\int_B -\Delta u(\nabla u\cdot x)\\
&\quad = -\int_{\partial B}(\nabla u\cdot x)(\nabla u\cdot\vec{n}) + \int_B \nabla u\cdot\big(\nabla^2 u\cdot x + I\cdot\nabla u\big)\\
&\quad = -\int_{\partial B}(\nabla u\cdot x)(\nabla u\cdot\vec{n}) + \int_B \nabla\left(\frac{1}{2}|\nabla u|^2\right)\cdot x + \int_B |\nabla u|^2\\
&\quad = -\int_{\partial B}(\nabla u\cdot x)(\nabla u\cdot\vec{n}) + \frac{1}{2}\int_{\partial B}|\nabla u|^2(x\cdot\vec{n}) + \frac{2-N}{2}\int_B|\nabla u|^2.
\end{aligned}\tag{2.4}$$

On the other side, integrating by parts we get

$$\int_B g(u)(\nabla u\cdot x) = \int_B \nabla G(u)\cdot x = \int_{\partial B} G(u)(x\cdot\vec{n}) - N\int_B G(u). \tag{2.5}$$

Combining (2.18) with (2.5) we obtain

$$\frac{N}{2^*}\int_B|\nabla u|^2 = N\int_B G(u) - \int_{\partial B}G(u)(x\cdot\vec{n}) - \int_{\partial B}(\nabla u\cdot x)(\nabla u\cdot\vec{n}) + \frac{1}{2}\int_{\partial B}|\nabla u|^2(x\cdot\vec{n}). \tag{2.6}$$

Finally, multiplying (2.3) for $-N/2^*$ and summing (2.6), we have (2.2). □

In the present case (i.e., $g(u) = |u|^{p-2}u + \lambda u$), (2.2) becomes

$$\left(\frac{N}{p} - \frac{N}{2^*}\right)\int_B |u|^p + \lambda \int_B |u|^2$$
$$= \frac{1}{p}\int_{\partial B} |u|^p (x \cdot \vec{n}) + \frac{\lambda}{2}\int_{\partial B} |u|^2 (x \cdot \vec{n}) + \int_{\partial B} (\nabla u \cdot x)(\nabla u \cdot \vec{n})$$
$$- \frac{1}{2}\int_{\partial B} |\nabla u|^2 (x \cdot \vec{n}) + \frac{N}{2^*}\int_{\partial B} (\nabla u \cdot \vec{n}) u. \tag{2.7}$$

By a translation, we can move the origin to any fixed point $x_0 \in \mathbb{R}^N$ and we can forget, being $p \leqslant 2^*$, the positive term $(\frac{N}{p} - \frac{N}{2^*})\int_B |u|^p$, in order to obtain the following "*Pohozaev-type*" inequality

$$\lambda \int_B |u|^2 \leqslant \frac{1}{p}\int_{\partial B} |u|^p \big((x - x_0) \cdot \vec{n}\big) + \frac{\lambda}{2}\int_{\partial B} |u|^2 \big((x - x_0) \cdot \vec{n}\big)$$
$$+ \int_{\partial B} \big(\nabla u \cdot (x - x_0)\big)(\nabla u \cdot \vec{n})$$
$$- \frac{1}{2}\int_{\partial B} |\nabla u|^2 \big((x - x_0) \cdot \vec{n}\big) + \frac{N}{2^*}\int_{\partial B} (\nabla u \cdot \vec{n}) u, \tag{2.8}$$

which we shall apply to the terms u_n of a balanced sequence, which enjoys (2.8) for $p = p_n$, on a suitable ball $B = B_n$.

2.1.2. *Decay tools.* The choice of the set B_n in (2.8) is a crucial point in order to produce the contradiction. We shall take as B_n a ball around the concentration point x_n trying to let B_n contain most of the concentrating mass, making the left-hand side of (2.8) consistent. On the other hand, we must force the right-hand side to be small and, to this aim, we have to take into account two opposite indications: (a) B_n must have a suitably small radius in order to keep the measure of the integration domain ∂B_n small; (b) B_n must have a suitably large radius to keep the points of ∂B_n far away from the concentration in order to make the term u and ∇u which appear in the integrals small. In order to guess a convenient choice of the radius, we can focus on the simple case in which the number k which appears in Definition 1.2 is 1, $u_n = \varphi_0 + \rho_n^1(\varphi_1)$ exactly and φ_1 is a Talenti function (defined at the end of Section 1.5). Since φ_0 is a smooth function, we easily find the bound

$$\forall n \in \mathbb{N},\ \forall x \in \Omega: \quad \big|u_n(x)\big| \leqslant c\left(1 + \left(\frac{\sigma_n}{1 + (\sigma_n |x - x_n|)^2}\right)^{(N-2)/N}\right). \tag{2.9}$$

So the values of $|u_n|$ are of the order of 1 when $|x - x_n|$ reaches the order of $\sigma_n^{-1/2}$. In such points we also have the bound

$$\forall n \in \mathbb{N},\ \forall x \in \Omega: \quad \big|u_n(x)\big| \leqslant c\sigma_n^{1/2}. \tag{2.10}$$

So a natural choice is to take ${\sigma_n}^{-1/2}$ as the radius of B_n and to bring (2.9) and (2.10) into the right-hand side of (2.8). In the general situation we must take care of several singularities corresponding to unknown functions φ_i and of the difference $u_n - \sum_{i=0}^{k} \rho_n^i(\varphi_i)$ which is infinitesimal only in H^1. So we shall work on the safe regions $\mathcal{A}_n^i$ which are annuli at a distance of the order of ${\sigma_n}^{-1/2}$ from the less concentrated mass and avoid the other concentrations. In the next subsection we shall prove the following lemma which essentially gives (2.9) in the most general setting.

PROPOSITION 2.1. *Let $(u_n)_{n\in\mathbb{N}}$ be a controlled concentrating sequence. Then there exists a constant $C > 0$ such that for any $n \in \mathbb{N}$ and for any $x \in \mathcal{A}_n^2$,*

$$u_n(x) \leqslant C.$$

Passing to the smaller annulus $\mathcal{A}_n^3$ we can also give (2.10) in an integral form.

PROPOSITION 2.2. *Let $(u_n)_{n\in\mathbb{N}}$ be a controlled concentrating sequence. Then there exists a constant $C > 0$ such that for any $n \in \mathbb{N}$,*

$$\int_{\mathcal{A}_n^3} |\nabla u_n|^2 \leqslant C\sigma_n^{(2-N)/2}. \tag{2.11}$$

A simple mean value argument allows us to deduce the following corollary which gives (2.10) in a boundary integral form.

COROLLARY 2.1. *Let $(u_n)_{n\in\mathbb{N}}$ be a controlled concentrating sequence. For any $n \in \mathbb{N}$ there exists $t_n \in [\overline{C} + 2, \overline{C} + 3]$ such that, denoting by B_n the set $B(x_n, t_n{\sigma_n}^{-1/2})$,*

$$\int_{\partial B_n} |\nabla u_n|^2 \leqslant C\sigma_n^{(3-N)/2}, \tag{2.12}$$

where C is the constant in the above proposition.

REMARK 2.2. The ball B_n appearing in the previous corollary is not yet, in general, the set on which (2.8) is going to be tested. Indeed, we are not sure that $B_n \subset \Omega$. This inclusion is not relevant as far as we work with a controlled sequence, whose terms can be assumed defined on the whole of $\mathbb{R}^N$, but must be forced if we want to deal with a balanced sequence in view of applying the Pohozaev inequality (2.8).

2.1.3. *Proof of Theorem 1.1.* In this subsection we shall use the local Pohozaev identity to prove that concentrations are not possible for balanced sequences in dimension $N \geqslant 7$.

LEMMA 2.2. *If $N \geqslant 7$ no concentrating sequence can be balanced.*

PROOF. Let a concentrating sequence $(u_n)_{n\in\mathbb{N}}$ be given and assume by contradiction that it is balanced. Let us fix $n \in \mathbb{N}$, we shall use (2.8) on $B_n = B(x_n, t_n{\sigma_n}^{-1/2}) \cap \Omega$, where

t_n is the same as in Corollary 2.1, and we shall split $\partial B_n = \partial_i B_n \cup \partial_e B_n$, where $\partial_e B_n$ (empty in the case in which the concentration point x_n of the basic rescaled function φ is sufficiently far from $\partial \Omega$) is $\partial \Omega \cap \overline{B}_n$. When $\partial_e B_n = \emptyset$, to the aim of applying (2.8), we shall take x_0 equal to the concentration point x_n. Otherwise we shall take x_0 out of Ω such that $d(x_0, x_n) \leqslant 2t_n \sigma_n^{-1/2}$ and

$$\forall x \in \partial_e B_n: \quad \vec{n} \cdot (x - x_0) < 0, \tag{2.13}$$

where $\vec{n}$ is the outward normal to ∂B_n (roughly speaking x_0 is the "symmetric" of x_n with respect to $\partial \Omega$). We want to show that (2.8) cannot hold true, in contradiction to the assumption that the sequence is balanced. To this aim, we give a lower bound to the left-hand side of (2.8) and a smaller upper bound to the right-hand side. For the first one, we shall restrict the integral on the ball $B_n' = B(x_n, \sigma_n^{-1})$, which is contained in Ω for n large. Then we have

$$\int_{B(x_n, \sigma_n^{-1})} u_n^2 = \sigma_n^{-2} \int_{B(x_n, 1)} \bigl((\rho_n^i)^{-1}(u_n)\bigr)^2 \geqslant \text{const},$$

since, by Remark 1.4, $(\rho_n^i)^{-1}(u_n) \to \varphi_i \neq 0$ and x_n is bounded in $\mathbb{R}^N$, we see that the left-hand side of (2.8) has a lower bound of the form $C\sigma_n^{-2}$, for a suitable constant C. Passing to the right-hand side, we firstly evaluate the possible contributions of $\partial_e B_n$. On such a set only two of the integrals must be taken into account because we have $u_n = 0$ on $\partial_e B_n \subset \partial \Omega$. For the same reason, ∇u_n has the direction of $\vec{n}$ and so the whole sum, from (2.13), can be written as

$$\frac{1}{2} \int_{\partial_e B_n} |\nabla u_n|^2 (x - x_0) \cdot \vec{n} \leqslant 0.$$

So we can focus our attention to the integrals extended to $\partial_i B_n$. Hence from Proposition 2.1, we get

$$\begin{aligned}
&\frac{\lambda}{2} \int_{\partial_i B_n} |u_n|^2 \bigl((x - x_0) \cdot \vec{n}\bigr) + \frac{1}{p} \int_{\partial_i B_n} |u_n|^p \bigl((x - x_0) \cdot \vec{n}\bigr) \\
&\quad \leqslant C \int_{\partial_i B_n} \bigl((x - x_0) \cdot \vec{n}\bigr) \leqslant C\sigma_n^{-N/2},
\end{aligned}$$

and from Corollary 2.1 and our choice of B_n,

$$\int_{\partial_i B_n} |\nabla u_n|^2 |x - x_0| \leqslant C\sigma_n^{(2-N)/2}.$$

Finally, from both Proposition 2.1 and Corollary 2.1, by the Hölder inequality,

$$\int_{\partial_i B_n} (\nabla u_n \cdot \vec{n}) u_n \leqslant \left(\int_{\partial_i B_n} |\nabla u_n|^2 \right)^{1/2} \left(\int_{\partial_i B_n} |u_n|^2 \right)^{1/2} \leqslant C\sigma_n^{(2-N)/2}.$$

Combining these estimates, we see that the right-hand side of (2.8) is therefore bounded by $C\sigma_n^{(2-N)/2}$. So (2.8) requires

$$\lambda\sigma_n^{-2} \leqslant C\sigma_n^{(2-N)/2}, \tag{2.14}$$

which when $N > 6$, since $\sigma_n \to \infty$, is clearly false for n large. □

Theorem 1.1 is an immediate consequence (essentially a restating which does not use the terminology introduced in this chapter) of the above lemma.

PROOF OF THEOREM 1.1. Let us suppose, by contradiction, that there exists a bounded balanced sequence $(u_n)_{n\in\mathbb{N}}$ such that

$$\sup_{n\in\mathbb{N}}\sup_{x\in\Omega}\left|u_n(x)\right| = +\infty.$$

A standard regularity argument [10] shows that u_n cannot be compact in H^1, so by Lemma 1.1 it has a balanced concentrating subsequence and this is excluded by Lemma 2.2. □

2.2. *Avoiding escaping masses*

We shall now take into exam the case of a balanced sequence related to problem (P) and we shall assume by contradiction that the sequence is broken, according to Definition 1.3. This means, roughly speaking, that there are some masses φ_i which are escaping to infinity. So we are going to study the variation of the functional under a small translation of one of such masses which brings it back to the origin.

2.2.1. *Local Pohozaev identity for translations.* The variation of the functional under the translation of a solution is evaluated by a Pohozaev-type formula. Since we only want to translate a part of the function, corresponding to one of the escaping masses, we must prove such a formula in a local version, namely without assuming boundary conditions, as stated in the next lemma.

We fix a general open smooth set B in $\mathbb{R}^N$ and shall consider, more in general, a semi-linear elliptic equation of the form

$$-\Delta u = g(x,u). \tag{2.15}$$

LEMMA 2.3. *Let u be a smooth solution to* (2.15) *on a smooth domain B and let $G(x,s)$ be a primitive with respect to s of the function $g(x,s)$. Then the following equation holds true*

$$\begin{aligned} &-\int_B \nabla_x G(x,u)\cdot\vec{t} \\ &\quad= \int_{\partial B}\left(\frac{1}{2}|\nabla u|^2 - G(x,u)\right)\left(\nu\cdot\vec{t}\right) - \int_{\partial B}(\nabla u\cdot\nu)\left(\nabla u\cdot\vec{t}\right). \end{aligned} \tag{2.16}$$

PROOF. We have

$$\int_B \big(-\Delta u + g(x,u)u\big)\big(\nabla u \cdot \vec{t}\,\big)\,\mathrm{d}x = 0. \tag{2.17}$$

Now integrating by parts, we obtain

$$\int_B -\Delta u\big(\nabla u \cdot \vec{t}\,\big)\,\mathrm{d}x = \int_B \big(\nabla u \cdot \nabla\big(\nabla u \cdot \vec{t}\,\big)\big)\,\mathrm{d}x - \int_{\partial B} (\nabla u \cdot \nu)\big(\nabla u \cdot \vec{t}\,\big)\,\mathrm{d}\sigma.$$

Then, taking into account that $\vec{t}$ does not depend on x, again using divergence theorem, we get

$$\begin{aligned}\int_B \big(\nabla u \cdot \nabla\big(\nabla u \cdot \vec{t}\,\big)\big)\,\mathrm{d}x &= \int_B \big(\nabla u \cdot \big(\nabla^2 u \cdot \vec{t}\,\big)\big)\,\mathrm{d}x \\ &= \frac{1}{2}\int_B \big(\nabla|\nabla u|^2 \cdot \vec{t}\,\big)\,\mathrm{d}x \\ &= \frac{1}{2}\int_{\partial B} |\nabla u|^2\big(\vec{t}\cdot\nu\big)\,\mathrm{d}\sigma\end{aligned}$$

and then

$$\begin{aligned}&\int_B -\Delta u\big(\nabla u \cdot \vec{t}\,\big)\,\mathrm{d}x \\ &\quad= \frac{1}{2}\int_{\partial B} \big(|\nabla u|^2\big(\vec{t}\cdot\nu\big)\big)\,\mathrm{d}\sigma - \int_{\partial B} (\nabla u \cdot \nu)\big(\nabla u \cdot \vec{t}\,\big)\,\mathrm{d}\sigma.\end{aligned} \tag{2.18}$$

Analogously we deduce

$$\int_{\partial B} G(x,u)\nu \cdot \vec{t} = \int_B \nabla G(x,u) \cdot \vec{t} = \int_B \nabla_x G(x,u) \cdot \vec{t} + \int_B g(x,u)\nabla u \cdot \vec{t}.$$

So

$$\int_B g(x,u)\nabla u \cdot \vec{t} = \int_{\partial B} G(x,u)\nu \cdot \vec{t} - \int_B \nabla_x G(x,u) \cdot \vec{t}.$$

Therefore

$$\begin{aligned}&\frac{1}{2}\int_{\partial B} \big(|\nabla u|^2\big(\vec{t}\cdot\nu\big)\big) - \int_{\partial B} (\nabla u \cdot \nu)\big(\nabla u \cdot \vec{t}\,\big) \\ &\quad= \int_{\partial B} G(x,u)\nu \cdot \vec{t} - \int_B \nabla_x G(x,u) \cdot \vec{t}.\end{aligned}$$

□

In the case of problem (P) we can take $G(x,s)=\frac{1}{p}|s|^p-\frac{a(x)}{2}s^2$.

COROLLARY 2.2. *Let $a(x)$ satisfy* (a$_1$) *and u be a solution to* (P). *Then the following identity*

$$\frac{1}{2}\int_B u^2\big(\nabla a(x)\cdot\vec{t}\,\big)\,\mathrm{d}x=\frac{1}{2}\int_{\partial B}\big(|\nabla u|^2+a(x)u^2\big)\big(\nu\cdot\vec{t}\,\big) -\int_{\partial B}(\nabla u\cdot\nu)\big(\nabla u\cdot\vec{t}\,\big)-\frac{1}{p}\int_{\partial B}|u|^p\big(\nu\cdot\vec{t}\,\big), \tag{2.19}$$

where ν is the outward normal to ∂B, holds.

2.2.2. *Drift estimates tools.* As for the case of (CP), we must now choose a convenient set B on which (1.10) leads to a contradiction. Again the contradiction will follow from the fact that the boundary integrals are too small with respect to the volume integral and this analysis is based on suitable decay estimates. Solutions to (P) have an exponential decay at infinity. One can guess that therefore we should find an uniform exponential decay on the terms of a balanced sequence of functions if we keep far away from the escaping masses. This means that the bound we are going to find is not of the type of $\mathrm{e}^{-\alpha|x|}$ but of $\mathrm{e}^{-\alpha\sigma_n(x)}$, where the function σ_n defined by

$$\sigma_n(x)=\inf_{0\leqslant i\leqslant k}\big|x-t_n^i\big|,\quad x\in\mathbb{R}^N, \tag{2.20}$$

will be called *drift distance* function and measures how much x is escaping from all the masses in which u_n gets broken. Note that $t_n^0=0$ so $\sigma_n(x)\leqslant|x|$ for all $n\in\mathbb{N}$. Indeed in Section 3 we shall prove the following exponential decay result.

PROPOSITION 2.3. *Let $a(x)$ satisfy* (a$_1$) *and* (a$_2$). *Let $(u_n)_{n\in\mathbb{N}}$ be a broken controlled sequence bounded in $H^1(\mathbb{R}^N)$. Then for any constant $\alpha\in(0,\sqrt{a_\infty})$, there exists a constant $c_\alpha>0$ such that for n large enough,*

$$u_n(x)\leqslant c_\alpha\mathrm{e}^{-\alpha\sigma_n(x)}\quad\forall x\in\mathbb{R}^n. \tag{2.21}$$

The above estimate suggests to apply (2.19), where $u=u_n$ is a term of a balanced sequence of functions, to a set $B=\mathcal{D}_n$ whose boundary is far away from all the masses, as it happens with the cones C_n and the other regions introduced in the end of Section 1.3. We must take into account that u_n solves the problem on B_{ρ_n}, so we have to take the trace of such cones on B_{ρ_n}. As in the case of the critical growth problem, the choice of B_n will follow from a gradient estimate in an integral form. Indeed, in Section 3 we shall also prove the following estimate.

PROPOSITION 2.4. *Let $a(x)$ satisfy* (a_1) *and* (a_2). *Let $(u_n)_{n\in\mathbb{N}}$ be a broken controlled sequence bounded in $H^1(\mathbb{R}^N)$. Then there exist constants $\alpha_* > 0$ and $c_* > 0$ such that for all $n \in \mathbb{N}$,*

$$\int_{\mathcal{S}_{1,n}} |\nabla u_n|^2 \,\mathrm{d}x \leqslant c_* \mathrm{e}^{-\alpha_*|t_n|}, \tag{2.22}$$

where $\mathcal{S}_{1,n}$ is as defined in (1.13).

Also in this case a mean value argument allows to pass to a boundary integral.

PROPOSITION 2.5. *Let $a(x)$ and $(u_n)_{n\in\mathbb{N}}$ be as in Proposition* 2.3. *Then there exist constants $\alpha^* > 0$, $c^* > 0$ and a sequence $(s_n)_{n\in\mathbb{N}}$, $s_n \in (-\frac{1}{2}, \frac{1}{2})$ such that for all $n \in \mathbb{N}$,*

$$\int_{\partial\mathcal{C}_{s_n,n}} |\nabla u_n|^2 \,\mathrm{d}x \leqslant c^* \mathrm{e}^{-\alpha^*|t_n|}, \tag{2.23}$$

where, for all n, $\mathcal{C}_{s_n,n}$ is as defined in (1.12).

Then, it makes sense setting

$$\mathcal{D}_n = \widetilde{\mathcal{C}}_n \cap B_{\rho_n}(0),$$

where $\widetilde{\mathcal{C}}_n$ denotes the cone $\mathcal{C}_{s_n,n}$. We remark that, for large n, the vertex $\frac{t_n}{2} \in \mathcal{D}_n$; indeed, even if $\rho_n < |t_n|$, for all n, $|t_n| - \rho_n \leqslant C$ for some constant C, otherwise $u_n(\cdot - t_n) \underset{n\to+\infty}{\rightharpoonup} 0$ contradicting the choice of t_n. Moreover, we remark that $\partial\mathcal{D}_n$ consists of an "internal part"

$$(\partial\mathcal{D}_n)_i = \partial\widetilde{\mathcal{C}}_n \cap B_{\rho_n}(0)$$

and an "external" one

$$(\partial\mathcal{D}_n)_e = \widetilde{\mathcal{C}}_n \cap \partial B_{\rho_n}(0).$$

We finally point out that by using (2.21) we easily get the following integral estimate, whose detailed proof is in Section 3.

PROPOSITION 2.6. *Let $a(x)$ satisfy* (a_1) *and* (a_2). *Let $(u_n)_{n\in\mathbb{N}}$ be a broken controlled sequence bounded in $H^1(\mathbb{R}^N)$ and $\mathcal{S}_n$ be as in* (1.14). *Then, for all $p \geqslant 2$, there exist constants $\tilde{\alpha} > 0$ and $\tilde{c} > 0$ such that for n large enough,*

$$\int_{\mathcal{S}_n} (u_n)^p \,\mathrm{d}x \leqslant \tilde{c}\mathrm{e}^{-\tilde{\alpha}|t_n|}. \tag{2.24}$$

2.2.3. *Proof of Theorem 1.3.* We are now ready to combine the local Pohozaev formula (2.19) with the exponential decay estimates in the previous propositions to the aim of proving Theorem 1.3. Let us begin with a lemma which is the only step in which we use (a_4).

LEMMA 2.4. *Let $a(x)$ satisfy* (a_1) *and* (a_4). *Let $(u_n)_{n\in\mathbb{N}}$ be a noncompact balanced sequence. Then, for large n, the inequality*

$$\int_{\mathcal{D}_n} \big(\nabla a(x)\cdot \vec{t}_n\big)u_n^2\,\mathrm{d}x \geqslant \frac{1}{2}\int_{\mathcal{D}_n} \frac{\partial a}{\partial \vec{x}}(x)u_n^2\,\mathrm{d}x \tag{2.25}$$

holds.

PROOF. Denoting by $(\vec{t}_n)_{\tau_x}$ the component of $\vec{t}_n$ lying in the space orthogonal to $\vec{x}$ and containing x, using (a_4) we get, for large n,

$$\begin{aligned}\big(\nabla a(x)\cdot \vec{t}_n\big) &= \big(\nabla a(x)\cdot \vec{x}\big)\big(\vec{t}_n\cdot\vec{x}\big) + \big(\nabla_{\tau_x} a(x)\cdot \big(\vec{t}_n\big)_{\tau_x}\big)\\ &\geqslant \frac{\partial a}{\partial \vec{x}}(x)\big(\vec{t}_n\cdot\vec{x}\big) - \bar{c}\frac{\partial a}{\partial \vec{x}}(x)\big|\big(\vec{t}_n\big)_{\tau_x}\big|\\ &= \frac{\partial a}{\partial \vec{x}}(x)\big[\big(\vec{t}_n\cdot\vec{x}\big) - \bar{c}\big|\big(\vec{t}_n\big)_{\tau_x}\big|\big].\end{aligned}$$

In order to evaluate $[(\vec{t}_n\cdot\vec{x}) - \bar{c}|(\vec{t}_n)_{\tau_x}|]$, let us first suppose $x\in B_{2R_n}(t_n)$, so that $|x-t_n| < 2R_n < \hat{\gamma}|t_n|$, with $\hat{\gamma}$ as in (1.11), then we have

$$\big(\vec{t}_n\cdot\vec{x}\big) = \left(\frac{t_n}{|t_n|}\cdot\frac{t_n + x - t_n}{|x|}\right) \geqslant \frac{|t_n| - |x-t_n|}{|x|} \geqslant \frac{|t_n| - |x-t_n|}{|t_n| + |x-t_n|} \geqslant \frac{1-\hat{\gamma}}{1+\hat{\gamma}} \tag{2.26}$$

and since

$$\begin{aligned}\vec{t}_n &= \frac{x}{|t_n|} + \frac{t_n - x}{|t_n|},\\ \big|\big(\vec{t}_n\big)_{\tau_x}\big| &\leqslant \frac{|t_n - x|}{|x|} < \hat{\gamma}.\end{aligned}\tag{2.27}$$

On the other hand, we can assert that, by homothety, (2.26) and (2.27) are also true for all x belonging to the cone $\mathcal{K}$ having as vertex the origin and generated by $B_{2R_n}(t_n)$. Then, in particular, (2.26) and (2.27) are true for all $x\in\mathcal{D}_n$, being $\mathcal{D}_n\subset\widetilde{\mathcal{C}}_n\subset\mathcal{K}$.

Thus (2.25) follows because we have, by the choice of $\hat{\gamma}$, $\frac{3}{4}\frac{1-\hat{\gamma}}{1+\hat{\gamma}} > \frac{1}{2}$ and $\frac{1-\hat{\gamma}}{1+\hat{\gamma}} - 4\bar{c}\hat{\gamma} > 0$. □

We can combine the previous lemma with Corollary 2.2 obtaining the following result.

LEMMA 2.5. *Let $a(x)$ and $(u_n)_{n\in\mathbb{N}}$ be as in Lemma* 2.4. *Then the inequality*

$$\begin{aligned}
&\frac{1}{4}\int_{\mathcal{D}_n}\frac{\partial a}{\partial\vec{x}}(x)u_n^2(x)\\
&\quad\leqslant\frac{1}{2}\int_{(\partial\mathcal{D}_n)_i}\big(|\nabla u_n|^2+a(x)u_n^2\big)\big(\nu_n\cdot\vec{t}_n\big)\\
&\qquad-\int_{(\partial\mathcal{D}_n)_i}(\nabla u_n\cdot\nu_n)\big(\nabla u_n\cdot\vec{t}_n\big)-\frac{1}{p}\int_{(\partial\mathcal{D}_n)_i}|u_n|^p\big(\nu_n\cdot\vec{t}_n\big)
\end{aligned}\tag{2.28}$$

holds.

PROOF. Combining (2.19) and (2.25) we obtain

$$\begin{aligned}
\frac{1}{4}\int_{\mathcal{D}_n}\frac{\partial a}{\partial\vec{x}}u_n^2\leqslant\frac{1}{2}\int_{\partial\mathcal{D}_n}&\big(|\nabla u_n|^2+a(x)u_n^2\big)\big(\nu_n\cdot\vec{t}_n\big)\\
&-\int_{\partial\mathcal{D}_n}(\nabla u_n\cdot\nu_n)\big(\nabla u_n\cdot\vec{t}_n\big)-\frac{1}{p}\int_{\partial\mathcal{D}_n}|u_n|^p\big(\nu_n\cdot\vec{t}_n\big).
\end{aligned}\tag{2.29}$$

Now, for all n, u_n solves $(\mathrm{P}_{B_{\rho_n}(0)})$, $u_n=0$ on $\partial B_{\rho_n}(0)\supset(\partial\mathcal{D}_n)_e$, so ∇u_n and ν_n have the same direction, moreover, on $(\partial\mathcal{D}_n)_e$ it is $(\nu_n\cdot\vec{t}_n)\geqslant 0$, thus we deduce

$$\int_{(\partial\mathcal{D}_n)_e}a(x)u_n^2\big(\nu_n\cdot\vec{t}_n\big)=0=\int_{(\partial\mathcal{D}_n)_e}|u_n|^p\big(\nu_n\cdot\vec{t}_n\big)\tag{2.30}$$

and

$$\begin{aligned}
&\frac{1}{2}\int_{(\partial\mathcal{D}_n)_e}|\nabla u_n|^2\big(\nu_n\cdot\vec{t}_n\big)-\int_{(\partial\mathcal{D}_n)_e}(\nabla u_n\cdot\nu_n)\big(\nabla u_n\cdot\vec{t}_n\big)\\
&\quad=\frac{1}{2}\int_{(\partial\mathcal{D}_n)_e}|\nabla u_n|^2\big(\nu_n\cdot\vec{t}_n\big)-\int_{(\partial\mathcal{D}_n)_e}(\nabla u_n\cdot\theta\nabla u_n)\bigg(\frac{1}{\theta}\nu_n\cdot\vec{t}_n\bigg)\\
&\quad=-\frac{1}{2}\int_{(\partial\mathcal{D}_n)_e}|\nabla u_n|^2\big(\nu_n\cdot\vec{t}_n\big)\leqslant 0.
\end{aligned}\tag{2.31}$$

Hence (2.28) follows inserting (2.30) and (2.31) in (2.29). □

Taking the decay estimate in (2.28) we can finally deduce the compactness of a balanced sequence.

LEMMA 2.6. *Let $a(x)$ satisfy* (a_1)–(a_4) *and $(u_n)_{n\in\mathbb{N}}$ be a bounded balanced sequence. Then $(u_n)_{n\in\mathbb{N}}$ is relatively compact*.

PROOF. We argue by contradiction and we assume that $(u_n)_{n\in\mathbb{N}}$ is not compact. Then, by Proposition 1.1, up to a subsequence, it is broken in L^p and, by Lemma 2.5, the inequality (2.28) must be true.

Let us consider, for n large, the right-hand side of (2.28). First of all, let us observe that, by (a_2), $a(x) \geqslant 0$ for all $x \in (\partial\mathcal{D}_n)_i$ so, taking into account that $(\nu_n \cdot \vec{t}_n) \leqslant 0$ on $(\partial\mathcal{D}_n)_i$, we have

$$\int_{(\partial\mathcal{D}_n)_i} \big(|\nabla u_n|^2 + a(x)u_n^2\big)\big(\nu_n \cdot \vec{t}_n\big) \leqslant 0. \tag{2.32}$$

Moreover, by using Proposition 2.5, we deduce

$$\begin{aligned} &-\int_{(\partial\mathcal{D}_n)_i} (\nabla u_n \cdot \nu_n)\big(\nabla u_n \cdot \vec{t}_n\big)\,\mathrm{d}\sigma \\ &\quad \leqslant \int_{(\partial\mathcal{D}_n)_i} |\nabla u_n|^2\,\mathrm{d}\sigma \leqslant \int_{\partial\widetilde{\mathcal{C}}_n} |\nabla u_n|^2\,\mathrm{d}\sigma \leqslant c^*\mathrm{e}^{-\alpha^*|t_n|}. \end{aligned} \tag{2.33}$$

Let us now show that there exist constants $\alpha' > 0$ and $c' > 0$, independent on n, so that

$$-\int_{(\partial\mathcal{D}_n)_i} |u_n|^p\big(\nu_n \cdot \vec{t}_n\big)\,\mathrm{d}\sigma \leqslant \int_{(\partial\mathcal{D}_n)_i} |u_n|^p\,\mathrm{d}\sigma \leqslant c'\mathrm{e}^{-\alpha'|t_n|}. \tag{2.34}$$

Since $(\partial\mathcal{D}_n)_i \subset \partial\widetilde{\mathcal{C}}_n$ and, for large n, $\partial\widetilde{\mathcal{C}}_n \subset \mathcal{S}_n$, using Proposition 2.3 we infer

$$\begin{aligned} &-\int_{(\partial\mathcal{D}_n)_i} |u_n|^p\,\mathrm{d}\sigma \\ &\quad \leqslant \int_{\partial\widetilde{\mathcal{C}}_n} |u_n|^p\,\mathrm{d}\sigma \leqslant c_\alpha \int_{\partial\widetilde{\mathcal{C}}_n} \mathrm{e}^{-\alpha\sigma_n(x)p}\,\mathrm{d}\sigma \leqslant c_\alpha \sum_{i=1}^{k} \int_{\partial\widetilde{\mathcal{C}}_n} \mathrm{e}^{-\alpha p|x-t_n^i|}\,\mathrm{d}\sigma, \end{aligned} \tag{2.35}$$

$\alpha \in (0, \sqrt{a_\infty}\,)$, $c_\alpha > 0$.

Setting, for $h \geqslant 1$ and $i = 0, 1, \ldots, k$,

$$A_{h,i} = \left\{x \in \partial\widetilde{\mathcal{C}}_n\colon\ 2^{h-1}\frac{r_n}{2} < \left|x - t_n^i\right| < 2^h\frac{r_n}{2}\right\} \tag{2.36}$$

and denoting by $|A_{h,i}|$ the $(N-1)$-dimensional (Hausdorff) measure of $A_{h,i}$, we have for $i = 0, 1, \ldots, k$,

$$|A_{h,i}| \leqslant C\left[2^h\frac{r_n}{2}\right]^{N-1}, \quad C \in \mathbb{R}, \tag{2.37}$$

because it is not difficult to understand that, $\forall h$, $|A_{h,i}|$ can be estimated by the surface of the cylinder having height and basis diameter measure equal to $\frac{r_n}{2}2^h$.

Thus, in view of (2.36) and (2.37), we deduce

$$\int_{\partial\widetilde{\mathcal{C}}_n} \mathrm{e}^{-\alpha p|x-t_n^i|}\,\mathrm{d}\sigma \leqslant \sum_{h=1}^{\infty}\int_{A_{h,i}} \mathrm{e}^{-\alpha p 2^{h-1} r_n/2}\,\mathrm{d}\sigma \leqslant C\sum_{h=1}^{\infty} \mathrm{e}^{-\alpha p 2^{h-1} r_n/2}\left[2^h \frac{r_n}{2}\right]^{N-1} \tag{2.38}$$

hence, inserting (2.38) in (2.35), we obtain as desired,

$$\int_{(\partial\mathcal{D}_n)_i} |u_n|^p\,\mathrm{d}\sigma \leqslant c'_\alpha k r_n^{N-1}\mathrm{e}^{-\alpha p r_n/2}\sum_{h=0}^{\infty}\mathrm{e}^{-\alpha p 2^h} 2^{h(N-1)} \leqslant c'\mathrm{e}^{-\alpha'|t_n|}. \tag{2.39}$$

On the other hand, denoting by $\tilde{\rho}_n = \max\{\rho_n, |t_n|\}$ and by

$$\widetilde{\mathcal{D}}_n = \widetilde{\mathcal{C}}_n \cap B_{\tilde{\rho}_n}(0),$$

we have, for large n,

$$\int_{\mathcal{D}_n} \frac{\partial a}{\partial \vec{x}}(x) u_n^2\,\mathrm{d}x \geqslant \inf_{\mathcal{D}_n}\left(\frac{\partial a}{\partial \vec{x}}(x)\right)\int_{\mathcal{D}_n} u_n^2\,\mathrm{d}x \geqslant \overline{C}\inf_{\widetilde{\mathcal{D}}_n}\left(\frac{\partial a}{\partial \vec{x}}(x)\right)\int_{\widetilde{\mathcal{D}}_n} u_n^2\,\mathrm{d}x, \tag{2.40}$$

$\overline{C} > 0$ constant, because, as remarked at the beginning of the section, $(||t_n| - \rho_n|)_{n\in\mathbb{N}}$ is bounded from above. Moreover, in view of Proposition 1.1 and of the choice of t_n, we infer

$$\liminf_{n\to+\infty}\int_{\widetilde{\mathcal{D}}_n} u_n^2\,\mathrm{d}x \geqslant \lambda > 0, \quad \lambda = \text{const}. \tag{2.41}$$

Then, combining (2.28) with (2.32)–(2.34), (2.40) and (2.41), we obtain

$$\frac{1}{4}\lambda\overline{C}\inf_{\widetilde{\mathcal{D}}_n}\left(\frac{\partial a}{\partial \vec{x}}(x)\right) \leqslant c^*\mathrm{e}^{-\alpha^*|t_n|} + \frac{c'}{p}\mathrm{e}^{-\alpha'|t_n|} \leqslant \bar{c}\mathrm{e}^{-\bar{\alpha}|t_n|},$$

$\bar{\alpha} = \min(\alpha^*, \alpha')$, and this is impossible by (a_3). □

PROOF OF THEOREM 1.3. If the statement is not true, we can extract from U a balanced sequence which is not precompact and therefore, by Proposition 1.1, has a broken subsequence. Then we get in contradiction to the previous lemma. □

3. Decay estimates

This section completes the previous one by proving the estimates stated in Sections 2.1.2 and 2.2.2.

3.1. *Decay estimates near concentration points*

We start by considering a balanced sequence related to problem (CP) and establishing the estimates in Section 2.1.2. We recall that we can always substitute the terms u_n with their absolute value, extended by 0 on all of $\mathbb{R}^N$, passing to the weaker assumption that the sequence is controlled but getting free from caring about the sign of the function or the shape of Ω.

3.1.1. *Integral estimates for controlled concentrating sequences.* So we shall work with a controlled concentrating sequence $(u_n)_{n\in\mathbb{N}}$. The boundedness of the sequence in H_0^1, and so in L^{2^*}, cannot hold in L^p for $p > 2^*$, because of the presence of concentrations. On the other hand, such concentrations are small in L^p for $p < 2^*$, so that, modulo an infinitesimal term, u_n can be split in a part which keeps bounded in L^p for large p and in a part which is infinitesimal in L^p for small p.

In order to guess what kind of estimate we are going to find, we can assume that $u_n = \varphi_0 + \rho_n^1(\varphi_1)$ exactly and that φ_1 is a Talenti function, as in the beginning of Section 2.1.2. In such a case, $\varphi_0 \in L^p$ for every p, while $\varphi_1 \in L^p$ only for $p > 2^*/2 = N/(N-2)$. If $p_1 > 2^*$, then $\|\varphi_0\|_{p_1} \leqslant \text{const}$ and if $2^*/2 < p_2 < 2^*$, then

$$\left\|\rho_n^1(\varphi_1)\right\|_{p_2} = {\sigma_n^1}^{N/2^*-N/p_2}\|\varphi_1\|_{p_2} = {\sigma_n^1}^{N/2^*-N/p_2}.$$

If one has several concentrating masses φ_i, then

$$\begin{aligned}\left\|\sum_{i=1}^k \rho_n^i(\varphi_i)\right\|_{p_2} &\leqslant \sum_{i=1}^k \left\|\rho_n^i(\varphi_i)\right\|_{p_2}\\ &\leqslant \sum_{i=1}^k {\sigma_n^i}^{N/2^*-N/p_2}\|\varphi_1\|_{p_2}\\ &\leqslant \text{const}\cdot\sigma_n^{N/2^*-N/p_2}. \end{aligned}\tag{3.1}$$

So we are lead to introduce the following definition.

DEFINITION 3.1. Let $p_1, p_2 \in]2, +\infty[$ be real numbers such that $p_2 < 2^* < p_1$, $\alpha > 0$ and $\sigma > 0$. We consider an inequalities system

$$\begin{cases}\|u_1\|_{p_1} \leqslant \alpha,\\ \|u_2\|_{p_2} \leqslant \alpha\sigma^{N/2^*-N/p_2},\end{cases}\tag{3.2}$$

which will let us introduce a norm depending on p_1, p_2 and σ, by setting

$$\|u\|_{p_1,p_2,\sigma} = \inf\bigl\{\alpha > 0 \mid \exists u_1, u_2 \text{ such that (3.2) is satisfied and } |u| \leqslant u_1 + u_2\bigr\}.$$

The above norm will be briefly denoted by $\|u\|_\sigma$ when p_1 and p_2 can be supposed to be given.

REMARK 3.1. Let $p_1, p_2 \in]2, +\infty[$ real numbers such that $p_2 < 2^* < p_1$ and $\sigma > 0$, then, by definition, for any function u, we get

$$\|u\|_\sigma \leqslant \|u\|_{p_1}, \qquad \|u\|_\sigma \leqslant \|u\|_{p_2} \sigma^{N/p_2 - N/2^*}.$$

We can easily see from (3.1) that for every p_1, p_2 such that $\frac{2^*}{2} < p_2 < 2^* < p_1$,

$$\left\| \varphi_0 + \sum_{i=1}^{k} \rho_n^i(\varphi_i) \right\|_{p_1, p_2, \sigma_n} \leqslant \text{const.}$$

The main goal of this section is to show that the same bound holds for u_n, which differs from $\varphi_0 + \sum_{i=1}^k \rho_n^i(\varphi_i)$ by an infinitesimal term in H_0^1 which is not a priori even bounded in the norm $\|\cdot\|_{p_1, p_2, \sigma_n}$. Thanks to the assumption that $(u_n)_{n\in\mathbb{N}}$ is also controlled, we shall be able to prove the following Brezis–Kato-type regularity result (see [10], Theorem 2.3):

PROPOSITION 3.1. *Let $(u_n)_{n\in\mathbb{N}}$ be a controlled concentrating sequence, then for any $p_1, p_2 \in]\frac{2^*}{2}, +\infty[$, $p_2 < 2^* < p_1$, there exists a constant $C(p_1, p_2)$ depending on the sequence and on the exponents p_1 and p_2, such that for any $n \in \mathbb{N}$,*

$$\|u_n\|_{\sigma_n} \leqslant C.$$

To this aim, we shall state three preliminary lemmas: a continuity lemma, a bootstrap lemma and the relative initialization lemma.

LEMMA 3.1. *Let u and $v \in H^1(\mathbb{R}^N)$ and $a \in L^{N/2}(\mathbb{R}^N)$ be three positive functions such that*

$$-\Delta u \leqslant a(x) v.$$

Then for each $p_1, p_2 \in]2, +\infty[$, there exists a constant $C(N, p_1, p_2)$, depending on the dimension N and on the exponents p_1 and p_2, such that for any $\sigma > 0$,

$$\|u\|_\sigma \leqslant C(N, p_1, p_2) \|a\|_{N/2} \|v\|_\sigma.$$

PROOF. Let u, v be as in the statement of the lemma and let fix $\sigma > 0$ and $\varepsilon > 0$. Let $v \leqslant v_1 + v_2$ such that v_1 and v_2 satisfy (3.2) for $\alpha = \|v\|_{p_1, p_2, \sigma} + \varepsilon$. Let us consider, for $i = 1, 2$, the solution $u_i \in H^1(\mathbb{R}^N)$ to $-\Delta u_i = a v_i$. Then

$$\|u_i\|_{p_i} \leqslant C(N, p_i) \|a\|_{N/2} \|v_i\|_{p_i}$$

and, being $-\Delta u_1 - \Delta u_2 = a v_1 + a v_2 \geqslant a v \geqslant -\Delta u$, by the maximum principle, we have $u \leqslant u_1 + u_2$. Since the functions u_i satisfy (3.2) with $\alpha = C(N, p_1, p_2) \|a\|_{N/2} \times (\|v\|_\sigma + \varepsilon)$, with $C(N, p_1, p_2) = \max(C(N, p_1), C(N, p_2))$, by the arbitrariness of ε we get the thesis. □

The bootstrap argument relies in the use of the following lemma.

LEMMA 3.2. *Let $p_1, p_2 \in \left]\frac{N+2}{N-2}, \frac{N}{2}\frac{N+2}{N-2}\right[$ such that $p_2 < 2^* < p_1$ and let q_i be defined, for $i = 1, 2$, by*

$$\frac{1}{q_i} = \frac{N+2}{N-2}\frac{1}{p_i} - \frac{2}{N}. \tag{3.3}$$

If u and v are two positive functions whose support is contained in a bounded set Ω and such that

$$-\Delta u \leqslant v^{2^*-1} + A,$$

then there exists a constant $C(N, p_1, p_2, \Omega)$ such that for any $\sigma > 0$,

$$\|u\|_{q_1,q_2,\sigma} \leqslant C(N, p_1, p_2, \Omega)\big(\big(\|v\|_{p_1,p_2,\sigma}\big)^{(N+2)/(N-2)} + 1\big). \tag{3.4}$$

PROOF. By proceeding as in the previous lemma, we consider $v = v_1 + v_2$ where the functions v_i satisfy (3.2) for $\alpha = \|v\|_{p_1,p_2,\sigma} + \varepsilon$ and ε is a real strictly positive number arbitrarily small. Let u_1 and u_2 be two functions in $H_0^1(\Omega)$ such that

$$\begin{aligned}
-\Delta u_1 &= 2^{4/(N-2)} v_1^{(N+2)/(N-2)} + A, \\
-\Delta u_2 &= 2^{4/(N-2)} v_2^{(N+2)/(N-2)}.
\end{aligned}$$

Since

$$\begin{aligned}
-\Delta u &\leqslant v^{(N+2)/(N-2)} + A \\
&\leqslant 2^{(N+2)/(N-2)-1} v_1^{(N+2)/(N-2)} + A + 2^{(N+2)/(N-2)-1} v_2^{(N+2)/(N-2)} \\
&= -\Delta u_1 - \Delta u_2,
\end{aligned}$$

by the maximum principle, $u \leqslant u_1 + u_2$ follows. Hence, we have to estimate $\|u_1\|_{q_1}$ and $\|u_2\|_{q_2}$. We have, using (3.3) and being $\frac{N+2}{N-2} < p_i < \frac{N}{2}\frac{N+2}{N-2}$,

$$\begin{aligned}
\|u_1\|_{q_1} &\leqslant C(N, p_1)\big\|v_1^{(N+2)/(N-2)} + A\big\|_{L^{p_1(N-2)/(N+2)}} \\
&\leqslant C(N, p_1)\big(\|v_1\|_{p_1}^{(N+2)/(N-2)} + A|\Omega|^{1/p_1(N+2)/(N-2)}\big) \\
&\leqslant C(N, p_1, \Omega)\big(\big(\|v\|_{p_1,p_2,\sigma} + \varepsilon\big)^{(N+2)/(N-2)} + 1\big).
\end{aligned}$$

Analogously, if we use the equality

$$\frac{N}{2^*} - \frac{N}{q_2} = \left(\frac{N}{2^*} - \frac{N}{p_2}\right)\frac{N+2}{N-2},$$

we get

$$\begin{aligned}\|u_2\|_{q_2} &\leqslant C(N,p_2)\big(\|v_2\|_{p_2}\big)^{(N+2)/(N-2)}\\ &\leqslant C(N,p_2)\big[\big(\|v\|_{p_1,p_2,\sigma}+\varepsilon\big)\sigma^{N/2^*-N/p_2}\big]^{(N+2)/(N-2)}\\ &= C(N,p_2)\big(\|v\|_{p_1,p_2,\sigma}+\varepsilon\big)^{(N+2)/(N-2)}\sigma^{(N/2^*-N/p_2)(N+2)/(N-2)}\\ &= C(N,p_2)\big(\|v\|_{p_1,p_2,\sigma}+\varepsilon\big)^{(N+2)/(N-2)}\sigma^{N/2^*-N/q_2}.\end{aligned}$$

So u_1 and u_2 solve (3.2) for $C = C(N,p_1,p_2,\Omega)((\|v\|_{p_1,p_2,\sigma}+\varepsilon)^{(N+2)/(N-2)}+1)$; this concludes the proof by the arbitrary choice of ε. □

Now we need to initialize the exponents through the following lemma.

LEMMA 3.3. *Let $(u_n)_{n\in\mathbb{N}}$ be a controlled concentrating sequence then there exists a constant C and exponents $p_1, p_2 \in]\frac{2^*}{2},+\infty[$, $p_2 < 2^* < p_1$, such that for any $n\in\mathbb{N}$,*

$$\|u_n\|_{\sigma_n} \leqslant C. \tag{3.5}$$

PROOF. This proof will follow a Brezis–Kato-type argument (see [10]) in order to get free from an infinitesimal term which is the only real obstacle to our estimates, as explained in the beginning of this section. For any $n\in\mathbb{N}$, we can consider using a homogeneous notation, $u_n = u_n^0 + u_n^1 + u_n^2$, where

- u_n^1 stands for the weak limit φ_0,
- u_n^2 stands for the sum of rescaled functions φ_i, $u_n^2 = \sum_{i=1}^k \rho_n^i(\varphi_i)$,
- $u_n^0 = u_n - u_n^2 - \varphi_0$ is, by definition of concentrating sequence, an infinitesimal term in L^{2^*} norm.

We shall overcome the difficulty due to the presence of u_n^0 by taking advantage of the assumption that we are dealing with a controlled concentrating sequence. Let u be one of the terms u_n, $u_i = u_n^i$ and $a_i = \max(1, 3^{(6-N)/(N-2)}) {u_i}^{4/(N-2)}$ for $i = 1,2,3$, and $\sigma = \sigma_n$. The infinitesimal character of u_n^0 shall allow us to consider a_0 as small as we want in the $L^{N/2}$ norm ((3.5) is easily checked on a finite number of terms, see [10] and [25]). Being

$$a = u^{2^*-2} \leqslant \max\big(1, 3^{(6-N)/(N-2)}\big)\big(|u_0|^{4/(N-2)} + u_1^{4/(N-2)} + u_2^{4/(N-2)}\big),$$

we can consider u as a solution to $-\Delta u \leqslant (a_0 + a_1 + a_2)u + A$, so by the monotonicity of the Green operator $\mathcal{G}$ ($\mathcal{G}: H^{-1}(\Omega) \to H_0^1(\Omega)$ denotes the inverse operator of $-\Delta$) we have

$$u \leqslant \mathcal{G}(a_0 u) + \mathcal{G}(a_1 u + A) + \mathcal{G}(a_2 u). \tag{3.6}$$

Since Ω is a bounded set and $a_1 \in L^\infty$, we get that $\mathcal{G}(a_1 u + A)$ is bounded in $W^{2,2^*} \hookrightarrow L^{p_1}$ for any p_1 such that

$$\frac{1}{p_1} \geqslant \frac{1}{2^*} - \frac{2}{N} = \frac{N-6}{2N}$$

and so (see Remark 3.1)

$$\left\|\mathcal{G}(a_1u + A)\right\|_\sigma \leqslant \left\|\mathcal{G}(a_1u + A)\right\|_{p_1} \leqslant C. \tag{3.7}$$

Now let $2^{*\prime} < p_2 < 2^*$ be given. We consider the index r such that

$$\frac{1}{p_2} = \frac{1}{r} + \frac{1}{2^*} - \frac{2}{N},$$

from $p_2 > 2^{*\prime}$ we get $r > \frac{N}{4}$. The decay speed of the solution $\varphi = \varphi_i$ (see [25]) allows us to say that $a_2 \in L^r$ and, if we want to estimate the L^r norm of a_2, we just have to take into account the less concentrated term, namely $\rho_n(\varphi)$, as follows from $r < \frac{N}{2}$, which is in turn a consequence of $p_2 < 2^*$. By easy computations we have

$$\|a_2\|_{L^r} \leqslant C\sigma^{2-N/r},$$

which, taking into account that $2 - N/r = N/2^* - N/p_2$, implies

$$\left\|\mathcal{G}(a_2u)\right\|_{p_2} \leqslant C\|a_2\|_{L^r}\|u\|_{L^{2^*}} \leqslant C\sigma^{N/2^*-N/p_2}, \tag{3.8}$$

therefore, from Remark 3.1,

$$\left\|\mathcal{G}(a_2u)\right\|_\sigma \leqslant \sigma^{N/p_2-N/2^*}\left\|\mathcal{G}(a_2u)\right\|_{p_2} \leqslant C. \tag{3.9}$$

Now we point out that with the above choice for p_1 and p_2 we get

$$\left\|\mathcal{G}(a_0u)\right\|_\sigma \leqslant \frac{1}{2}\|u\|_\sigma. \tag{3.10}$$

Indeed, by Lemma 3.1, we get

$$\left\|\mathcal{G}(a_0u)\right\|_\sigma \leqslant C\|a_0\|_{N/2}\|u\|_\sigma \leqslant \frac{1}{2}\|u\|_\sigma, \tag{3.11}$$

under a suitable choice of the bound on the norm of a_0. So by (3.6), (3.10) and the triangular inequality, we finally obtain

$$\|u\|_\sigma \leqslant 2\left\|\mathcal{G}(a_1u + A)\right\|_\sigma + 2\left\|\mathcal{G}(a_2u)\right\|_\sigma,$$

which, combined with (3.7) and (3.9), gives the thesis. □

PROOF OF PROPOSITION 3.1. Let $(u_n)_{n\in\mathbb{N}}$ be a controlled concentrating sequence. By applying the initialization Lemma 5.2, we can find a constant $C > 0$ and two exponents, p_1 and $p_2 \in]\frac{N+2}{N-2}, \frac{N}{2}\frac{N+2}{N-2}[$, $p_2 < 2^* < p_1$, such that (3.5) holds. Using the bootstrap Lemma 3.2 we can repeatedly enlarge the interval $]p_2, p_1[$ to $]q_2, q_1[$, where the exponents q_i are given by (3.3), obtaining (3.4). This procedure allows us to manage, in a finite number of steps, every exponent $p_1, p_2 \in]\frac{2^*}{2}, +\infty[$. □

3.1.2. *Local uniform bounds on controlled concentrating sequences.* We are now going to establish the local uniform bound on the terms of a controlled concentrating sequence on the safe regions $\mathcal{A}_n^2$ stated in Proposition 2.1, whose proof is the main goal of this section.

The proof is a simple variant of the argument used in [41] and in [25] and shall require some preliminary steps. We begin by establishing a weaker estimate.

PROPOSITION 3.2. *Let $(u_n)_{n\in\mathbb{N}}$ a controlled concentrating sequence. Then there exists a constant $C > 0$ such that, for any $n \in \mathbb{N}$ and for any $x \in \mathcal{A}_n^1$,*

$$u_n(x) \leqslant C\sigma_n^{(N-2)/4}.$$

PROOF. We shall proceed by contradiction: let $(y_n)_{n\in\mathbb{N}}$ be a sequence such that $y_n \in \mathcal{A}_n^1$ for any $n \in \mathbb{N}$ and

$$\lim_{n\to+\infty} u_n(y_n)\sigma_n^{(2-N)/4} = +\infty, \tag{3.12}$$

and let us scale the functions u_n in such a way to carry the point y_n in the origin and normalize the value of the functions. The required scaling sends u_n in $\tilde{u}_n$ defined as

$$\tilde{u}_n(x) = \rho_n^{N/2^*} u_n(\rho_n x + y_n),$$

where

$$\rho_n = \big(u_n(y_n)\big)^{2/(2-N)} = \big(u_n(y_n)\big)^{-2^*/N},$$

so that $\tilde{u}_n(0) = 1$. Note that, using (3.12), we have

$$\lim_{n\to+\infty} \frac{\rho_n}{\sigma_n^{-1/2}} = 0. \tag{3.13}$$

Therefore, since $y_n \in \mathcal{A}_n^1$, there is no concentration point which approximates y_n at a distance less than or equal to $\sigma_n^{-1/2}$ and so of the order of ρ_n, we can deduce that $\tilde{u}_n \rightharpoonup \tilde{u} = 0$. The contradiction will be archived by showing that we can choose the points y_n in such a way to have $\tilde{u} \neq 0$. This shall possibly force us to work on a $(\varepsilon\sigma_n^{-1/2})$-neighborhood of $\mathcal{A}_n^1$, but this change will obviously not make any relevant difference in the above argument. The choice will consist in forcing the property

$$\tilde{u}_n(y) \leqslant 2 \quad \big(= 2\tilde{u}_n(0)\big) \quad \forall y \in B_\rho(0), \tag{3.14}$$

for some given $\rho > 0$. Then by using that $\tilde{u}_n$ still satisfies (EI) and by estimating the variation of the mean value of u_n, we have for $0 < r \leqslant \rho$,

$$\fint_{\partial B_r} \tilde{u}_n = \tilde{u}_n(0) + \int_0^r \frac{1}{N b_N t^{N-1}} \left(\int_{B_t} \Delta \tilde{u}_n \right) \mathrm{d}t$$
$$\geqslant 1 - C \int_0^r \frac{1}{t^{N-1}} \int_{B_t} \left(2^{2^*-1} + A\right) \mathrm{d}t = 1 - Cr^2 \geqslant \frac{1}{2},$$

where b_N stands for the $(N-1)$-dimensional measure of the unit sphere in $\mathbb{R}^N$, provided we choose r conveniently small. So the weak limit $\tilde{u}$ cannot be zero. Therefore we only have to prove (3.14). To this aim, let us fix $\rho > 0$ and assume that, for a given $n \in \mathbb{N}$, y_n does not satisfy (3.14). Then we must fire y_n and look for a better point to hire for the same job. Since (3.14) is false, we can find $z_n \in B_\rho(0)$ such that

$$\tilde{u}_n(z_n) = \rho_n^{(N-2)/2} u_n(\rho_n z_n + y_n) \geqslant 2. \tag{3.15}$$

The first candidate to replace y_n is

$$y_n^{(1)} = \rho_n z_n + y_n$$

which leads us to replace ρ_n by

$$\rho_n^{(1)} = \left[u_n\left(y_n^{(1)}\right)\right]^{2/(2-N)} \leqslant 2^{2/(2-N)} \rho_n. \tag{3.16}$$

We can be sure that $y_n^{(1)}$ is at least as good as y_n to let (3.12) hold since (3.15) implies that

$$u_n\left(y_n^{(1)}\right) \geqslant 2 u_n(y_n). \tag{3.17}$$

Moreover, being $z_n \in B_\rho(0)$, we get

$$\left|y_n^{(1)} - y_n\right| = |z_n \rho_n| \leqslant \rho \rho_n. \tag{3.18}$$

We can define $\tilde{u}_n$ as before by substituting y_n and ρ_n with $y_n^{(1)}$ and $\rho_n^{(1)}$, respectively. If this new $\tilde{u}_n$ satisfies (3.14) we do not have to look for other choices. Otherwise, we repeat the same argument and we choose a second candidate $y_n^{(2)}$ by arguing in the same way. For any fixed $n \in \mathbb{N}$, we proceed recursively finding a sequence $y_n^{(1)}, y_n^{(2)}, \ldots, y_n^{(k)}, \ldots$ as far as we do not find a successful choice, which lets us claim (3.14). We can easily see that this process cannot go on indefinitely. Indeed (3.16) becomes in the general case, for $i > 0$,

$$\rho_n^{(i+1)} \leqslant 2^{2/(2-N)} \rho_n^{(i)}$$

and (3.18),

$$\left|y_n^{(i+1)} - y_n^{(i)}\right| \leqslant \rho \rho_n^{(i)}.$$

Then one easily sees, by taking the sum of a geometric sequence, that $y_n^{(i)}$ converges to a point $y_n^{(\infty)}$ as $i \to +\infty$ but, by construction, we have $u_n(y_n^{(i)}) \to +\infty$, in contradiction to the smoothness of u_n. Finally, for every $i > 0$, we have

$$\left|y_n^{(i)} - y_n\right| \leqslant \rho\rho_n \sum_{j=0}^{+\infty} 2^{2/(2-N)j} < \varepsilon\sigma_n^{-1/2}$$

for n large. So all the points $y_n^{(i)}$ are in the $(\varepsilon\sigma_n^{-1/2})$-neighborhood of $\mathcal{A}_n^1$ and so can be used to replace y_n. □

PROPOSITION 3.3. *Let $(u_n)_{n\in\mathbb{N}}$ be a controlled concentrating sequence, then there exists a constant $C > 0$ such that, for any $n \in \mathbb{N}$ and for any $r \in [\overline{C}\sigma_n^{-1/2}, (\overline{C}+5)\sigma_n^{-1/2}]$,*

$$⨍_{\partial B_r(x_n)} u_n \leqslant C.$$

PROOF. By continuity, being $(u_n)_{n\in\mathbb{N}}$ bounded in $L^{2^*} \subset L^1$, we can suppose

$$\int_{B_1(x_n)} u_n \leqslant C$$

with a constant C independent from n. So, for any $n \in \mathbb{N}$, there exist $r_n \in [\frac{1}{2}, 1]$, such that

$$⨍_{\partial B_{r_n}(x_n)} u_n = C.$$

We are going to use Proposition 3.1 for $p_1 = N\frac{N+2}{N-2}$ and $p_2 = \frac{N+2}{N-2}$, so, for any $n \in \mathbb{N}$, we choose $u_1 = u_{1,n}$ and $u_2 = u_{2,n}$ such that (3.2) is satisfied for $\sigma = \sigma_n$ and with a constant α that does not depend on n. Estimating the spherical mean variation from r_n to r and taking into account that $(\overline{C}+5)\sigma_n^{-1/2} < 1/2$, i.e., $r < r_n$ for n large, we find

$$\begin{aligned}
⨍_{\partial B_r(x_n)} u_n &= C + \int_{r_n}^{r} \frac{\mathrm{d}}{\mathrm{d}t} ⨍_{\partial B_t(x_n)} u_n \,\mathrm{d}t = C + \int_r^{r_n} \frac{1}{Nb_N t^{N-1}} \int_{B_t(x_n)} -\Delta u_n \,\mathrm{d}t \\
&\leqslant C + \int_{\overline{C}\sigma_n^{-1/2}}^{1} \frac{1}{Nb_N t^{N-1}} \int_{B_t(x_n)} \left(u_n^{2^*-1} + A\right) \mathrm{d}t \\
&\leqslant C + \int_{\overline{C}\sigma_n^{-1/2}}^{1} 2^{4/(N-2)} \frac{1}{Nb_N t^{N-1}} \int_{B_t(x_n)} u_{1,n}^{(N+2)/(N-2)} \,\mathrm{d}t \\
&\quad + \int_{\overline{C}\sigma_n^{-1/2}}^{1} 2^{4/(N-2)} \frac{1}{Nb_N t^{N-1}} \int_{B_t(x_n)} u_{2,n}^{(N+2)/(N-2)} \,\mathrm{d}t + \frac{A}{N}\int_0^1 t \,\mathrm{d}t \\
&= C + \frac{2^{4/(N-2)}}{Nb_N}(A_1 + A_2) + \frac{A}{2N},
\end{aligned}$$

where, for $i = 1, 2$,

$$A_i = \int_{\overline{C}\sigma_n^{-1/2}}^1 \frac{1}{t^{N-1}} \int_{B_t(x_n)} u_{i,n}^{(N+2)/(N-2)} \, \mathrm{d}t.$$

Being $u_{1,n} \in L^{N(N+2)/(N-2)}$, by the Hölder inequality, we get

$$A_1 \leqslant C \int_0^1 \frac{1}{t^{N-1}} \left(t^N\right)^{1-1/N} \|u_{1,n}\|_{L^{N(N+2)/(N-2)}}^{(N+2)/(N-2)} \, \mathrm{d}t \leqslant C\alpha \leqslant C.$$

On the other side, being $u_{2,n} \in L^{(N+2)/(N-2)}$, i.e., $u_{2,n}^{(N+2)/(N-2)} \in L^1$ we have

$$\begin{aligned} A_2 &\leqslant \int_{\overline{C}\sigma_n^{-1/2}}^1 \frac{1}{t^{N-1}} \left[\alpha \sigma_n^{(N/2^* - N(N-2)/(N+2))}\right]^{(N+2)/(N-2)} \mathrm{d}t \\ &= \alpha^{(N+2)/(N-2)} \sigma_n^{(2-N)/2} \int_{\overline{C}\sigma_n^{-1/2}}^1 \frac{1}{t^{N-1}} \, \mathrm{d}t \leqslant C, \end{aligned}$$

and this concludes the proof. □

From Proposition 3.3 we see, by integrating with respect to r, that

$$⨍_{\mathcal{A}_n^1} u_n \leqslant C. \tag{3.19}$$

Since, $\forall x \in \mathcal{A}_n^2$: $B_{\sigma_n^{-1/2}(x)} \subset \mathcal{A}_n^1$ and the measure of the two sets are of the same order, from (3.19) we deduce that

$$\forall x \in \mathcal{A}_n^2: \quad ⨍_{B_{\sigma_n^{-1/2}}(x)} u_n \leqslant C. \tag{3.20}$$

Since

$$u_n(x) = \lim_{\rho \to 0} ⨍_{B_\rho(x)} u_n,$$

Proposition 2.1 follows from (3.20) if we estimate the variation of $⨍_{B_\rho(x)} u_n$ for $0 \leqslant \rho \leqslant \sigma_n^{-1/2}$.

PROOF OF PROPOSITION 2.1. Let us fix an index $n \in \mathbb{N}$ and a point $x \in \mathcal{A}_n^2$. If $u_n(x) \leqslant 2 ⨍_{B_{\sigma_n^{-1/2}}(x)} u_n$, by (3.20) we have done. Otherwise, setting for any $\rho > 0$,

$$m(\rho) = ⨍_{\partial B_\rho(x)} u_n \quad \text{and} \quad m(0) = u_n(x),$$

we deduce that

$$\exists\bar\rho \leqslant \sigma_n^{-1/2} \quad \text{such that} \quad m(\bar\rho) \leqslant \frac{1}{2}m(0) = \frac{1}{2}u_n(x).$$

Then we take ρ_1 and $\rho_2 \in [0,\bar\rho]$ such that $m(\rho)$ attains its maximum in ρ_1, and ρ_2 is the least value of $\rho \geqslant \rho_1$ such that $m(\rho) \leqslant \frac{1}{2}m(\rho_1)$.

Being u_n solution to (EI), and $B_{\rho_2}(x) \subset \mathcal{A}_n^1$, we have on such a set, by Proposition 3.2, $u_n^{4/(N-2)} \leqslant C\sigma_n$. So we find, for n sufficiently large,

$$\begin{aligned}
\frac{1}{2}m(\rho_1) &= \int_{\rho_2}^{\rho_1}\left(\frac{\mathrm{d}}{\mathrm{d}\rho}\fint_{\partial B_\rho(x)} u_n\right)\mathrm{d}\rho = \int_{\rho_1}^{\rho_2}\frac{1}{Nb_N\rho^{N-1}}\int_{B_\rho(x)} -\Delta u_n \,\mathrm{d}\rho \\
&\leqslant \int_{\rho_1}^{\rho_2}\frac{1}{Nb_N\rho^{N-1}}\int_{B_\rho(x)}\big(u_n^{4/(N-2)}u_n + A\big)\,\mathrm{d}\rho \\
&\leqslant \frac{1}{Nb_N}\int_{\rho_1}^{\rho_2}\frac{1}{\rho^{N-1}}\left(\left(\sup_{B_\rho(x)} u_n{}^{4/(N-2)}\right)\left(\int_{B_\rho(x)} u_n\right) + Ab_N\rho^N\right)\mathrm{d}\rho \\
&\leqslant C\int_{\rho_1}^{\rho_2}\frac{1}{\rho^{N-1}}\left(\sigma_n\int_{B_\rho(x)} u_n + A\rho^N\right)\mathrm{d}\rho \\
&\leqslant C\big(m(\rho_1)\sigma_n + A\big)\int_{\rho_1}^{\rho_2}\rho\,\mathrm{d}\rho \leqslant Cm(\rho_1)\sigma_n\big(\rho_2^2-\rho_1^2\big),
\end{aligned}$$

therefore $(\rho_2^2-\rho_1^2) > C\sigma_n^{-1}$ and so $\rho_2-\rho_1 > C\sigma_n^{-1/2}$. Denoting by $\mathcal{A}$ the annulus centered in x of radii ρ_1 and ρ_2 we have that measure of $\mathcal{A}$ is of the order of $\sigma_n^{-N/2}$, i.e., of the same order of $\mathcal{A}_n^1$ and so as in (3.20) we have

$$\fint_{\mathcal{A}} u_n \leqslant C.$$

On the other hand,

$$\fint_{\mathcal{A}} u_n \geqslant m(\rho_2) = \frac{1}{2}m(\rho_1)$$

and so

$$u_n(x) = m(0) \leqslant m(\rho_1) \leqslant C. \qquad \square$$

3.1.3. *Gradient estimates.* In this subsection we shall prove the integral bound for the derivatives of every term u_n of a controlled concentrating sequence in its safe regions $\mathcal{A}_n^3$ stated in Proposition 2.2. One can easily guess that, since u_n and Δu_n are uniformly bounded on $\mathcal{A}_n^2$ and the width of $\mathcal{A}_n^2$ is of the order of $\sigma_n^{-1/2}$, ∇u_n can be expected to be of the order of $\sigma_n^{1/2}$ as stated in an integral form in (2.11). Such an estimate can be easily proved in a rigorous way by a Caccioppoli-type inequality.

PROOF OF PROPOSITION 2.2. Let us fix $n \in \mathbb{N}$ and consider $\varphi_n : \mathbb{R}^N \to [0,1]$ a smooth positive mollifier radially symmetric around x_n such that

(1) $\varphi_n = 1$ on $\mathcal{A}_n^3$,
(2) $\varphi_n = 0$ out of $\mathcal{A}_n^2$,
(3) $\Delta\varphi_n \leqslant C\sigma_n$.

By (2) we have $\varphi_n = 0$ and $\nabla\varphi_n = 0$ on $\partial\mathcal{A}_n^2$, and so, integrating by parts, by (1) we get

$$\begin{aligned}
\int_{\mathbb{R}^N} -\Delta u_n u_n \varphi_n &= \int_{\mathcal{A}_n^2} |\nabla u_n|^2 \varphi_n + \int_{\mathcal{A}_n^2} \nabla u_n \cdot \nabla\varphi_n u_n \\
&\geqslant \int_{\mathcal{A}_n^3} |\nabla u_n|^2 + \int_{\mathcal{A}_n^2} \nabla\left(\frac{1}{2}u_n^2\right) \cdot \nabla\varphi_n \\
&= \int_{\mathcal{A}_n^3} |\nabla u_n|^2 - \frac{1}{2}\int_{\mathcal{A}_n^2} \Delta\varphi_n u_n^2.
\end{aligned}$$

Therefore, being u_n solution to (EI), by Proposition 2.1 and (3), we have

$$\int_{\mathcal{A}_n^3} |\nabla u_n|^2 \leqslant \int_{\mathcal{A}_n^2} \left(|u_n|^{2^*} + Au_n\right)\varphi_n + \frac{1}{2}\int_{\mathcal{A}_n^2} \Delta\varphi_n u_n^2 \leqslant C(1+\sigma_n)\left|\mathcal{A}_n^2\right|. \tag{3.21}$$

Since $\sigma_n \geqslant 1$ for n large, one has (2.11). □

3.2. *Decay estimates at drift points*

The purpose of this second part is to establish the decay estimates and integral bounds, concerning bounded controlled sequences contained in Propositions 2.3, 2.4 and 2.6.

3.2.1. *Uniform estimates.* The first step is proving a lemma that allows to obtain an uniform upper bound on the values of the Laplacian on a controlled sequence.

LEMMA 3.4. *Let $a(x)$ satisfy* (a$_1$) *and* (a$_2$). *Let $(u_n)_{n\in\mathbb{N}}$ be a controlled sequence bounded in $H^1(\mathbb{R}^N)$. Then $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(\mathbb{R}^N)$.*

PROOF. By (a$_1$) and (a$_2$), there exist a constant $\tilde{a} \in (0, a_\infty)$ and a positive function $c(x) \in \mathcal{C}_0(\mathbb{R}^N)$ such that $a(x) \geqslant \tilde{a} - c(x)\ \forall x \in \mathbb{R}^N$. Therefore u_n weakly solves

$$-\Delta u_n + \tilde{a}u_n \leqslant u_n^{p-1} + c(x)u_n \quad \text{in } \mathbb{R}^N,$$

moreover, by the maximum principle, for any weak positive solution $v_n \in H^1(\mathbb{R}^N)$ to

$$-\Delta v + \tilde{a} v = u_n^{p-1} + c(x) u_n \quad \text{in } \mathbb{R}^N, \tag{3.22}$$

the relation

$$u_n(x) \leqslant v_n(x) \quad \text{in } \mathbb{R}^N \tag{3.23}$$

holds.

Now, let us consider a sequence $(v_n)_{n\in\mathbb{N}}$, $v_n \in H^1(\mathbb{R}^N)$, such that for all $n \in \mathbb{N}$, v_n solves (3.22). By (3.23), the claim follows proving that $(|v_n|_\infty)_{n\in\mathbb{N}}$ is bounded.

Since $u_n \in H^1(\mathbb{R}^N)$ and $c(x) \in \mathcal{C}_0(\mathbb{R}^N)$, we can assume $u_n^{p-1} + c(x)u_n \in L^{2^*/(p-1)}$, so by regularity results $v_n \in W^{2,2^*/(p-1)}(\mathbb{R}^N)$. Now, the space $W^{2,2^*/(p-1)}(\mathbb{R}^N)$ embeds continuously in $L^{\hat{q}}(\mathbb{R}^N)$, where $\frac{1}{\hat{q}} = \frac{p-1}{2^*} - \frac{2}{N}$ and, since $\frac{2^*}{p-1} > \frac{2^*}{2^*-1} = \frac{2N}{N+2} = 2^{*\prime}$, $\hat{q} > \frac{N2^{*\prime}}{N-2\cdot 2^{*\prime}} = 2^*$. Then, by (3.23), $u_n \in L^{\hat{q}}(\mathbb{R}^N)$ with $\frac{\hat{q}}{2^*} > 1$, and

$$|u_n|_{\hat{q}} \leqslant |v_n|_{\hat{q}} \leqslant k_1 \left| u_n^{p-1} + c(x)u_n \right|_{2^*/(p-1)} < k_2.$$

By iterating the same argument, we gradually increase the regularity properties of u_n and v_n, obtaining also uniform bounds to the norms in the respective spaces. After a finite number of steps we obtain $v_n \in W^{2,\tilde{q}}(\mathbb{R}^N)$ with $\tilde{q} > \frac{N}{2}$ and $\|v_n\|_{W^{2,\tilde{q}}} < k_3$, k_3 not depending on n.

Then the Sobolev embedding theorem gives $v_n \in \mathcal{C}^{0,\mu}(\mathbb{R}^N)$ for some $\mu \in (0,1)$, and $\|v_n\|_{\mathcal{C}^{0,\mu}(\mathbb{R}^N)} < k_4$.

This last relation with the L^2 summability allows to obtain an L^∞ uniform bound on $(v_n)_{n\in\mathbb{N}}$ and, in turn, on $(u_n)_{n\in\mathbb{N}}$ as desired. □

COROLLARY 3.1. *Let $(u_n)_{n\in\mathbb{N}}$ and $a(x)$ be as in Lemma* 2.3. *Then there exists a constant $c_1 > 0$ such that for all $n \in \mathbb{N}$, the relation*

$$-\Delta u_n \leqslant c_1 \tag{3.24}$$

weakly holds.

The proof of Proposition 2.3 is carried out through some estimates, on bounded controlled sequences, proved in a slightly general setting. In order to do this we introduce the following definition.

DEFINITION 3.2. Given a sequence of functions $(u_n)_{n\in\mathbb{N}}$, $u_n \in H^1(\mathbb{R}^N)$, and a sequence $(x_n)_{n\in\mathbb{N}}$, $x_n \in \mathbb{R}^N$, we say that $(x_n)_{n\in\mathbb{N}}$ is a sequence of *drift points* for $(u_n)_{n\in\mathbb{N}}$ if

$$u_n(\cdot - x_n) \rightharpoonup 0 \quad \text{weakly in } H^1\big(\mathbb{R}^N\big). \tag{3.25}$$

REMARK 3.2. If $u_n \not\rightharpoonup 0$, a sequence of points $(x_n)_{n\in\mathbb{N}}$ is a drift sequence if and only if $\sigma_n(x_n) \to +\infty$ and, since $\sigma_n(x_n) \leqslant |x|$, $(x_n)_{n\in\mathbb{N}}$ is unbounded. In general, $\sigma_n(x_n) \to +\infty$ is equivalent to $(x_n)_{n\in\mathbb{N}}$ being an unbounded drift sequence.

The following lemma guarantees that the values that a controlled bounded sequence takes around the drift points x_n of a sequence are small.

LEMMA 3.5. *Let $a(x)$ satisfy* (a$_1$) *and* (a$_2$) *and let $(u_n)_{n\in\mathbb{N}}$ be a controlled sequence bounded in $H^1(\mathbb{R}^N)$. Let $(x_n)_{n\in\mathbb{N}}$ be a drift points sequence for $(u_n)_{n\in\mathbb{N}}$ and let $\delta_n = \sigma_n(x_n)$. Then for all $h \in (0,1)$,*

$$\lim_{n\to+\infty} \sup_{B_{h\delta_n}(x_n)} u_n(x) = 0. \tag{3.26}$$

PROOF. We argue by contradiction and we assume that there exist real numbers $h \in (0,1)$, $\eta > 0$ and a sequence $(y_n)_{n\in\mathbb{N}}$, $y_n \in B_{h\delta_n}(x_n)$, such that for large n,

$$u_n(y_n) > \Big(\sup_{B_{h\delta_n}(x_n)} u_n(x)\Big) - \frac{1}{n} > \eta.$$

The above relation, combined with (3.24), allows to conclude that, for large n and ρ small enough,

$$⨍_{B_\rho(y_n)} u_n \,\mathrm{d}x = \frac{1}{|B_\rho(y_n)|} \int_{B_\rho(y_n)} u_n \,\mathrm{d}x > \frac{\eta}{2},$$

where $|B_\rho(y_n)|$ denotes the Lebesque N-dimensional measure of $B_\rho(y_n)$.

Hence $u_n(\cdot - y_n) \rightharpoonup v \neq 0$ as $n \to +\infty$. This is impossible because, by the choice of δ_n, h and y_n and by Definition 3.2 $u_n(\cdot - y_n) \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$. □

Next lemma contains the key estimate for proving Proposition 2.3.

LEMMA 3.6. *Let $a(x)$ satisfy* (a$_1$) *and* (a$_2$) *and let $(u_n)_{n\in\mathbb{N}}$ be a controlled sequence bounded in $H^1(\mathbb{R}^N)$. Let $(x_n)_{n\in\mathbb{N}}$ be an unbounded drift points sequence for $(u_n)_{n\in\mathbb{N}}$. Then, for all $\alpha \in (0, \sqrt{a_\infty})$, there exists a constant $\hat{c}_\alpha > 0$ such that for all n,*

$$u_n(x_n) \leqslant \hat{c}_\alpha \mathrm{e}^{-\alpha\sigma_n(x_n)}. \tag{3.27}$$

PROOF. Let $\alpha \in \mathbb{R}$, $0 < \alpha < \sqrt{a_\infty}$, be fixed, and let us choose $h \in (\frac{\alpha}{\sqrt{a_\infty}}, 1)$ and $\bar{\alpha} \in (\alpha, \sqrt{a_\infty}h)$.

Then, by using Lemma 3.5, we obtain that, for any n large enough, u_n weakly satisfies

$$\Delta u_n \geqslant a(x)u_n - u_n^{p-1} > \bar{\alpha}^2 h^{-2} u_n \geqslant 0 \quad \text{in } B_{h\delta_n}(x_n), \tag{3.28}$$

where $\delta_n = \sigma_n(x_n)$. Thus, since $h\delta_n > 1$ for large n, we have

$$u_n(x_n) \leqslant \int_{\partial B_r(x_n)} u_n \, \mathrm{d}\sigma \quad \forall r\colon 0 < r \leqslant 1, \tag{3.29}$$

and we deduce

$$u_n(x_n) \leqslant \int_0^1 \left[\int_{\partial B_r(x_n)} u_n \, \mathrm{d}\sigma \right] \mathrm{d}r = \int_{B_1(x_n)} u_n \, \mathrm{d}x.$$

So, in order to obtain (3.27) for large n, it is enough to show that a constant $\bar{c}_\alpha > 0$ exists such that

$$\int_{B_1(x_n)} u_n \, \mathrm{d}x \leqslant \bar{c}_\alpha \mathrm{e}^{-\alpha\delta_n}. \tag{3.30}$$

To do this, let us consider the functions

$$v_n(\rho) = \int_{B_\rho(x_n)} u_n \, \mathrm{d}x, \qquad w_n(\rho) = \frac{(h\delta_n)^N \omega_N}{\mathrm{e}^{\bar{\alpha}\delta_n}} \mathrm{e}^{\bar{\alpha}\rho/h},$$

where ω_N is the Lebesgue measure of the unitary ball in $\mathbb{R}^N$, and let us remark that $v_n(1)$ is just the left-hand side of (3.30), while

$$w_n(1) = \frac{(h\delta_n)^N \omega_N}{\mathrm{e}^{\bar{\alpha}\delta_n}} \mathrm{e}^{\bar{\alpha}/h} \leqslant h\omega_N \mathrm{e}^{\sqrt{a_\infty}} \frac{\delta_n^N}{\mathrm{e}^{\bar{\alpha}\delta_n}} \leqslant \bar{c}_\alpha \mathrm{e}^{-\alpha\delta_n}$$

for n large enough. So (3.27) follows, by proving $v_n(1) \leqslant w_n(1)$ and taking into account that for any finite set $u_n(x_n)$, $n < \bar{n}$, (3.27) is obviously true for a suitable choice of the constant $\hat{c}_\alpha$.

Let us then show that for n large

$$v_n(\rho) \leqslant w_n(\rho) \quad \forall \rho \in [0, h\delta_n]. \tag{3.31}$$

First, let us observe that

$$v_n(0) \leqslant w_n(0) \quad \forall n \in \mathbb{N},$$

and that for n large, by Lemma 3.5,

$$v_n(h\delta_n) \leqslant \left| B_{h\delta_n}(x_n) \right| \sup_{B_{h\delta_n}(x_n)} u_n(x) \leqslant \omega_N (h\delta_n)^N = w_n(h\delta_n).$$

Now, if for some point in $[0, h\delta_n]$ (3.31) were false, then the function $(v_n - w_n)(\rho)$ should have a maximum point, $\bar\rho_n \in (0, h\delta_n)$, for which $(v_n - w_n)(\bar\rho_n) > 0$ and, of course, $v''(\bar\rho_n) - w''(\bar\rho_n) \leqslant 0$. Let us show that this is impossible. Indeed, since

$$v_n(\rho) = \int_{B_\rho(x_n)} u_n(x)\,\mathrm{d}x = \int_0^\rho \left[\int_{\partial B_r(x_n)} u_n\,\mathrm{d}\sigma\right]\mathrm{d}r,$$

we have

$$v_n'(\rho) = \frac{\mathrm{d}}{\mathrm{d}\rho} v_n(\rho) = \int_{\partial B_\rho(x_n)} u_n\,\mathrm{d}\sigma,$$

moreover,

$$⨍_{\partial B_\rho(x_n)} u_n\,\mathrm{d}\sigma = \frac{1}{N\omega_N\rho^{N-1}}\int_{\partial B_\rho(x_n)} u_n\,\mathrm{d}\sigma = \frac{v_n'(\rho)}{N\omega_N\rho^{N-1}}$$

and, by using divergence theorem,

$$\begin{aligned} ⨍_{\partial B_\rho(x_n)} u_n\,\mathrm{d}\sigma &= \int_0^\rho \frac{1}{N\omega_N r^{N-1}}\left[\int_{\partial B_r(x_n)} \frac{\partial}{\partial\nu}u_n\,\mathrm{d}\sigma\right]\mathrm{d}r \\ &= \int_0^\rho \frac{1}{N\omega_N r^{N-1}}\left[\int_{B_r(x_n)} \Delta u_n\,\mathrm{d}x\right]\mathrm{d}r. \end{aligned}$$

So

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\frac{v_n'(\rho)}{N\omega_N\rho^{N-1}} = \frac{1}{N\omega_N\rho^{N-1}}\int_{B_\rho(x_n)} \Delta u_n\,\mathrm{d}x,$$

from which, using (3.28), we obtain

$$\begin{aligned} &\frac{v_n''(\rho)}{\rho^{N-1}} + (1-N)\frac{v_n'(\rho)}{\rho^N} \\ &\quad = \frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{v_n'(\rho)}{\rho^{N-1}}\right) = \frac{1}{\rho^{N-1}}\int_{B_\rho(x_n)} \Delta u_n\,\mathrm{d}x \geqslant \frac{\alpha^2 h^{-2}}{\rho^{N-1}} v_n(\rho). \end{aligned}$$

Hence, taking into account that $v_n'(\rho) > 0$ and $N > 1$,

$$v_n''(\rho) \geqslant \alpha^2 h^{-2} v_n(\rho) \quad \forall\rho \in (0, h\delta_n) \tag{3.32}$$

follows.

Let now $\bar\rho_n \in (0, h\delta_n)$ be a maximum point for $(v_n - w_n)(\rho)$ for which $(v_n - w_n)(\bar\rho_n) > 0$ then by (3.32), we get

$$v_n''(\bar\rho_n) - w_n''(\bar\rho_n) \geqslant \alpha^2 h^{-2}\big(v_n(\bar\rho_n) - w_n(\bar\rho_n)\big) > 0,$$

and we are in contradiction. □

PROOF OF PROPOSITION 2.3. Arguing by contradiction, we assume that there are $\alpha \in (0, \sqrt{a_\infty})$, $(x_n)_{n\in\mathbb{N}}$ and a sequence of integers $k_n \in \mathbb{N}$ such that

$$u_{k_n}(x_n) > n\mathrm{e}^{-\alpha\sigma_{k_n}(x_n)}$$

or, by replacing $(u_n)_{n\in\mathbb{N}}$ by the subsequence $(x_{k_n})_{n\in\mathbb{N}}$,

$$u_n(x_n) > n\mathrm{e}^{-\alpha\sigma_n(x_n)}.$$

By Lemma 3.4 we get $\sigma_n(x_n) \to +\infty$ so we get in contradiction to Lemma 3.6 whose assumptions are fulfilled by $(x_n)_{n\in\mathbb{N}}$. □

PROOF OF PROPOSITION 2.6. By using Proposition 2.3, we deduce that, for n large enough and $\alpha \in (0, \sqrt{a_\infty})$,

$$\begin{aligned}\int_{\mathcal{S}_n} (u_n)^p \,\mathrm{d}x &\leqslant c_\alpha \int_{\mathcal{S}_n} \mathrm{e}^{-\alpha p\sigma_n(x)}\,\mathrm{d}x \\ &\leqslant c_\alpha \sum_{i=0}^{k} \int_{\mathcal{S}_n} \mathrm{e}^{-\alpha p|x-t_n^i|}\,\mathrm{d}x \\ &\leqslant c_\alpha k \int_{\frac{\hat{\gamma}}{4k}|t_n|}^{+\infty} \mathrm{e}^{-\alpha pt} t^{N-1}\,\mathrm{d}t \leqslant \tilde{c}\mathrm{e}^{-\tilde{\alpha}|t_n|}.\end{aligned}$$

□

3.2.2. *Gradient estimates.* Also in this case, a Caccioppoli-type inequality allows to pass to an integral estimate of the gradient.

PROOF OF PROPOSITION 2.4. For any fixed $n \in \mathbb{N}$, let $\varphi_n \in \mathcal{C}^\infty(\mathbb{R}^N, [0,1])$ be a function fulfilling the following conditions:

$$\begin{cases} \text{(i) } \varphi_n = 1 \quad \text{on } \mathcal{S}_{1,n}, \\ \text{(ii) } \operatorname{supp}(\varphi_n) \subset \mathcal{S}_{2,n}, \\ \text{(iii) } \Delta\varphi_n \leqslant C, \quad C \in \mathbb{R}. \end{cases} \tag{3.33}$$

Since u_n weakly solves (EI) and $\varphi_n = 0$ in $\mathbb{R}^N \setminus \mathcal{S}_{2,n}$, we have

$$\begin{aligned}\int_{\mathcal{S}_{2,n}} (-\Delta u_n)(u_n\varphi_n) &= \int_{\mathbb{R}^N} (-\Delta u_n)(u_n\varphi_n) \\ &\leqslant \int_{\mathbb{R}^N} \big(u_n^p - a(x)u_n^2\big)\varphi_n \qquad (3.34)\\ &= \int_{\mathcal{S}_{2,n}} \big(u_n^p - a(x)u_n^2\big)\varphi_n. \qquad (3.35)\end{aligned}$$

On the other hand, taking into account that by (3.33)(ii), $\varphi_n = 0$ and $\nabla\varphi_n = 0$ on $\partial\mathcal{S}_{2,n}$ and using (3.33)(i), we get

$$\begin{aligned}\int_{\mathcal{S}_{2,n}} (-\Delta u_n)(u_n\varphi_n) &= \int_{\mathcal{S}_{2,n}} |\nabla u_n|^2\varphi_n + \int_{\mathcal{S}_{2,n}} (\nabla u_n \cdot \nabla\varphi_n)u_n \\ &\geqslant \int_{\mathcal{S}_{1,n}} |\nabla u_n|^2 + \int_{\mathcal{S}_{2,n}} \left(\nabla\left(\frac{1}{2}u_n^2\right)\cdot\nabla\varphi_n\right) \\ &= \int_{\mathcal{S}_{1,n}} |\nabla u_n|^2 - \frac{1}{2}\int_{\mathcal{S}_{2,n}} (\Delta\varphi_n)u_n^2. \end{aligned} \tag{3.36}$$

So, inserting (3.34) in (3.36), using (3.33)(iii) and taking into account that, if n is large enough, $\mathcal{S}_{2,n} \subset \mathcal{S}_n$ and $a(x) > 0$ on $\mathcal{S}_n$, we deduce for large n,

$$\begin{aligned}\int_{\mathcal{S}_{1,n}} |\nabla u_n|^2 &\leqslant \int_{\mathcal{S}_{2,n}} \big(u_n^p - a(x)u_n^2\big)\varphi_n + \frac{1}{2}\int_{\mathcal{S}_{2,n}} (\Delta\varphi_n)u_n^2 \\ &\leqslant \int_{\mathcal{S}_n} u_n^p\varphi_n + \frac{1}{2}\int_{\mathcal{S}_n} (\Delta\varphi_n)u_n^2 \\ &\leqslant \int_{\mathcal{S}_n} u_n^p + \frac{1}{2}C\int_{\mathcal{S}_n} u_n^2.\end{aligned}$$

Then, applying Proposition 2.6, taking also into account that for any finite set of indexes (2.22) is true for a suitable choice of the constant c_*, we obtain the thesis. □

4. Multiplicity results

This section is devoted to the proof of the two multiplicity theorems (Theorems 1.2 and 1.4). The proofs will be achieved by using the compactness results proved in Section 2 and well-known variational tools, employed for the approximating problems, based on the use of Krasnoselskii genus. Traditionally this approach is used for searching constrained critical points, however we shall use a recent variant of this method which works with unconstrained min–max classes [43]. This will make working with several functionals at the same time easier, will simplify the use of the Morse index and will let us make a final application in the next section in a case in which the constrained approach is made difficult by a lack of regularity of the constraint manifold.

So we shall begin by introducing the classical Krasnoselskii genus, then we shall pass to the variant introduced in [43], we shall show how the min–max approaches on the natural constraint $\mathcal{V}$ are equivalent to an unconstrained min–max, we shall introduce the double natural constraint $\mathcal{W}$ showing that it enjoys the same property and finally we shall give an estimate on the Morse index. After this introductory parts, in the last two parts of this section we shall give the proof of the two multiplicity theorems.

4.1. *Krasnoselskii genus*

We recall some well-known facts about the Krasnoselskii genus in view of its application to semilinear elliptic equations of the type addressed here, moreover, we introduce some generalizations of those concepts introduced in [43] which will be useful in the following. We refer to [26] and [46] for a more comprehensive treatment of the subject. Throughout this subsection we shall denote by E any given Banach space.

Let $A \subset E$ be a symmetric subset of E, i.e., $A = -A$, we define $\Lambda_k(A)$ as the space of Krasnoselskii test maps on A of dimension n as follows

$$\Lambda_k(A) = \left\{\varphi \in C\left(A, \mathbb{R}^n\right) \mid \varphi(x) = -\varphi(-x)\right\}.$$

We shall call Krasnoselskii genus of A the number $\gamma(A)$ so defined

$$\gamma(A) = \inf\left\{n \in \mathbb{N} \mid \exists \varphi \in \Lambda_k(A)\colon\ 0 \notin \varphi(A)\right\}.$$

We notice that, from the above definitions, for any given $\varphi \in \Lambda_k(A)$, if $k < \gamma(A)$ then $0 \in \varphi(A)$, whereas if $k \geqslant \gamma(A)$ then there exists $\varphi \in \Lambda_k(A)$ such that $0 \notin \varphi(A)$. The main properties of the Krasnoselskii genus, which will be useful for the subsequent arguments, rely on the following statements.

PROPOSITION 4.1. *Let A be any given symmetric subset of a Banach space and let $\eta : A \to E$ be a given odd map. Then $\gamma(\eta(A)) \geqslant \gamma(A)$.*

PROOF. Let $k < \gamma(A)$ and let $\varphi \in \Lambda_k(\eta(A))$. Since $\varphi \circ \eta \in \Lambda_k(A)$, then we have $0 \in \varphi(\eta(A))$ and therefore $\gamma(\eta(A)) > k$. The thesis follows from the arbitrariness of k. □

PROPOSITION 4.2. *Let $S \subset E$ be any closed subspace of co-dimension $k \in \mathbb{N}$. If $A \subset E$ is any symmetric subset such that $\gamma(A) > k$, then $A \cap S \neq \emptyset$.*

PROOF. Let $S^{\perp}$ be the orthogonal complement of S in E, which is isomorphic to $\mathbb{R}^k$, and let $P : E \to S^{\perp}$ be the orthogonal projection map. Then $P \in \Lambda_k(A)$ and $0 \in P(A)$ so, since $P^{-1}(0) = S$, we have that $S \cap A \neq \emptyset$. □

PROPOSITION 4.3. $\gamma(S^{k-1}) = k$.

PROOF. Let $\varphi \in \Lambda_{k-1}(S^{k-1})$, by the Borsuk theorem we have $0 \in \varphi(S^{k-1})$ and so $\gamma(S^{k-1}) > k - 1$. On the contrary, the canonical injection $i : S^{k-1} \to \mathbb{R}^k$ belongs to $\Lambda_k(S^{k-1})$ and $0 \notin i(S^{k-1})$, therefore $\gamma(S^{k-1}) \leqslant k$. □

The next proposition states a trace property for the Krasnoselskii genus.

PROPOSITION 4.4. *Let $A \subset E$ be any given symmetric subset and let $\varphi \in \Lambda_k(A)$, with $k < \gamma(A)$. Then $\gamma(\varphi^{-1}(0)) \geqslant \gamma(A) - k$.*

PROOF. Firstly, let us notice that $\varphi^{-1}(0)$ is a symmetric set. We take $h < \gamma(A) - k$ and $\psi \in \Lambda_h(\varphi^{-1}(0))$. By virtue of the Tietze extension theorem, ψ may be extended to an odd map $\bar{\psi} \in C(A, \mathbb{R}^k)$. We introduce the map $\varphi \times \bar{\psi} : x \mapsto (\varphi(x), \bar{\psi}(x)) \in \mathbb{R}^{k+h}$, with $k + h < \gamma(A)$. Since $\varphi \times \bar{\psi}$ is a Krasnoselskii test map we have that $0 \in (\varphi \times \bar{\psi})(A)$ and this means that $0 \in \psi(\varphi^{-1}(0))$. So $\gamma(\varphi^{-1}(0)) > h$ and by arbitrariness of h we get the thesis. □

4.2. *Genus of a symmetric set*

For any $k \in \mathbb{N}$, we adopt the following notation: $Q_k = [-1, 1]^k$, $F^i_\pm = \{x \in Q_k \mid x_i = \pm 1\}$. Given $n, k \in \mathbb{N}$, for every $x \in \mathbb{R}^{n+k}$, we shall split x as $x = (x_0, x_1, x_2, \dots, x_k)$ with $x_0 \in \mathbb{R}^n$ and $x_i \in \mathbb{R}$ for $i = 1, \dots, k$. Analogously, if $\varphi(x) \in \mathbb{R}^{n+k}$, we shall write $\varphi(x) = (\varphi_0(x), \varphi_1(x), \dots, \varphi_k(x))$ with $\varphi_0(x) \in \mathbb{R}^n$ and $\varphi_i(x) \in \mathbb{R}$ for $i = 1, \dots, k$. Sometimes, we shall also use the notation $x = (x_0, x')$, $\varphi(x) = (\varphi_0(x), \varphi'(x))$ instead of the previous one, by assuming $x', \varphi'(x) \in \mathbb{R}^k$. We shall set $B_n = \{x \in \mathbb{R}^n \mid |x| \leqslant 1\}$, $S^n = \partial B_{n+1} = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$ and we shall denote by E any given Banach space.

For any given $k \in \mathbb{N}$, we set $I_k = \{1, 2, \dots, k\}$ and we denote by $\mathcal{P}_k$ the set of all the involutive permutations on I_k, i.e., $\pi \in \mathcal{P}_k$ if and only if $\pi : I_k \to I_k$ and $\pi \circ \pi = \mathrm{id}$. Given any $\pi \in \mathcal{P}_k$, we introduce the map $\hat{\pi} : \mathbb{R}^k \to \mathbb{R}^k$ so defined

$$\hat{\pi}(x_1, x_2, \dots, x_k) = (x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(k)}).$$

We have

$$\hat{\pi}\bigl(\hat{\pi}(x)\bigr) = \hat{\pi}(x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(k)}) = (x_{\pi(\pi(1))}, x_{\pi(\pi(2))}, \dots, x_{\pi(\pi(k))}) = x,$$

which means $\hat{\pi} \circ \hat{\pi} = \mathrm{id}$. Furthermore, given any Banach space E, let us define the map $\pi_E : E \times \mathbb{R}^k \to E \times \mathbb{R}^k$ such that $\pi_E : (x_0, x') \mapsto (-x_0, \hat{\pi}(x'))$. It is easily seen that π_E is involutive too, indeed

$$\pi_E\bigl(\pi_E(x)\bigr) = \pi_E\bigl(-x_0, \hat{\pi}(x')\bigr) = \bigl(x_0, \hat{\pi}\bigl(\hat{\pi}(x')\bigr)\bigr) = \bigl(x_0, x'\bigr) = x$$

and so we also have $\pi_E \circ \pi_E = \mathrm{id}$. We shall set $\tilde{\pi} = \pi_E$ if $E = \mathbb{R}^n$. Now, let $A \subset E$ be any symmetric subset and let $\varphi : A \times Q_k \to \mathbb{R}^{n+k}$ and $\pi \in \mathcal{P}_k$ be given, we set

$$\varphi_\pi = \tilde{\pi} \circ \varphi \circ \pi_A : x \mapsto \tilde{\pi}\bigl(\varphi\bigl(\pi_E(x)\bigr)\bigr),$$

where $\pi_A : A \times Q_k \to A \times Q_k$ is the restriction of π_E. We shall refer to φ_π as to the π-*symmetric* of φ. We shall say that φ is π-*symmetric* if $\varphi = \varphi_\pi$ and that φ is *symmetric* if there exists π such that $\varphi = \varphi_\pi$. We note that $(\varphi_\pi)_\pi = \tilde{\pi} \circ (\tilde{\pi} \circ \varphi \circ \pi_A) \circ \pi_A = \varphi$.

REMARK 4.1. We observe that if $\pi = \mathrm{id}$ φ is π-symmetric if

(S1) $\varphi_0(-x_0, x') = -\varphi_0(x_0, x')$ $\forall x \in A \times Q_k$,

(S2) $\varphi'(-x_0, x') = \varphi'(x_0, x')$ $\forall x \in A \times Q_k$.

DEFINITION 4.1. Let $A \subset E$ be any given symmetric subset, for any $k \in \mathbb{N}$ we shall say that a symmetric function $\varphi \in C(A \times Q_k, \mathbb{R}^{n+k})$ is a *test function of dimension* n for A if the following condition holds

(T) $\pm\varphi_i(x) \geqslant 0\ \forall x \in A \times F^i_{\pm}, \forall i = 1, \dots, k$.

We shall denote by $\Lambda^*_n(A)$ the set of the n-dimensional test functions for A, obtained for any value of k. Then we are ready to define the *genus* of a set A as the number

$$\gamma^*(A) = \inf\bigl\{n \in \mathbb{N} \bigm| \exists\varphi \in \Lambda^*_n(A) \bigm| \text{ s.t. } 0 \notin \varphi(A \times Q_k)\bigr\}.$$

Firstly, we remark that, since $\Lambda^*_n(A)$ is a larger set than the set $\Lambda_n(A)$ of the test maps related to the Krasnoselskii genus (indeed the last one coincides with the subset of the former one which contains the test functions constructed by taking $k = 0$), we have that, in general, $\gamma^*(A) \leqslant \gamma(A)$. By definition, given any $\varphi \in \Lambda^*_n(A)$ with $n < \gamma^*(A)$, $0 \in \varphi(A)$.

In view of proving the analogous statement of Propositions 4.1–4.4 for the genus γ^*, we prove some useful lemmas stating some properties of the test functions.

LEMMA 4.1. *Let $\varphi : A \times Q_k \to \mathbb{R}^{n+k}$ satisfy* (T) *and let $\pi \in \mathcal{P}_k$ be given, then φ_π also satisfies* (T).

PROOF. Let $x = (x_0, x') \in A \times F^{\pm}_i$ for $i = 1, \dots, k$. Then $\hat{\pi}(x') \in F^{\pm}_{\pi(i)}$ and, since φ satisfies (T), we have

$$\pm\varphi_{\pi(i)}\bigl(\pi_E\bigl(x_0, x'\bigr)\bigr) = \pm\varphi_{\pi(i)}\bigl(-x_0, \hat{\pi}\bigl(x'\bigr)\bigr) \geqslant 0.$$

By the previous definitions we know that $\varphi_{\pi(i)} = (\tilde{\pi} \circ \varphi)_i$ and so $\pm(\tilde{\pi} \circ \varphi)_i(\pi_E(x)) \geqslant 0$, that is, $\pm(\varphi_\pi)_i(x) \geqslant 0$, as stated in the thesis. □

LEMMA 4.2. *Let $A \subset E$ be any given symmetric subset. Then every test function $\varphi \in \Lambda^*_n(A)$ can be extended to a map in $\Lambda^*_n(E)$.*

PROOF. Firstly, by virtue of the Tietze–Dugundji theorem (see [22]) we take an extension of the components φ_i on $E \times F^{\pm}_i$ valued in $\mathbb{R}_{\pm}$ keeping the sign property (T), then we extend them and φ_0 continuously on all of $E \times Q_k$ so getting a map $\bar{\varphi} : E \times Q_k \to \mathbb{R}^{n+k}$ which, obviously, satisfies (T). By Lemma 4.1, the map $\bar{\varphi}_\pi$, which is an extension of φ_π, satisfies (T) and the same happens for $\varphi_s = \frac{1}{2}\bar{\varphi} + \frac{1}{2}\bar{\varphi}_\pi : E \times Q_k \to \mathbb{R}^{n+k}$, where $\pi \in \mathcal{P}_k$ is such that φ is π-symmetric. Obviously, φ_s is symmetric since

$$(\varphi_s)_\pi = \frac{1}{2}\bar{\varphi}_\pi + \frac{1}{2}(\bar{\varphi}_\pi)_\pi = \frac{1}{2}\bar{\varphi}_\pi + \frac{1}{2}\bar{\varphi} = \varphi_s,$$

so $\varphi_s \in \Lambda^*_n(E)$. If $x \in A \times Q_k$ then $\varphi(x) = \varphi_\pi(x)$, hence $\varphi(x) = \bar{\varphi}(x) = \bar{\varphi}_\pi(x)$ and so $\varphi(x) = \varphi_s(x)$. Therefore φ_s extends φ to all of E. □

The property in Proposition 4.1 also holds for γ^*.

PROPOSITION 4.5. *Let $A \subset E$ be any given symmetric subset and let $\eta : A \to E$ be a given odd continuous map. Then $\gamma^*(\eta(A)) \geqslant \gamma^*(A)$.*

PROOF. Let $n < \gamma^*(A)$, $\varphi \in \Lambda_n(\eta(A))$ and let $\bar\eta(x_0, x') = (\eta(x_0), x')$, $\bar\eta : A \times Q_k \to \eta(A) \times Q_k$. Given $\pi \in \mathcal{P}_k$, we have

$$\begin{aligned}\pi_E\big(\bar\eta\big(x_0, x'\big)\big) &= \big(-\eta(x_0), \hat\pi\big(x'\big)\big) = \big(\eta(-x_0), \hat\pi\big(x'\big)\big)\\ &= \bar\eta\big(-x_0, \hat\pi\big(x'\big)\big) = \bar\eta\big(\pi_E\big(x'\big)\big)\end{aligned}$$

and so $\pi_{\eta(A)} \circ \bar\eta = \bar\eta \circ \pi_A$. We take $\bar\varphi = \varphi \circ \bar\eta : A \times Q_k \to \mathbb{R}^{n+k}$, then

$$\bar\varphi_\pi = \tilde\pi \circ \bar\varphi \circ \pi_A = \tilde\pi \circ \varphi \circ \bar\eta \circ \pi_A = \tilde\pi \circ \varphi \circ \pi_{\eta(A)} \circ \bar\eta = \varphi_\pi \circ \bar\eta.$$

Therefore, if we fix π such that φ is π-symmetric, that is, $\varphi = \varphi_\pi$, then $\bar\varphi_\pi = \varphi \circ \bar\eta = \bar\varphi$, hence also $\bar\varphi$ is symmetric. Since $\bar\eta(A \times F_i^{\pm}) = \eta(A) \times F_i^{\pm}$, we deduce that $\bar\varphi$ satisfies (T). Thus $\bar\varphi \in \Lambda_n^*(A)$, hence

$$0 \in \bar\varphi(A \times Q_k) = \varphi\big(\bar\eta(A \times Q_k)\big) = \varphi\big(\eta(A) \times Q_k\big).$$

It follows that $n < \gamma^*(\eta(A))$ by the arbitrariness of φ and, consequently, $\gamma^*(A) \leqslant \gamma^*(\eta(A))$ by the arbitrariness of n. $\square$

Proposition 4.2 directly applies to γ^* since $\gamma^*(A) \leqslant \gamma(A)$.

In order to prove the analogous of Proposition 4.3 we need a topological lemma which states a more general property than the Borsuk theorem.

THEOREM 4.1. *Let $f : B_n \times Q_k \to \mathbb{R}^{n+k}$ be a continuous map symmetric on $\partial B_n \times Q_k$ (i.e., such that for some $\pi \in \mathcal{P}_k$, $f(x) = f_\pi(x)\ \forall x \in S^{n-1} \times Q_k$) and assume that* (T) *holds for $\varphi = f$ and $A = B_n$. Then there exists $x \in B_n \times Q_k$ such that $f(x) = 0$.*

PROOF. The first step consists in introducing a suitable change of variables which will allow to deal with the symmetry properties involved in the most convenient way.

For given $k \in \mathbb{N}$ and $\pi \in \mathcal{P}_k$, let us introduce the sets

$$I_0 = \big\{i \in I_k \mid i = \pi(i)\big\}, \qquad I_1 = \big\{i \in I_k \mid i < \pi(i)\big\}.$$

We note that, if $k_i = \sharp I_i$, $k_0 + 2k_1 = k$. Let us introduce the functions p_1 and p_2, both defined on $\mathbb{R}^k$ as

$$p_1(x) = (x_i + x_{\pi(i)})_{i \in I_1} + (x_i)_{i \in I_0}, \qquad p_2(x) = (x_i - x_{\pi(i)})_{i \in I_1}.$$

We have $\mathbb{R}^k \cong p_1(\mathbb{R}^k) \oplus p_2(\mathbb{R}^k)$. Then for every $x \in B_n \times Q_k$, we set

$$\mathbf{x}_1 = \big(x_0, p_2\big(x'\big)\big), \qquad \mathbf{x}_2 = p_1\big(x'\big),$$

and so we have $x = (x_0, x') \cong (\mathbf{x}_1, \mathbf{x}_2)$. Analogously, for any $f = (f_0, f') \in \mathbb{R}^{n+k}$, we define the functions

$$\mathbf{f}_1 = \big(f_0, p_2 \circ f'\big), \qquad \mathbf{f}_2 = p_1 \circ f',$$

and so $f \cong (\mathbf{f}_1, \mathbf{f}_2)$. Now we observe that $\mathbf{x}_1 = 0$ if and only if $x_0 = 0$ and, for every $i \in I_k$, $x_i = x_{\pi(i)}$, that $(-\mathbf{x}_1, \mathbf{x}_2) \cong (-x_0, \hat{\pi}(x')) = \tilde{\pi}(x)$ and that, consequently, f π-symmetric means

$$f(-\mathbf{x}_1, \mathbf{x}_2) = \tilde{\pi}\big(\varphi(\mathbf{x}_1, \mathbf{x}_2)\big) = \big(-\mathbf{f}_1(\mathbf{x}_1, \mathbf{x}_2), \mathbf{f}_2(\mathbf{x}_1, \mathbf{x}_2)\big). \tag{4.1}$$

We assume by contradiction that

$$0 \notin f\big(\partial(B_n \times Q_k)\big) = f\left(\big(S^{n-1} \times Q_k\big) \cup \bigcup_{i=1}^{k} \big(B_n \times \big(F_i^+ \cup F_i^-\big)\big)\right).$$

In such a case, we shall show that the topological degree of f in zero is different from zero, that is, $\deg(B_n \times Q_k, f, 0) \neq 0$. The first step in this direction consists in forcing the assumption (T) to be satisfied with strict inequalities, by adding to f' the function $\varepsilon x'$ with $\varepsilon > 0$ suitably small in order to keep the value of the topological degree. We just remark that the function $g : (x_0, x') \mapsto (0, x')$ is π-symmetric, as it can be easily seen. Then, through a linear homotopy, we can pass from f to $\frac{1}{2}(f + f_\pi)$, which we will continue to denote by f. By Lemma 4.1 we know that the symmetrization does not change the degree since f is not modified on $\partial B_n \times Q_k$ and f_i and $(f_\pi)_i$ have a fixed sign on $B_n \times F_i^\pm$.

By a standard perturbation argument we can also assume $f \in C^1(B_n \times Q_k, \mathbb{R}^{n+k})$. We set $X = \{x \in \mathbb{R}^{n+k} \mid \mathbf{x}_1 = 0\}$. We shall introduce further modifications of f which make $f^{-1}(0) \cap X$ *contain only regular points*. Firstly we know that, by the oddness of $\mathbf{f}_1$ with respect to the variable $\mathbf{x}_1$ stated in (4.1), $\mathbf{f}_1 = 0$ on X and so the zeros of f on X are the zeros of $\mathbf{f}_2$. Then we observe that, since after the previous symmetrization $\mathbf{f}_2$ is even with respect to the variable $\mathbf{x}_1$, so the partial Jacobian matrix $J_{\mathbf{x}_1}\mathbf{f}_2(x)$ is identically zero for every $x \in X$. Then, for any $x \in X$, by Laplace rule we get $|Jf(x)| = |J_{\mathbf{x}_1}\mathbf{f}_1(x)||J_{\mathbf{x}_2}\mathbf{f}_2(x)|$. We can force $|J_{\mathbf{x}_2}\mathbf{f}_2| \neq 0$ on every $x \in X$ such that $\mathbf{f}_2(x) = 0$ by subtracting from $\mathbf{f}_2$ a small regular value $h \in p_1(\mathbb{R}^k)$, given by the Sard theorem. Note that this perturbation does not affect the symmetry properties of f and, if h is taken sufficiently small, does not change the value of the topological degree. In particular, we get that $f^{-1}(0) \cap X$ is a finite set and therefore the set L which contains all the eigenvalues of $J_{\mathbf{x}_1}\mathbf{f}_1(x)$ in the points of $f^{-1}(0) \cap X$ is also finite. Then we can force the determinant $|J_{\mathbf{x}_1}\mathbf{f}_1(x)|$ to be different from zero in such points by adding to $\mathbf{f}_1$ the function $-\lambda\mathbf{x}_1$ with $\lambda \in \mathbb{R} \setminus L$. Again, this new perturbation preserves the symmetry properties of f and, if λ is sufficiently small, does not change the value of $\deg(B_n \times Q_k, f, 0)$ and the zeros of $\mathbf{f}_2$ remain of course the same. So we can be sure that f has only regular zeros on X.

Now, since $\mathbf{f}_2$ satisfies the hypotheses of the Miranda theorem (see [33]) on $X \cong Q_{k_0+k_1}$, we can state that $f^{-1}(0) \cap X$ is composed by an odd number of regular zeros. By continuity, we have that there exists a small $\varepsilon > 0$ such that the closed tubular neighborhood

$X_\varepsilon = \{x \in B_n \times Q_k \mid d(x, X) \leqslant \varepsilon\}$ contains only regular zeros. In order to force 0 to be a regular value for f we have to deal with the set $(B_n \times Q_k) \setminus X_\varepsilon$ and, to this aim, we argue as follows.

Let $A_i^\pm = \{x \in B_n \times Q_k \mid \pm\mathbf{x}_1^i \geqslant \frac{\varepsilon}{n}\}$, where, for $i = 1, \dots, n + k_1$, $\mathbf{x}_1^i$ is the component of $\mathbf{x}_1$ of index i. One can easily see that

$$(B_n \times Q_k) \setminus X_\varepsilon \subset \bigcup_{i=1}^{n+k_1} \left(A_i^+ \cup A_i^-\right).$$

We are going to perform a new perturbation of f, which keeps the symmetry properties and is too small to change the topological degree or to introduce singular zeros in X_ε, in order to exclude the presence of singular zeros of f also on $A_1^\pm$. Let S be the set of singular values of f. By the Sard theorem $S \cup \tilde{\pi}(S)$ is a negligible set, so we can take $h_1 \in \mathbb{R}^{n+k} \setminus (S \cup \tilde{\pi}(S))$ arbitrarily small. Let $g : \mathbb{R} \to [0, 1]$ be a smooth function such that $g(x) = 0$ for $x \leqslant 0$ and $g(x) = 1$ for $x \geqslant 1$. Let $\psi : \mathbb{R}^n \to \mathbb{R}^{n+k}$ be defined as $\psi(x) = g(\frac{n}{\varepsilon}\mathbf{x}_1^1)h_1 + g(-\frac{n}{\varepsilon}\mathbf{x}_1^1)\tilde{\pi}(h_1)$. One easily sees that $\psi(x) = h_1$ if $x \in A_1^+$ and $\psi(x) = \tilde{\pi}(h_1)$ if $x \in A_1^-$. So the function $\bar{f} : x \mapsto f(x) - \psi(x)$ has no singular zeros on $A_1^+ \cup A_1^-$. Moreover, $\bar{f}$ keeps the symmetry properties of f. The value of the degree and the regularity of the zeros in X_ε are preserved by stability, provided h_1 is taken suitably small.

We proceed in this construction through $n + k_1$ steps, from $i = 1$ to $i = n + k_1$. Thanks to the stability property of the regular points, we are sure that at each step the regularity gained at the previous step on $X_\varepsilon \cup \bigcup_{j=1}^{i-1}(A_j^+ \cup A_j^-)$ is kept, provided the perturbation term h_i given by the Sard theorem is chosen sufficiently small. Therefore, we can conclude that we have regularity everywhere on $B_n \times Q_k$ and so 0 is a regular value for f. Finally, we know that $f^{-1}(0) \cap X$ is made by an odd number of points and, since $f^{-1}(0) = \tilde{\pi}(f^{-1}(0))$, $f^{-1}(0) \setminus X$ is made by an even number of points, indeed $\tilde{\pi}(x) \neq x$ for $x \notin X$. Then we can conclude that $\deg(B_n \times Q_k, f, 0)$ is odd and so we get the thesis. □

REMARK 4.2. It is worth to notice that the previous theorem reduces to the Borsuk theorem when $k = 0$ and to the Miranda theorem when $n = 0$ and $\pi = \mathrm{id}$. If it is easy to see that Miranda theorem can be deduced from the Borsuk theorem, nevertheless reconducing the above statement to the Borsuk theorem does not seem to be an obvious task.

PROPOSITION 4.6. $\gamma^*(S^n) = n + 1$.

PROOF. We know that $\gamma^*(S^n) \leqslant \gamma(S^n) \leqslant n + 1$. On the other hand, let $\varphi \in \Lambda_n^*(S^n)$ and let $S_+^n = \{(x_1, \dots, x_{n+1}) \in S^n \mid x_{n+1} \geqslant 0\}$. S_+^n is homeomorphic to B_n and so, by Theorem 4.1, there exists $x \in S_+^n \times Q_k$ such that $\varphi(x) = 0$. So $n < \gamma^*(S^n)$. □

Also the trace property in Proposition 4.4 is enjoyed by γ^*.

PROPOSITION 4.7. *Let $A \subset E$ be any given symmetric subset and let $\varphi \in \Lambda_k(A)$, with $k < \gamma^*(A)$. Then $\gamma^*(\varphi^{-1}(0)) \geqslant \gamma^*(A) - k$.*

PROOF. Let $k \leqslant n < \gamma^*(A)$ and $\psi \in \Lambda^*_{n-k}(\varphi^{-1}(0))$, $\psi : \varphi^{-1}(0) \times Q_h \to \mathbb{R}^{n-k+h}$. By Lemma 4.2, ψ can be extended to $\bar{\psi}$ defined on all of E (and so on A) and such that $\bar{\psi} \in \Lambda^*_{n-k}(A)$. Let us consider the function $\varphi \times \bar{\psi} : A \times Q_h \to \mathbb{R}^{n+h}$ defined as

$$(\varphi \times \bar{\psi})(x) = (\varphi(x_0), \bar{\psi}(x)) = ((\varphi(x_0), \bar{\psi}_0(x)), \bar{\psi}'(x)).$$

It is easily seen that $\varphi \times \bar{\psi}$ satisfies (T), since the components involved in such condition only belong to $\bar{\psi}$. Moreover, if $\pi \in \mathcal{P}_h$ then

$$\begin{aligned} \tilde{\pi}\big((\varphi \times \bar{\psi})(\pi_E(x))\big) &= \tilde{\pi}\big((\varphi(-x_0), \bar{\psi}_0(\pi_E(x))), \bar{\psi}'(\pi_E(x))\big) \\ &= \big((-\varphi(-x_0), -\bar{\psi}_0(\pi_E(x))), \hat{\pi}(\bar{\psi}'(\pi_E(x)))\big) \\ &= \big((\varphi(x_0), (\tilde{\pi} \circ \bar{\psi})_0(\pi_E(x))), (\tilde{\pi} \circ \bar{\psi})' \pi_E(x)\big) \\ &= (\varphi \times \bar{\psi}_\pi)(x). \end{aligned}$$

Therefore $(\varphi \times \bar{\psi})_\pi = \varphi \times \bar{\psi}_\pi$, hence since $\bar{\psi}$ is symmetric, then also $\varphi \times \bar{\psi}$ satisfies the same condition. Thus $\varphi \times \bar{\psi} \in \Lambda_n(A)$ and then $0 \in (\varphi \times \bar{\psi})(A \times Q_h)$, that is, $0 \in \bar{\psi}(\varphi^{-1}(0) \times Q_h)$.

By the arbitrariness of $\bar{\psi}$, we get $n - k < \gamma^*(\varphi^{-1}(0))$ and by the arbitrariness of n, the thesis follows. □

A different trace property, which actually is the main motivation for passing to the present variant of the notion of genus, can be now also proved.

PROPOSITION 4.8. *Let $A \subset E$ be any symmetric subset and let $\sigma : A \times Q_k \to E$ and $\varphi : E \to \mathbb{R}^k$ be given. If there exists $\pi \in \mathcal{P}_k$ such that*

(a) $\forall x \in A \times Q_k$: $\sigma(\pi_E(x)) = -\sigma(x)$,

(b) $\forall u \in E$: $\varphi(-u) = \hat{\pi}(\varphi(u))$,

(c) $\forall x \in A \times F_i^\pm$: $\pm\varphi_i(\sigma(x)) \geqslant 0$ $\forall i = 1, \ldots, k$,

then $\gamma^*(\sigma(A \times Q_k) \cap \varphi^{-1}(0)) \geqslant \gamma^*(A)$.

PROOF. Firstly we observe that conditions (a) and (b) allow to state respectively that $\sigma(A \times Q_k)$ and $\varphi^{-1}(0)$ are symmetric subsets of E and so the assertion makes sense. We fix $n < \gamma^*(A)$ and a test function ψ of dimension n defined on $(\sigma(A \times Q_k) \cap \varphi^{-1}(0))$ and extended by Lemma 4.2 on E. Then $\psi : E \times Q_h \to \mathbb{R}^{n+h}$ satisfies (T) and is symmetric for some $\pi_1 \in \mathcal{P}_h$. Let us define $\rho \in \mathcal{P}_{k+h}$ such that $\rho(i) = \pi_1(i)$ if $i \leqslant h$ and $\rho(i) = h + \pi(i - h)$ if $i > h$. In this way, if $x' \in \mathbb{R}^{h+k}$ is decomposed as $x' = (z', y')$ with $z' \in \mathbb{R}^h$ and $y' \in \mathbb{R}^k$, we have $\hat{\rho}(x') = (\hat{\pi}_1(z'), \hat{\pi}(y'))$. We define $\vartheta : A \times Q_{h+k} \to \mathbb{R}^{n+h+k}$ by setting

$$\vartheta(x_0, x') = \vartheta(x_0, z', y') = \big(\psi(\sigma(x_0, y'), z'), \varphi(\sigma(x_0, y'))\big),$$

that is, $\vartheta_0 = \psi_0$ and $\vartheta' = (\psi', \varphi)$. We are going to prove that $\vartheta \in \Lambda^*_n(A)$. Since ψ satisfies (T), we have that, for $i \leqslant h$, $\pm\vartheta_i = \pm\psi_i \geqslant 0$ if $z' \in F_i^\pm$, whereas condition (c) states that $\pm\vartheta_{h+i} = \pm\varphi_i \geqslant 0$ if $y' \in F_i^{\pm}$ and so ϑ satisfies (T). Moreover, by (a), (b) and the symmetry of ψ,

$$\begin{aligned}
\vartheta\big(\rho_E(x)\big) &= \vartheta\big(-x_0, \hat\pi_1\big(z'\big), \hat\pi\big(y'\big)\big) \\
&= \big(\psi\big(\sigma\big(-x_0, \hat\pi\big(y'\big)\big), \hat\pi_1\big(z'\big)\big), \varphi\big(\sigma\big(-x_0, \hat\pi\big(y'\big)\big)\big)\big) \\
&= \big(\psi\big(\sigma\big(\pi_E\big(x_0, y'\big)\big), \hat\pi_1\big(z'\big)\big), \varphi\big(\sigma\big(\pi_E\big(x_0, y'\big)\big)\big)\big) \\
&= \big(\psi\big(-\sigma\big(x_0, y'\big), \hat\pi_1\big(z'\big)\big), \varphi\big(-\sigma\big(x_0, y'\big)\big)\big) \\
&= \big(\psi\big((\pi_1)_E\big(\sigma\big(x_0, y'\big), z'\big)\big), \hat\pi\big(\varphi\big(\sigma\big(x_0, y'\big)\big)\big)\big) \\
&= \big(\tilde\pi_1\big(\psi\big(\sigma\big(x_0, y'\big), z'\big)\big), \hat\pi\big(\varphi\big(\sigma\big(x_0, y'\big)\big)\big)\big) \\
&= \big(-\psi_0\big(\sigma\big(x_0, y'\big), z'\big), \hat\pi_1\big(\psi'\big(\sigma\big(x_0, y'\big), z'\big)\big), \hat\pi\big(\varphi\big(\sigma\big(x_0, y'\big)\big)\big)\big) \\
&= \big(-\psi_0\big(\sigma\big(x_0, y'\big), z'\big), \hat\rho\big(\psi'\big(\sigma\big(x_0, y'\big), z'\big)\big), \varphi\big(\sigma\big(x_0, y'\big)\big)\big) \\
&= \big(-\vartheta_0(x), \hat\rho\big(\vartheta'(x)\big)\big) \\
&= \tilde\rho\big(\vartheta(x)\big).
\end{aligned}$$

Then ϑ is symmetric and so $\vartheta \in \Lambda_n^*(A)$ as claimed above and, since $n < \gamma^*(A)$, $0 \in \vartheta(A \times Q_{h+k})$. This means that there exist $x_0 \in A$, $y' \in Q_k$ and $z' \in Q_h$ such that $\psi(\sigma(x_0, y'), z') = 0$ and $\varphi(\sigma(x_0, y')) = 0$. So $0 \in \psi(((\sigma(A \times Q_k) \cap \varphi^{-1}(0))) \times Q_h)$. By the arbitrariness of ψ, one gets $n \leqslant \gamma^*(\sigma(A \times Q_k) \cap \varphi^{-1}(0))$ and by the arbitrariness of n, one gets the thesis. □

4.3. *Min–max classes on the natural constraint*

Let $I : H \to \mathbb{R}$ be one of the functionals related to the problem (CP) or to the problem (P), with $H = H_0^1(\Omega)$ or $H = H^1(\mathbb{R}^N)$, respectively. Let $\mathcal{V}$ be the natural constraint defined in (1.16). For any fixed n we introduce the class of sets

$$\Gamma_n^{\mathcal{V}} = \big\{A \subset \mathcal{V} \mid A \text{ compact}, \gamma^*(A) \geqslant n\big\}$$

and denote by c_n the corresponding min–max level, that is,

$$c_n = \inf_{A \in \Gamma_n^{\mathcal{V}}} \sup_{u \in A} I(u).$$

We assume $c_n > 0$. Such an assumption will let us find a neighborhood C^- of 0 such that for every $u \in C^-$,

$$I(u) \leqslant \frac{c_n}{2}, \quad \nabla I(u) \cdot u \geqslant 0,$$

in conjunction to which we take a neighborhood of infinity C^+ such that for every $u \in C^+$,

$$I(u) \leqslant 0 \quad \text{and therefore} \quad \nabla I(u) \cdot u \leqslant 0.$$

Let us observe that, given any $u \in \mathcal{V}$, the function $\alpha \mapsto I(\alpha u)$ is strictly increasing between 0 and 1, strictly decreasing after 1 and tends to $-\infty$ as $\alpha \to +\infty$. So two constants $\varepsilon, c > 0$ such that $\varepsilon u \in C^-$ and $cu \in C^+$ always exist. Moreover, if A is any compact subset of $\mathcal{V}$, ε and c can be uniformly fixed for $u \in A$, also ε can be fixed in maximal way and c can be fixed in minimal way such that both the values turn out to be considered as functions of A. We introduce the function $\sigma_A : A \times [-1, 1] \to H$, defined as

$$\sigma_A(u, \alpha) = \left(\varepsilon + (\alpha + 1)\frac{c - \varepsilon}{2}\right)u,$$

satisfying the conditions

$$\sigma_A(u, -1) = \varepsilon u, \qquad \sigma_A(u, 1) = cu \quad \forall u \in A$$

and the set

$$\Sigma_A = \sigma_A\big(A \times [-1, 1]\big)$$

which will be named *simple homothetic expansion of A*. Let us notice that if $A \in \Gamma_n^{\mathcal{V}}$ then σ_A belongs to the class of continuous functions defined by

$$\begin{aligned}\mathcal{F}_n = \big\{\sigma\colon\ A \times [-1, 1] \to H \,\big|\, A \in \Gamma_n^{\mathcal{V}},\\ \forall u \in A\colon\ \sigma(-u, \alpha) = -\sigma(u, \alpha),\\ \sigma(u, -1) \in C^-, \sigma(u, 1) \in C^+\big\}\end{aligned}$$

and Σ_A belongs to the class of sets defined by

$$\begin{aligned}\Gamma_n = \big\{X \subset H \setminus \{0\} \,\big|\, \exists A \in \Gamma_n^{\mathcal{V}},\\ \exists \sigma : A \times [-1, 1] \to H, \sigma \in \mathcal{F}_n,\ \text{s.t. } X = \sigma\big(A \times [-1, 1]\big)\big\}.\end{aligned}$$

Let us introduce two further classes of continuous functions and sets, namely

$$\begin{aligned}\mathcal{F}_n^* = \big\{\varphi : H \to \mathbb{R} \,\big|\, \varphi(-u) = -\varphi(u),\\ \varphi(u) \leqslant 0\ \forall u \in H \cap C^-, \varphi(u) \geqslant 0\ \forall u \in H \cap C^+\big\},\end{aligned}$$

$$\Gamma_n^* = \big\{X \subset H \setminus \{0\} \,\big|\, X \text{ compact}, X = -X,\ \forall \varphi \in \mathcal{F}_n^*\colon\ \gamma^*\big(X \cap \varphi^{-1}(0)\big) \geqslant n\big\}.$$

LEMMA 4.3. $\Gamma_n \subset \Gamma_n^*$.

PROOF. Let $\varphi \in \mathcal{F}^*$, $A \in \Gamma_n^{\mathcal{W}}$, $\sigma : A \times Q_1 \to H$, $\sigma \in \mathcal{F}_n$ be fixed. By virtue of Theorem 4.8 we have $\gamma^*(\sigma(A \times Q_1) \cap \varphi^{-1}(0)) \geqslant n$. By the arbitrariness of $\varphi \in \mathcal{F}^*$, we get $\sigma(A \times Q_1) \in \Gamma_n^*$. □

LEMMA 4.4. *For every $A \in \Gamma_n^*$, we have $A \cap \mathcal{V} \in \Gamma_n^{\mathcal{V}}$.*

PROOF. To prove the statement it suffices to note that the function $\varphi : u \mapsto \nabla I(u) \cdot u$ belongs to $\mathcal{F}_n^*$ and hence, by the definition of Γ_n^*, the thesis follows. □

LEMMA 4.5. *If* $A \in \Gamma_n^{\mathcal{V}}$ *then* $\sup_{\Sigma_A} I = \sup_A I$.

PROOF. The assertion trivially follows from the definition of Σ_A. □

LEMMA 4.6. $\inf_{A\in\Gamma_n} \sup_{u\in A} I(u) = \inf_{A\in\Gamma_n^*} \sup_{u\in A} I(u) = c_n$.

PROOF. By virtue of the inclusion $\Gamma_n \subset \Gamma_n^*$, we have

$$\inf_{A\in\Gamma_n^*} \sup_{u\in A} I(u) \leqslant \inf_{A\in\Gamma_n} \sup_{u\in A} I(u).$$

Moreover, for every $X \in \Gamma_n^*$, from Lemma 4.4 we have

$$c_n \leqslant \sup_{u\in X\cap\mathcal{V}} \leqslant \sup_{u\in X} I(u)$$

and so, by the arbitrariness of X, $c_n \leqslant \inf_{A\in\Gamma_n^*} \sup_{u\in A} I(u)$. Finally, by Lemma 4.5 we have that, for every $X \in \Gamma_n^{\mathcal{V}}$, since $\Sigma_X \in \Gamma_n$,

$$\inf_{A\in\Gamma_n} \sup_{u\in A} I(u) \leqslant \sup_{u\in\Sigma_X} I(u) = \sup_{u\in X} I(u).$$

By the arbitrariness of X in $\Gamma_n^{\mathcal{V}}$, we have

$$\inf_{A\in\Gamma_n} \sup_{u\in A} I(u) \leqslant \inf_{A\in\Gamma_n^{\mathcal{V}}} \sup_{u\in A} I(u) = c_n.$$

□

PROPOSITION 4.9. Γ_n *and* Γ_n^* *are two admissible min–max classes.*

PROOF. Since $c_n > 0$, we can find deformations η reducing the level c_n and leaving unchanged the sublevel of $\frac{c_n}{2}$. Obviously such deformations do not change the sets C^- and C^+. It follows that if $\sigma \in \mathcal{F}_n$ and $\varphi \in \mathcal{F}_n^*$ then $\eta \circ \sigma \in \mathcal{F}_n$ and so if $A \in \Gamma_n$ then $\eta(A) \in \Gamma_n$ and if $A \in \Gamma_n^*$ then $\eta(A) \in \Gamma_n^*$. □

By the above results we can state that, for every n, the min–max levels of the classes in $\Gamma_n^{\mathcal{V}}$ are all critical levels provided they satisfy the Palais–Smale condition, regardless any regularity property of the constraint since they are levels of the unconstrained admissible min–max classes Γ_n and Γ_n^*. The corresponding critical points have also a characterization of unconstrained min–max points with the relative properties like estimates of the Morse index or others. Moreover, the three classes are equivalent from the point of view of the localization of the critical points near the maximum points of the terms of a minimizing sequence of sets, as stated in the next proposition.

PROPOSITION 4.10. *If $(A_n)_{n\in\mathbb{N}}$ is any minimizing sequence in $\Gamma_n^{\mathcal{V}}$, i.e., for every $n \in \mathbb{N}$, $A_n \in \Gamma_n^{\mathcal{V}}$ and $\lim_n(\sup_{u\in A_n} I(u)) = c_n$, then the sequence of the homothetic expansions $(\Sigma_{A_n})_{n\in\mathbb{N}}$ is a minimizing sequence in Γ_n (and so, in particular, in Γ_n^*).*

PROOF. The assertion easily follows by Lemma 4.5 and by Proposition 4.9. □

PROPOSITION 4.11. *If $(A_n)_{n\in\mathbb{N}}$ is any minimizing sequence in Γ_n^* (and so, in particular, in Γ_n) then the sequence of the traces on $\mathcal{V}(A_n \cap \mathcal{V})_{n\in\mathbb{N}}$ is a minimizing sequence in $\Gamma_n^{\mathcal{V}}$.*

PROOF. The assertion easily follows by Lemma 4.4 and by Proposition 4.9. □

4.4. *Min–max classes on the double natural constraint*

In this subsection we shall apply the above introduced notion of genus to the study of the min–max classes on the double natural constraint. To this aim we fix $\pi \in \mathcal{P}_2$, $\pi \neq \mathrm{id}$, so that we simply have $\pi(1) = 2$, $\pi(2) = 1$, and consequently we fix the maps $\hat{\pi}$ and π_E as defined in the previous subsection. Let $\Omega \subset \mathbb{R}^N$ be given and let $I : H \to \mathbb{R}$ be the functional defined in (1.15), related to problem (SP), with $H = H_0^1(\Omega)$. For every $x \in \Omega$, we set, as usual, $u^+(x) = \max(u(x), 0)$, $u^-(x) = \max(-u(x), 0)$. Let λ_1 be the first eigenvalue of $-\Delta$ on Ω, we suppose $\lambda < \lambda_1$. Let $\mathcal{W}$ be the double natural constraint so defined

$$\mathcal{W} = \{u \mid u^\pm \neq 0, \nabla I(u) \cdot u^\pm = 0\} \subset \mathcal{V}, \tag{4.2}$$

we set for $n \geqslant 1$,

$$\Gamma_n^{\mathcal{W}} = \{A \subset \mathcal{W} \mid A \text{ compact}, \gamma^*(A) \geqslant n\}$$

and

$$c_n = \inf_{A\in\Gamma_n^{\mathcal{W}}} \sup_{u\in A} I(u).$$

Since $\lambda < \lambda_1$ then there exists a ground state level $\bar{c} > 0$ so, for every n, $c_n > 2\bar{c}$ since, for every $u \in \mathcal{W}$, $I(u^\pm) \geqslant \bar{c}$ (see [14]). Let us introduce the sets

$$C^- = \left\{u \in H \setminus \{0\} \,\middle|\, I(u) \leqslant \frac{\bar{c}}{3}, \nabla I(u) \cdot u \geqslant 0\right\},$$

$$C^+ = \{u \in H \setminus \{0\} \mid I(u) \leqslant 0 \ (\text{and so } \nabla I(u) \cdot u \leqslant 0)\},$$

$$L_n = \left\{u \in H \setminus \{0\} \,\middle|\, I(u) \leqslant c_n - \frac{\bar{c}}{3}\right\}.$$

Let us observe that, being $\lambda < \lambda_1$, given any $u \in H \setminus \{0\}$, the function $\alpha \mapsto I(\alpha u)$ grows with a positive derivative for α small as far as it reaches its maximum, then it has a negative derivative and tends to $-\infty$ as $\alpha \to +\infty$. So two constants $\varepsilon, c > 0$ such that $\varepsilon u \in C^-$ and $cu \in C^+$ always exist. Moreover, if $A \subset H \setminus \{0\}$ is any compact set, ε and c can be uniformly fixed for $u \in A$, as well as, if $A \subset \mathcal{W}$, for $u \in A^\pm = \{u^\pm \mid u \in A\}$, since $A^\pm$ turn out to be also compact sets which do not contain 0. Furthermore, ε can be fixed in a maximal way and c can be fixed in a minimal way so that both the two values can be considered as functions of A. We introduce the function $\sigma_A : A \times Q_2 \to H$, defined as

$$\sigma_A(u,\alpha,\beta) = \left(\varepsilon + (\alpha+1)\frac{c-\varepsilon}{2}\right)u^+ - \left(\varepsilon + (\beta+1)\frac{c-\varepsilon}{2}\right)u^-$$

and the sets

$$C_A = \sigma_A(A \times \partial Q_2) \quad \text{and} \quad \Sigma_A = \sigma_A(A \times Q_2).$$

We shall refer to Σ_A as to the *double homothetic expansion of A*. Let us notice that if $A \in \Gamma_n^{\mathcal{W}}$ and $\sup_A I < c_n + \frac{\bar{c}}{3}$, then σ_A belongs (see Lemma 4.9) to the class of functions defined by

$$\mathcal{F}_n = \left\{\sigma : A \times Q_2 \to H \;\middle|\; A \in \Gamma_n^{\mathcal{W}} \text{ and } 1, 2, 3 \text{ hold}\right\},$$

where

(1) $\sigma(\pi_H(x)) = -\sigma(x)\ \forall x \in A \times Q_2$,
(2) $\sigma(u,\alpha,\beta) \in L_n$ and $\sigma(u,\alpha,\beta)^+ \in C^\pm$ if $\alpha = \pm 1$,
(3) $\sigma(u,\alpha,\beta) \in L_n$ and $\sigma(u,\alpha,\beta)^- \in C^\pm$ if $\beta = \pm 1$.

Therefore Σ_A belongs to the class of sets defined by

$$\Gamma_n = \big\{X \subset H \setminus \{0\} \mid \exists A \in \Gamma_n^{\mathcal{W}},\ \exists \sigma : A \times Q_2 \to H, \sigma \in \mathcal{F}_n, \text{ s.t. } X = \sigma(A \times Q_2)\big\}.$$

Let us introduce two further classes of continuous functions and sets, namely

$$\mathcal{F}^* = \left\{\varphi : H \to \mathbb{R}^2 \mid 1^*, 2^*, 3^* \text{ hold}\right\},$$

where

(1*) $\varphi(-u) = \hat{\pi}(\varphi(u))$,
(2*) $\pm\varphi_1(u) \geqslant 0$ if $u \in L_n$ and $u^+ \in C^\pm$,
(3*) $\pm\varphi_2(u) \geqslant 0$ if $u \in L_n$ and $u^- \in C^\pm$,

and

$$\Gamma_n^* = \big\{X \subset H \setminus \{0\} \mid X \text{ compact},\ X = -X \text{ and } \forall \varphi \in \mathcal{F}_n^*\colon\ \gamma^*\big(X \cap \varphi^{-1}(0)\big) \geqslant n\big\}.$$

LEMMA 4.7. $\Gamma_n \subset \Gamma_n^*$.

PROOF. Let $\varphi \in \mathcal{F}^*$, $A \in \Gamma_n^{\mathcal{W}}$, $\sigma : A \times Q_2 \to H$, $\sigma \in \mathcal{F}_n$ be fixed. By virtue of Theorem 4.8, we have $\gamma^*(\sigma(A \times Q_2) \cap \varphi^{-1}(0)) \geqslant n$. By the arbitrariness of $\varphi \in \mathcal{F}^*$, we get $\sigma(A \times Q_2) \in \Gamma_n^*$. □

LEMMA 4.8. *For every $A \in \Gamma_n^*$, we have $A \cap \mathcal{W} \in \Gamma_n^{\mathcal{W}}$.*

PROOF. Let $g(u) = \nabla I(u) \cdot u$ cut off by a positive constant near 0 so that $g(0) > 0$. To prove the statement it suffices to note that the function $\varphi_\pm : u \mapsto -g(u^\pm) \in \mathbb{R}^2$ belongs to $\mathcal{F}^*$ and hence by the definition of Γ_n^* the thesis follows. □

LEMMA 4.9. *If $A \in \Gamma_n^{\mathcal{W}}$ then $\sup_{\Sigma_A} I = \sup_A I$ and $\sup_{C_A} I \leqslant \sup_A I - \frac{2}{3}\bar{c}$.*

PROOF. Of course, $A \subset \Sigma_A$ so $\sup_A I \leqslant \sup_{\Sigma_A} I$. Conversely, if $u \in \Sigma_A$, u has the form $u = \mu\bar{u}^+ + \nu\bar{u}^-$ with $\bar{u} \in A$. Since $\bar{u}^\pm \in \mathcal{V}$, we have

$$I(u) = I\big(\mu\bar{u}^+\big) + I(\nu\bar{u}^-) \leqslant I\big(\bar{u}^+\big) + I(\bar{u}^-) = I(\bar{u}) \leqslant \sup_A I.$$

If $u \in C_A$, then $\{\mu, \nu\} \cap \{\varepsilon, c\} \neq \emptyset$ so $I(\bar{u}^\pm) - I(u^\pm) \geqslant \bar{c} - \frac{\bar{c}}{3}$ for at least one of the "+" and "−" sign. □

LEMMA 4.10. *The min–max levels of the above defined classes are the same, i.e.,*

$$\inf_{A \in \Gamma_n} \sup_{u \in A} I(u) = \inf_{A \in \Gamma_n^*} \sup_{u \in A} I(u) = c_n.$$

PROOF. By virtue of the inclusion $\Gamma_n \subset \Gamma_n^*$, we have

$$\inf_{A \in \Gamma_n^*} \sup_{u \in A} I(u) \leqslant \inf_{A \in \Gamma_n} \sup_{u \in A} I(u).$$

Moreover, for every $X \in \Gamma_n^*$, from Lemma 4.8 we have

$$c_n \leqslant \sup_{u \in X \cap \mathcal{W}} \leqslant \sup_{u \in X} I(u)$$

and so, by the arbitrariness of X, $c_n \leqslant \inf_{A \in \Gamma_n^*} \sup_{u \in A} I(u)$. Finally, by Lemma 4.9 we have that, for every $X \in \Gamma_n^{\mathcal{W}}$, since $\Sigma_X \in \Gamma_n$,

$$\inf_{A \in \Gamma_n} \sup_{u \in A} I(u) \leqslant \sup_{u \in \Sigma_X} I(u) = \sup_{u \in X} I(u).$$

By the arbitrariness of X in $\Gamma_n^{\mathcal{W}}$, we have

$$\inf_{A \in \Gamma_n} \sup_{u \in A} I(u) \leqslant \inf_{A \in \Gamma_n^{\mathcal{W}}} \sup_{u \in A} I(u) = c_n.$$ □

PROPOSITION 4.12. *Γ_n and Γ_n^* are two admissible min–max classes.*

PROOF. Since $c_n > 0$, we can find a cut-off function φ such that $\varphi(c_n) = 1$ and $\varphi(s) = 0$ for $s \leqslant c_n - \frac{\bar{c}}{3}$ (see [44,46]). By multiplying the gradient $\nabla I(u)$ by $\varphi(I(u))$ we obtain a cut-off gradient flow $\eta : \mathbb{R} \times H \to H$ which lets the points at level c_n move along the reverse gradient direction and leaves the points in L_n fixed. It follows that, setting $\eta_t : x \mapsto \eta(t, x)$, if $\sigma \in \mathcal{F}_n$ and $\varphi \in \mathcal{F}_n^*$ then $\eta_t \circ \sigma \in \mathcal{F}_n$ and $\varphi \circ \eta_t \in \mathcal{F}_n^*$. So if $A \in \Gamma_n$ then $\eta_t(A) \in \Gamma_n$ and if $A \in \Gamma_n^*$ then $\eta_t(A) \in \Gamma_n^*$. □

By the above results we can state that, for every n, the min–max levels of the classes in $\Gamma_n^{\mathcal{W}}$ are all critical levels provided they satisfy the Palais–Smale condition, regardless any regularity property of the constraint since they are levels of the unconstrained admissible min–max classes Γ_n and Γ_n^*. The corresponding critical points also have a characterization of unconstrained min–max points with the relative properties like, for instance, the direct estimates of the Morse index in the whole space. Moreover, the three classes are also equivalent from the point of view of the localization of the critical points near the maximum points of the terms of a minimizing sequence of sets, as stated in the next proposition.

PROPOSITION 4.13. *If $(A_i)_{i \in \mathbb{N}}$ is any minimizing sequence in $\Gamma_n^{\mathcal{W}}$, i.e., for every $i \in \mathbb{N}$, $A_i \in \Gamma_n^{\mathcal{W}}$ and $\lim_i (\sup_{u \in A_i} I(u)) = c_n$, then the sequence of the double homothetic expansions $(\Sigma_{A_i})_{i \in \mathbb{N}}$ is a minimizing sequence in Γ_n (and so, in particular, in Γ_n^*) for i large.*

PROOF. The assertion easily follows from Lemmas 4.9 and 4.10. □

PROPOSITION 4.14. *If $(A_i)_{i \in \mathbb{N}}$ is any minimizing sequence in Γ_n^* (and so, in particular, in Γ_n) then the sequence of the traces on $\mathcal{W}(A_i \cap \mathcal{W})_{i \in \mathbb{N}}$ is a minimizing sequence in $\Gamma_n^{\mathcal{W}}$.*

PROOF. The assertion easily follows from Lemmas 4.8 and 4.10. □

It is worth to remark that the two previous propositions imply, in particular, that the admissible min–max classes Γ_n and Γ_n^* produce Palais–Smale sequences in $\mathcal{W}$.

4.5. *An estimate on the Morse index*

As an example of the utility of working without constraints and in view of an application in remaining part of this subsection, we shall briefly show a rough estimate which states that at the level c_n we can find critical points in which the augmented Morse index is greater or equal to $n - 1$ (the sharp estimate would give: greater or equal to n). The proofs follow the ideas in [27], to which we refer for more details. Firstly we shall take into account the case in which at level c_n we only have nondegenerate critical points. In such a case we shall show that each one of such points can be avoided, in the sense that every minimizing sequence can be modified in order to have an empty intersection with a neighborhood of such a point, if it has a Morse index smaller than $n - 1$. If all the points are in this situation, we can find a minimizing sequence of sets far away from all the critical points at level c_n, getting in contradiction in case of compactness. This argument is based on a simple topological lemma, see [27].

LEMMA 4.11. *Let $C \subset \mathbb{R}^{N-1}$ be a compact set and let $f : C \to \mathbb{R}^N \setminus \{0\}$ be a continuous map. Then f has a continuous extension from $\mathbb{R}^{N-1}$ to $\mathbb{R}^N \setminus \{0\}$.*

PROOF. By the Dugundji theorem [22] f has a continuous extension $\bar{f}$ from $\mathbb{R}^{N-1}$ to $\mathbb{R}^N$ which is C^1 out of C. By the Sard theorem $\bar{f}(\mathbb{R}^{N-1} \setminus C)$ is a negligible set because it cannot contain regular values. By compactness we know that a positive number $r > 0$ such that $B_r(0) \cap f(C) = \emptyset$ can be chosen. Then

$$B_r(0) \not\subset f(C) \cup \bar{f}\big(\mathbb{R}^{N-1} \setminus C\big) = \bar{f}\big(\mathbb{R}^{N-1}\big).$$

So we can find $\bar{y} \in B_r(0) \setminus \bar{f}(\mathbb{R}^{N-1})$. We denote by p the projection of $B_r(0)$ from $\bar{y}$ on $\partial B_r(0)$ extended by the identity out of $B_r(0)$. Then $p \circ \bar{f}$ is the desired extension. □

Now, let $\bar{u}$ be a regular critical point at level c_n and let E_- and E_+, respectively, denote the subspaces of H_0^1 spanned by the eigenvectors corresponding to the negative and positive eigenvalues of $\nabla^2 I(\bar{u})$, so that we can find a constant $m > 0$ such that

$$\forall u_\pm \in E_\pm: \quad \pm\nabla^2 I(\bar{u})(u_\pm) \cdot u_\pm \geqslant m\|u_\pm\|^2. \tag{4.3}$$

For two given $r_\pm > 0$, let $B_\pm = B_{r_\pm}(0) \cap E_\pm$, $S_\pm = \partial B_\pm$ in $E_\pm$, $B' = \bar{u} + B_- + B_+$ and $B = \bar{u} + (B_{\frac{1}{2}r_-}(0) \cap E_-) + B_+$. We shall split $u \in B'$ as $\bar{u} + u_+ + u_-$ with $u_\pm \in B_\pm$. Let us divide B' in two parts,

$$B_1 = \big\{u \in B' \mid \nabla^2 I(\bar{u})(u_+) \cdot u_+ \geqslant -2\nabla^2 I(\bar{u})(u_-) \cdot u_-\big\},$$
$$B_2 = B' \setminus B_1.$$

Note that by (4.3),

$$\forall u_\pm \in B_1: \quad \pm\nabla^2 I(\bar{u})(u) \cdot u_+ \geqslant \frac{m}{2}\|u_+\|^2 \geqslant \text{const} \cdot \|u\|^2, \tag{4.4}$$

$$\forall u_\pm \in B_2: \quad \pm\nabla^2 I(\bar{u})(u) \cdot u_- \leqslant -m\|u_-\|^2 \leqslant -\text{const} \cdot \|u\|^2. \tag{4.5}$$

So, if we fix $r_\pm$ conveniently small, we can respectively deduce that

$$\forall u_\pm \in B_1: \quad \pm\nabla I(u) \cdot u_+ \geqslant 0, \tag{4.6}$$

$$\forall u_\pm \in B_2: \quad \pm I(u) \leqslant I(\bar{u}) = c_n. \tag{4.7}$$

Furthermore, by also choosing r_- suitably smaller than r_+ we can also assume that

$$\inf_{\bar{u}+B_-+S_+} I \geqslant s_n > c_n. \tag{4.8}$$

In a more regular context the following lemma can be proved in a slightly easier way by using Morse lemma instead of the previous construction.

LEMMA 4.12. *Let $\bar{u}$ be a nondegenerate critical point of I at level c_n. Then for every $A \in \Gamma_n^*$ such that $\sup_A I < s_n$ there exists $A' \in \Gamma_n^*$ such that*

$$A' \setminus (B' \cup -B') = A \setminus (B' \cup -B'), \tag{4.9}$$

$$A' \cap \overline{B} \subset \bar{u} + E, \tag{4.10}$$

$$\sup_A I \leqslant \sup_{A'} I. \tag{4.11}$$

PROOF. Let $\eta : B' \to B'$ be defined by

$$\eta(u) = \eta(\bar{u} + u_- + u_+) = \bar{u} + u_- + \alpha(u)u_+, \tag{4.12}$$

where $\alpha(u) = (\frac{2}{r_-}\|u_-\| - 1)^+$. So $\eta = \mathrm{id}$ on $\bar{u} + S_- + B_+$. We define η in a symmetric way near the symmetric critical point $-\bar{u}$ and we extend it by identity on all of H_0^1. We can assume that B' does not touch $C_\pm$, so η leaves the two sets fixed. The map η so extended turns out to be discontinuous only on the sets $\pm(\bar{u} + B_- + S_+)$ where the functional I takes values greater or equal to s_n. So $A \cap ((\bar{u} + B_- + S_+) \cup -(\bar{u} + B_- + S_+)) = \emptyset$ and we can modify η in order to restore the discontinuity without modifying the values on A. Let $A' = \eta(A)$. Since η is odd and leaves the sets $C_\pm$ fixed, $\eta(A) \in \Gamma_n^*$. Moreover, (4.9) follows by easy considerations since $\eta = \mathrm{id}$ on $A \setminus (B' \cup -B')$, (4.10) holds because if $u \in \overline{B}$ then $\alpha(u) = 0$ and $\eta(u) = \bar{u} + u_- \in \bar{u} + E_-$. Finally, (4.11) trivially follows from

$$\forall u \in B': \quad I(\eta(u)) \leqslant \max(I(u), c_n), \tag{4.13}$$

as we are going to check. Indeed by (4.6),

$$I(u) - I(\eta(u)) = \int_{\alpha(u)}^1 \nabla I(\bar{u} + u_- + su_+) \cdot u^+ \,\mathrm{d}s \geqslant 0,$$

if $\bar{u} + u_- + \alpha(u)u_+ = \eta(u)$ is in B_1. Otherwise by (4.7), $I(\eta(u)) \leqslant c_n$. □

The above result can be improved by the use of Lemma 4.11, if we assume that the Morse index of $\bar{u}$, namely $\dim E_-$, is smaller than $n - 1$.

PROPOSITION 4.15. *Let $\bar{u}$ be a nondegenerate critical point at level c_n and let $\dim E_- < n - 1$. Then we can find two arbitrarily small neighborhoods B and B' of $\bar{u}$ such that* (4.9) *and* (4.11) *and*

$$A' \cap B = \emptyset \tag{4.14}$$

hold.

PROOF. Let A', B, B' be as in the previous lemma. We shall show that $A' \setminus (B \cup -B)$ still belongs to Γ_n^* and therefore it satisfies the properties in the thesis. Indeed, assume to have

$\varphi \in \mathcal{F}^*$ and $\psi \in \Lambda^{n-1}((A' \setminus (B \cup -B)) \cap \varphi^{-1}(0))$ such that $0 \notin \psi(((A' \setminus (B \cup -B)) \cap \varphi^{-1}(0)) \times Q_k)$. By (4.10),

$$T = \big(A' \setminus (B \cup -B)\big) \cap \big(A' \cap \overline{B}\big) \cap \varphi^{-1}(0) \subset \bar{u} + E.$$

We have $\psi : ((A' \setminus (B \cup -B) \cap \varphi^{-1}(0)) \times Q_k \to \mathbb{R}^{n-1+k} \setminus \{0\}$. We want to extend ψ to $((A' \cap B) \cap \varphi^{-1}(0)) \times Q_k$ and we must extend it from $T \times Q_k$ which is the common boundary of the set on which ψ is already defined and the set to which it must be extended. We can firstly extend, by the Dugundji theorem, the components $\pm\psi_i$, $i = 1, \ldots, k$, to $((A' \cap \overline{B}) \cap \varphi^{-1}(0)) \times F_i^{\pm}$ to positive values and then all the other components in order to have

$$\psi : (T \times Q_k) \cup \big(\big(A' \cap \overline{B} \cap \varphi^{-1}(0)\big) \times \partial Q_k\big) \to \mathbb{R}^{n-1+k} \setminus \{0\}.$$

Since

$$(T \times Q_k) \cup \big(\big(A' \cap \overline{B} \cap \varphi^{-1}(0)\big) \times \partial Q_k\big) \subset (\bar{u} + E_-) \times \mathbb{R}^k$$

and $(\bar{u} + E_-) \times \mathbb{R}^k$ is a linear space of dimension strictly smaller than $n - 1 + k$, by Lemma 4.11 we can extend ψ to $(T \times Q_k) \cup ((A' \cap B \cap \varphi^{-1}(0)) \times Q_k)$ with values in $\mathbb{R}^{n-1+k} \setminus \{0\}$. Symmetrically, we can also extend ψ to $((A' \cap -B \cap \varphi^{-1}(0)) \times Q_k)$ and therefore to all of $(A' \cap \varphi^{-1}(0)) \times Q_k$ with values in $\mathbb{R}^{n-1+k} \setminus \{0\}$, in contradiction to the condition $A \in \Gamma_n^*$. □

By iterating this flattening and excision procedure, if I has only (a finite number of) nondegenerate critical points at level c_n and all of them have the Morse index strictly smaller than $n - 1$, we can easily find a minimizing sequence $(A_i)_{i \in \mathbb{N}}$ in Γ_n^* whose terms have an empty intersection with a fixed neighborhood V of the set $\mathcal{K}_{c_n}$ of the critical points at level c_n, getting in contradiction in case of compactness.

In the general case, one can take advantage of the Marino–Prodi perturbation argument, (see [27,32,40] and [41], Chapter 3, Section 5) which allows one to reduce $\mathcal{K}_{c_n}$ to a set of nondegenerate critical points and to consequently deduce the following result

PROPOSITION 4.16. *If I satisfies PS at level c_n there exists $\bar{u}$ in $\mathcal{K}_{c_n}$ which has an augmented Morse index greater or equal to $n - 1$.*

4.6. *Multiple solutions to the problem at critical growth*

We can now give the proof of Theorem 1.2. Let us choose a sequence $(p_n)_{n \in \mathbb{N}}$ in $]2, 2^*[$ such that $p_n \to 2^*$ and introduce the functionals

$$I_\lambda^n(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \frac{\lambda}{2}\int_\Omega |v|^2 - \frac{1}{p_n}\int_\Omega |v|^{p_n},$$

whose critical points are solutions to problems (SP) for $p = p_n$.

Let $\mathcal{V}$ be as defined in Section 1.4 and $\mathcal{V}_n$ the analogous constraint for $I = I_\lambda^n$. Let

$$c_k = \inf_{A \in \Gamma_k^{\mathcal{V}}} \sup_{v \in A} I_\lambda(v)$$

and

$$c_k^n = \inf_{A \in \Gamma_k^{\mathcal{V}_n}} \sup_{v \in A} I_k^n(v).$$

By the results of Section 4.3, we see that every c_k can be also regarded as the min–max level of I on the class of sets Γ_k or Γ_k^*. The same conclusion holds for c_k^n and, by a suitable choice of the sets C^- and C^+ which can be uniformly fixed with respect to n, c_k^n can be seen as the min–max level I_λ^n on the same classes Γ_k and Γ_k^* as before which, therefore, do not depend on n.

LEMMA 4.13. *If* $\lambda < \lambda_k$ *then* $c_k > 0$ *and, for every* $n \in \mathbb{N}$, $c_k^n > 0$.

PROOF. Let $A \in \Gamma_k^{\mathcal{V}}$, by Proposition 4.2 $A \cap E_{k-1}^\perp \neq \emptyset$. Let $u \in A \cap E_{k-1}^\perp \subset \mathcal{V} \cap E_{k-1}^\perp$. By the constraint equation,

$$I_\lambda(u) = \frac{1}{N}\left(\int_\Omega |\nabla u|^2 - \lambda \int_\Omega |u|^2\right) \geqslant \frac{1}{N}\left(1 - \frac{\lambda}{\lambda_k}\right) \int_\Omega |\nabla u|^2. \tag{4.15}$$

The constraint equation also implies

$$\int_\Omega |u|^{2^*} = \int_\Omega |\nabla u|^2 - \lambda \int_\Omega |u|^2 \geqslant \left(1 - \frac{\lambda}{\lambda_k}\right) \int_\Omega |\nabla u|^2,$$

which implies

$$\left(\|u\|_{2^*}\right)^{2^*-2} \geqslant \left(1 - \frac{\lambda}{\lambda_k}\right) S$$

and therefore

$$\sup_A I \geqslant I(u) = \frac{1}{N} \int_\Omega |u|^{2^*} \geqslant \frac{1}{N}\left(\left(1 - \frac{\lambda}{\lambda_k}\right) S\right)^{N/2}.$$

Taking the infimum for $A \in \Gamma_k^{\mathcal{V}}$ we finally have

$$c_k \geqslant \frac{1}{N}\left(\left(1 - \frac{\lambda}{\lambda_k}\right) S\right)^{N/2} > 0.$$

The second part of the thesis can be proved in the same way. □

The proof of Theorem 1.2 will follow from the following two lemmas.

LEMMA 4.14. $\lim_{n\to+\infty} c_k^n = c_k$ *for any* k *as in the previous lemma.*

PROOF. Let us fix $k \in \mathbb{N}$ and $A \in \Gamma_k$, then for any $u \in A$, $I_\lambda^n(u) \to I_\lambda(u)$. Being A compact and the functionals equicontinuous,

$$\sup_{u\in A} I_\lambda^n(u) \to \sup_{u\in A} I_\lambda(u).$$

Then $\limsup_{n\in\mathbb{N}} c_k^n \leqslant \sup_{u\in A} I_\lambda(u)$ and, being A an arbitrary set in Γ_k, we get

$$\limsup_{n\to+\infty} c_k^n \leqslant c_k.$$

Since for $s > 0$ the function $h(s) = \frac{1}{p_n} s^{p_n} - \frac{1}{2^*} s^{2^*}$ gets its maximum value in $s = 1$, we have $h(s) \leqslant \frac{1}{p_n} - \frac{1}{2^*}$ for all $s > 0$. Therefore for every $u \in H_0^1$,

$$I_\lambda(u) \leqslant I_\lambda^n(u) + \left(\frac{1}{p_n} - \frac{1}{2^*}\right)|\Omega|$$

so, for any $k \in \mathbb{N}$,

$$c_k \leqslant \liminf_{n\to+\infty} c_k^n$$

and we have the thesis. □

LEMMA 4.15. $\lim_{k\to+\infty} c_k = +\infty$.

PROOF. We want to remark that such a result is not based on compactness properties because, if we prove the statement for $\lambda > 0$, i.e., when we have compactness, the statement itself is obviously true when $\lambda \leqslant 0$ and compactness fails. Moreover, this lemma is also true in lower dimension, as we can see, in the same way, by adding a suitably big subcritical term. On the other side, the use of compactness techniques takes advantage of the previous results in this chapter. Let us suppose, by contradiction, that the sequence $(c_k)_{k\in\mathbb{N}}$ is bounded, hence it converges to a real number c. For any $k \in \mathbb{N}$, by Lemma 4.14 there exists $n_k > k$ such that $|c_k^{n_k} - c_k| < \frac{1}{k}$; hence

$$\lim_{k\to+\infty} c_k^{n_k} = \lim_{k\to+\infty} c_k = c \tag{4.16}$$

and the sequence $(n_k)_{k\in\mathbb{N}}$ is diverging, i.e.,

$$\lim_{k\to+\infty} n_k = +\infty.$$

Let u_{n_k} be a solution of (SP) at level $c_k^{n_k}$. Using the Morse index estimates on min–max points as in Section 4.5, we can select the sequence $(u_{n_k})_{k\in\mathbb{N}}$ such that every u_{n_k} has an augmented Morse index greater or equal to $k-1$. By our assumptions, we can claim that the sequence $(u_{n_k})_{k\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$. Indeed, being u_{n_k} a solution to (SP) we have

$$I_\lambda^{n_k}(u_{n_k}) = \left(\frac{1}{2} - \frac{1}{p_{n_k}}\right)\int_\Omega |u_{n_k}|^{p_{n_k}} \to c, \tag{4.17}$$

which gives the boundedness of $-\Delta u_{n_k}$ in H^{-1} and, in turn, the boundedness of u_{n_k} on H_0^1. So the sequence $(u_{n_k})_{k\in\mathbb{N}}$ is uniformly bounded by Theorem 1.1 and therefore the Morse index of u_{n_k} must keep bounded, in contradiction to our construction. □

PROOF OF THEOREM 1.2. Fixed $k \in \mathbb{N}$, we take, for any $n \in \mathbb{N}$, $u_n = u_k^n$ a critical point at level c_k^n for the functional I_λ^n. First we use Lemma 4.14 and the analogous of (4.17) to have $(u_n)_{n\in\mathbb{N}}$ bounded in H_0^1. The sequence $(u_n)_{n\in\mathbb{N}}$, by Theorem 1.1, is then uniformly bounded, so by standard compactness arguments we can find a converging subsequence to a solution $\bar{u}_k$ to (CP) at level c_k, as follows from Lemma 4.14. By Lemma 4.15, we have infinitely many distinct values of c_k for $k \in \mathbb{N}$ and so the proof is concluded. □

4.7. *Multiple solutions to the problem on the whole domain*

This part is devoted to the proof of Theorem 1.4 which states the existence of infinitely many solutions to problem (P).

Let us fix a sequence $(r_n)_{n\in\mathbb{N}}$, $r_n \in \mathbb{R}^+$ such that $r_n \to +\infty$ and consider the problems $(\mathrm{P}_n) = (\mathrm{AP}_{r_n})$ defined in Section 1.2.

Beside the functional I defined in Section 1.2, we introduce the functional

$$I_n(u) = \frac{1}{2}\int_{B_{r_n}(0)} \left(|\nabla u|^2 + a(x)u^2\right)\mathrm{d}x - \frac{1}{p}\int_{B_{r_n}(0)} |u|^p\,\mathrm{d}x,$$

and the min–max levels

$$c_k = \inf_{A\in\Gamma_k^{\mathcal{V}}} \sup_{v\in A} I(v)$$

and

$$c_k^n = \inf_{A\in\Gamma_k^{\mathcal{V}_n}} \sup_{v\in A} I_n(v),$$

where $\mathcal{V}$ is the natural constraint related to the functional I and $\mathcal{V}_n$ is the natural constraint related to I_n or, equivalently, $\mathcal{V}_n = \mathcal{V} \cap H_0^1(B_{r_n}(0))$.

Since (a_1) and (a_2) widely imply that $(a - \frac{1}{2}a_\infty)^- = \sup(-(a - \frac{1}{2}a_\infty), 0) \in L^{N/2}$, we know that the eigenvalue problem

$$\begin{cases} -\Delta u = \mu\left(a - \frac{1}{2}a_\infty\right)^-(x)u & \text{in } \mathbb{R}^N, \\ u \in H^1\left(\mathbb{R}^N\right), \end{cases} \tag{LP}$$

has a diverging sequence of eigenvalues $\mu_1 < \mu_2 \leqslant \mu_3 \leqslant \cdots \leqslant \mu_n \leqslant \cdots$, see [30]. So we can find k such that $\mu_k > 1$. Also in this situation we have the analogous statement of Lemma 4.13.

LEMMA 4.16. *If* $\mu_k > 1$ *then* $c_k > 0$.

PROOF. The proof is formally equal to that of Lemma 4.13, provided we take the functions φ_i as the eigenvectors given by (LP) corresponding to the eigenvalues μ_i and $E_i = \mathrm{Span}\{\varphi_1, \varphi_2, \ldots, \varphi_i\}$. By the constraint equation, if $u \in \mathcal{V} \cap E_{k-1}^{\perp}$,

$$\begin{aligned} I(u) &= \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathbb{R}^N} \left(|\nabla u|^2 + a(x)u^2\right) \\ &\geqslant \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathbb{R}^N} \left(|\nabla u|^2 + \frac{1}{2}a_\infty u^2 - \left(a - \frac{1}{2}a_\infty\right)^{-}(x)u^2\right) \\ &\geqslant \left(\frac{1}{2} - \frac{1}{p}\right)\left(\left(1 - \frac{1}{\mu_k}\right) \int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} \frac{a_\infty}{2} u^2\right). \end{aligned} \tag{4.18}$$

So

$$\begin{aligned} \|u\|^p &\geqslant \left(\frac{1}{2} - \frac{1}{p}\right)\left(1 - \frac{1}{\mu_k}\right) \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{a_\infty}{2} \int_{\mathbb{R}^N} u^2 \\ &\geqslant \left(\frac{1}{2} - \frac{1}{p}\right)\left(\left(1 - \frac{1}{\mu_k}\right) S\|u\|_{2^*}^2 + \frac{a_\infty}{2}\|u\|_2^2\right). \end{aligned} \tag{4.19}$$

On the other hand, by the Hölder and Young inequalities,

$$\begin{aligned} \|u\|_p^2 &\leqslant \|u\|_{2^*}^{2(2^*(p-2))/(p(2^*-2))} \|u\|_2^{2(2(2^*-p))/(p(2^*-2))} \\ &\leqslant \frac{2^*(p-2)}{p(2^*-2)}\|u\|_{2^*}^2 + \frac{2(2^*-p)}{p(2^*-2)}\|u\|_2^2. \end{aligned} \tag{4.20}$$

Therefore, combining (4.19) and (4.20), one easily gets

$$\|u_p\|^2 \leqslant \mathrm{const} \cdot \|u\|_p^p$$

and finally,

$$I(u) = \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathbb{R}^N} |u|^p \geqslant \mathrm{const}.$$

Since, for every $A \in \Gamma_k^{\mathcal{V}}$, one has by Proposition 4.2 that $A \cap E_{k-1}^{\perp} \neq \emptyset$, by fixing $u \in A \cap E_{k-1}^{\perp} \subset \mathcal{V} \cap E_{k-1}^{\perp}$, we have

$$\sup_A I \geqslant I(u) \geqslant \mathrm{const} \tag{4.21}$$

and, taking the infimum, $c_k \geqslant \text{const} > 0$. □

In this second case we always have

$$c_k \leqslant c_k^n \tag{4.22}$$

because $\mathcal{V}_n = \mathcal{V} \cap H_0^1(B_{r_n}) \subset \mathcal{V}$. Moreover, the property of Lemma 4.13 still holds.

LEMMA 4.17. *If* $\mu_k > 1$ *then* $c_k = \lim_n c_k^n$.

PROOF. Let k be given and let $A \in \Gamma_k^{\mathcal{V}_n}$. Since A is a compact set, for every $\varepsilon > 0$ A has a finite ε-net F contained in $\mathcal{D}(\Omega)$. For n large, $F \subset H_0^1(B_{r_n})$. Let P_n be the orthogonal projection of $H^1(\mathbb{R}^N)$ onto $H_0^1(B_{r_n})$. We know that, for every $x \in A$, $\|x - P_n(x)\| < \varepsilon$. Since $A \subset \mathcal{V}$, for $v \in P_n(A)$,

$$\alpha(v) = \left(\frac{\int_{\mathbb{R}^N} |\nabla v|^2 - \int_{\mathbb{R}^N} a(x) v^2 \,\mathrm{d}x}{\int_{\mathbb{R}^N} |v|^p \,\mathrm{d}x} \right)^{1/(p-2)} > 0, \tag{4.23}$$

provided ε is taken sufficiently small. Then the map $\eta_n : A \to H_1^0(B_{r_n}) \cap \mathcal{V} = \mathcal{V}_n$ defined by

$$\eta_n : u \mapsto \alpha\bigl(P_n(u)\bigr) P_n(u) \tag{4.24}$$

is an odd map. So $\gamma^*(\eta_n(A)) \geqslant \gamma^*(A) \geqslant k$ and therefore, $\eta_n(A) \in \Gamma_k^{\mathcal{V}_n}$. One can also easily see that for any $\delta > 0$, by taking a conveniently small value of ε to have $\eta_n(A)$ close enough to A,

$$c_k^n \leqslant \sup_{\eta_n(A)} I \leqslant \sup_A I + \delta \tag{4.25}$$

for n large. Taking the infimum for $A \in \Gamma_k^{\mathcal{V}}$ and $\delta > 0$ we finally find

$$\limsup_n c_k^n \leqslant c_k, \tag{4.26}$$

which, combined with (4.22), gives the thesis. □

We can also formally use the same proof of Lemma 4.13 to see that, also in this second case, the sequence c_k is diverging. The same argument used for the proof of Theorem 1.2 gives now Theorem 1.4.

PROOF OF THEOREM 1.4. Let k be as in Lemma 4.16. For every n we can find a solution u_n to (P_n) at level c_k^n. Since

$$c_k^n = I_n(u_n) = \left(\frac{1}{2} - \frac{1}{p}\right) \|u_n\|_p^p \to c_k$$

as $n \to +\infty$, then $(u_n)_{n\in\mathbb{N}}$ is bounded in L^p, so also

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 + \int_{\mathbb{R}^N} a(x)u_n^2 \leqslant \text{const}$$

and finally,

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u_n|^2 + \frac{1}{2}a_\infty \int_{\mathbb{R}^N} u_n^2 &\leqslant \text{const} + \int_{\mathbb{R}^N} \left(a - \frac{1}{2}a_\infty\right)^- u_n^2 \\ &\leqslant \text{const} + \left\| \left(a - \frac{1}{2}a_\infty\right)^- \right\|^{(p/2)'} \|u_n\|_p^2 \\ &\leqslant \text{const},\end{aligned}$$

so $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^N)$. Then by Theorem 1.3, $(u_n)_{n\in\mathbb{N}}$ has a converging subsequence to a solution u_k to (P) at level c_k. Since $(c_k)_{k\in\mathbb{N}}$ is diverging, (P) has infinitely many solutions. □

5. Concluding remarks

We conclude this exposition with some comments on the assumptions employed for the two problems and indicating what one knows or what one can expect without them. In the second part of the section we shall show what kind of multiplicity results are already known for (SP) in nonsymmetric cases when the fundamental restriction $N \geqslant 7$ on the dimension is not assumed.

5.1. *Review on the assumptions and open problems*

We have considered the critical growth problem only from the point of view of taking into exam the presence of subcritical terms which make the concentrations to be not convenient. The techniques discussed here have no application (so far) to problems in which the concentrations are avoided thanks to assumptions on the shape of the domain or other geometric conditions. Easy extensions are possible to similar equations involving other operators as, for instance, the p-Laplacian or more general elliptic operators. The most delicate point is concerned with the bound on the dimension. The dimension seven has no particular meaning as happens, for instance, with the minimal surface problems, but depends on the linearity of the subcritical perturbation term λu. A nonlinear subcritical term would require a different bound on the dimension. Let us point out that $N \geqslant 4$ is enough for the existence of a nontrivial solution for every λ but three more dimensions are needed for the multiplicity results. The reason is due to the fact that as we have seen in Section 1.5, starting on dimension 4, the gain due to the quadratic term $-\lambda \int_\Omega u^2$ in the functional is more consistent than the cost of a cut-off term which brings a Talenti function to zero within the boundary of Ω. On the other hand, if one wants to perform a similar test for the multiplicity problem, one must cut off the function in order to reach a value of opposite sign and so the cost of the cut-off is considerably more expensive.

As we have seen in Section 2, the existence of infinitely many solutions to (CP) is assured for every $\Omega \subset \mathbb{R}^N$ only for $N \geqslant 7$ and relies on a compactness theorem which cannot be extended to lower dimensions. This circumstance does not mean that the existence of infinitely many solutions cannot be proved through different tools as it happens, for instance, in the case of symmetric domains (see [24]). Such a result is false if one looks for particular solutions as, if Ω is a ball, the radial ones (see [2]), which certainly exist if $N \geqslant 7$, and this suggests, in the general case, to search for solutions which change sign near the boundary, so excluding the radial symmetric functions. This approach was pursued in [14] in order to show the existence of two (pairs of) distinct solutions for $N \geqslant 4$ and $\lambda < \lambda_1$.

A result in this sense will be shown in the second part of this section.

In the case of problem (P) the assumptions (a_1) and (a_2) let the superquadratic term of the functional bounded in terms of the quadratic part and allow the proof of Proposition 1.1. The smoothness asked in (a_1) is more than what we need to this aim ($a^- \in L^{N/2}(\mathbb{R}^N)$ would be more than enough) but it is required for the other assumptions. Condition (a_3) is the equivalent of the bound $N \geqslant 7$ for (CP). The proof of the multiplicity results without this assumption would probably require completely different arguments. One can possibly show even uniqueness results in some particular cases, as happens for (CP) with the radial symmetry. On the contrary (a_4) has been only used for the proof of Lemma 2.4 to the aim of passing from an integral of the function $\frac{\partial a}{\partial t}$ to an integral of $\frac{\partial a}{\partial x}$. It can certainly be weakened, the question if some assumption of this kind is necessary for the multiplicity result or what is the minimal condition is open.

5.2. *Estimates in lower dimension*

Now we shall prove, following [20], that the estimate in [14] increases up to at least $\frac{N}{2} + 1$ solutions or even to $N + 1$ if λ is suitably close to zero. This result has been recently extended in [15] to the case $\lambda \geqslant \lambda_1$. In any case, there is no reason to suppose that such a result may be optimal and the problem of proving the existence of infinitely many solutions, or even of getting an optimal estimate on the number of solutions, remain, as far as we know, largely open for $N \leqslant 6$. We shall work with the double natural constraint introduced in Section 4.4.

LEMMA 5.1. *Let $(u_n)_{n\in\mathbb{N}}$ be a PS sequence in $\mathcal{W}$ at level $c < \frac{2}{N} S^{N/2}$, then the sequence cannot converge weakly to zero.*

PROOF. Since $(u_n)_{n\in\mathbb{N}}$ is a PS sequence in $\mathcal{W}$ and $\lambda \in]0, \lambda_1[$, we have $\|\nabla u_n^{\pm}\| \geqslant \text{const}$ therefore an eventual strong limit of the two sequences $(u_n^{\pm})_{n\in\mathbb{N}}$ cannot be zero. We have to extend this claim to the case of a weak limit, which brings a weaker information when the sequence is not compact. We know that the "bad" levels c for noncompact PS sequences detected by Theorem 1.6 in $]0, \frac{2}{N} S^{N/2}[$ are $c = c' + \frac{1}{N} S^{N/2}$ with c' critical level for I_λ and the weak limit φ_0 of the sequence is a solution u to (P) at level c'. The obstruction to the compactness of $(u_n)_{n\in\mathbb{N}}$ is given by scaled copies of a global solution φ_1 which has a constant sign and disappears if we restrict ourselves to $u_n^\pm$ for one of the $+$ or $-$ signs. So, in one of the two cases, we have $u_n^\pm \to u^\pm$ strongly and so $u \neq 0$, according to the assertion in the beginning of the proof. Therefore or c or $c - \frac{1}{N}S^{N/2}$ is a nonnull critical level which corresponds to the strong or respectively to the weak limit of the sequence $(u_n)_{n\in\mathbb{N}}$. □

We set

$$c_0 = \inf_{u\in\mathcal{V}} I_\lambda(u).$$

LEMMA 5.2. *For all $k \in \{1, \dots, N+1\}$, we have*

$$2c_0 \leqslant c_k < \frac{2}{N}S^{N/2}.$$

PROOF. The finite sequence of sets $(\Gamma_k)_{k\in\{1,\dots,N+1\}}$ is decreasing and therefore we must only prove that $2c_0 < c_1$ and $c_{N+1} < \frac{2}{N}S^{N/2}$. The first inequality is obvious since, for all $u \in \mathcal{W}$, $u^+, u^- \in \mathcal{V}$ and $I_\lambda(u) = I_\lambda(u^+) + I_\lambda(u^-) \geqslant 2c_0$. We start by determining a constant $\bar{c} < \frac{1}{N}S^{N/2}$ such that $c_N < 2\bar{c}$, namely we can find a set $A \in \Gamma_N$ such that $\sup_A I_\lambda \leqslant 2\bar{c} < \frac{2}{N}S^{N/2}$. Let $B = B_R$ be a ball with radius $R > 0$ contained in Ω which, in order to use a simpler notation, will be assumed, without any restriction, to be centered in the origin. For any $\nu \in S^{N-1} = \partial B$, we consider two balls B_1^ν and B_2^ν of radius $\frac{R}{2}$ contained in B and tangent to ∂B respectively in ν and in $-\nu$. Let us consider the function $\varphi : \partial B \to \mathcal{W}$ such that for any $\nu \in \partial B$, $\varphi(\nu) = u^\nu : \Omega \to \mathbb{R}$, where $(u^\nu)^+$ and $(u^\nu)^-$ are respectively the function u_σ^* found in Section 1.5 in B_1^ν and B_2^ν and let us call $\bar{c} = I_\lambda((u^\nu)^+) = I_\lambda((u^\nu)^-)$. In this way, we map ∂B into a set $A \subset \mathcal{W}$ in an odd continuous way, therefore $\gamma(A) \geqslant \gamma(\partial B) = N$. Moreover, $\forall \nu \in \partial B$: $I_\lambda(u^\nu) = I_\lambda((u^\nu)^+) + I_\lambda((u^\nu)^-) = 2\bar{c} < \frac{2}{N}S^{N/2}$.

From the bound $c_N < 2\bar{c}$ we shall now pass to show that c_{N+1} is below $\frac{2}{N}S^{N/2}$. Actually we shall find in this new case the bound $c_{N+1} \leqslant \bar{c} + \frac{1}{N}S^{N/2}$, by extending the above introduced map φ from ∂B to an N-dimensional sphere. From the construction in Section 1.5, one can easily see that $\bar{u}_1$ can be modified continuously to the analogous function $\bar{u}_1^s$, which has a support on the ball $B^\nu(s)$ concentric with B_1^ν and radius s, belongs to $\mathcal{V}$ and keeps the property $I_\lambda(\bar{u}_1^s) < \frac{1}{N}S^{N/2}$. Of course $I_\lambda(\bar{u}_1^s) \to \frac{1}{N}S^{N/2}$ as $s \to 0$.

We shall firstly extend φ to B, namely we shall find a continuous homotopy with values in $\mathcal{W}$ from φ to a constant map. This homotopy will be performed in two steps.

Firstly we shall shrink the radius of B_1^ν to a conveniently small radius ρ by taking

$$H_1(s, \nu) = \bar{u}_1^s - \bar{u}_2$$

with $s \in [0, \frac{R}{2}]$. Then we shall translate the balls $B^\nu(\rho)$ and B_2^ν bringing the two centers on the origin, obtaining the homotopy

$$H_2(t, \nu) = x \mapsto \bar{u}_1^\rho\left(x + t\frac{\nu}{2}\right) - \bar{u}_2\left(x - t\frac{\nu}{2}\right)$$

for $t \in [0, 1]$.

The very different scales of $\bar{u}_1^\rho$ and $\bar{u}_2$ make infinitesimal the mixed terms in the evaluation of I_λ, so we have, for all $t \in [0, 1]$,

$$I_\lambda\big(H_2(t, \nu)\big) = I_\lambda\big(\bar{u}_1^\rho\big) + I_\lambda(\bar{u}_2) + \varepsilon_1(\rho) = \bar{c} + \frac{1}{N} S^{N/2} + \varepsilon_2(\rho), \tag{5.1}$$

where for $i = 1, 2$, $\varepsilon_i(\rho) \to 0$ when $\rho \to 0$. The definition of H_2 should still be changed by multiplying the positive and the negative part of $H_2(t, \nu)$ by coefficients $\alpha_\pm(t)$ in such a way to provide that $H_2(t, \nu) \in \mathcal{W}$ for all $t \in [0, 1]$. Since the function $\alpha_\pm(t)$ are clearly continuous and $\alpha_\pm(t) \to 1$ as $\rho \to 0$, as we can see by arguing as in (5.1), we still keep (5.1) after this small correction. $H_2(1, \nu)$ gives the same function whatever is $\nu \in \partial B$, so the homotopy connects φ to a constant map.

We have in this way an extension of φ to B and so to a hemisphere in dimension $N + 1$. By an odd extension we define φ on the whole sphere.

Then the set of the functions obtained via this transformation is a compact symmetric set contained in $\mathcal{W}$ whose genus is greater or equal to $N + 1$, moreover, on this set the functional I_λ is bounded by $\bar{c} + \frac{1}{N} S^{N/2} + \varepsilon_2(\rho) < \frac{2}{N} S^{N/2}$ and the thesis follows. □

REMARK 5.1. If we work on the larger constraint $\mathcal{V}$, we can add one more step to the previous construction showing that even the constrained level c_{N+2} is in such a case lower than $\frac{2}{N} S^{N/2}$. Indeed, one can easily find a homotopy from the map φ, extended to the sphere in $N + 1$ dimension to a constant map, by arguing as follows. For $s \in [0, 1]$ one performs the same construction as before by multiplying by s the function u_σ^* taken in B_2^ν and then normalizing the whole u^ν by multiplying it by a normalization constant $\alpha(s)$ which lets $\alpha(s)u^\nu \in \mathcal{V}$. Analogously, one normalizes $H_i(s, \nu)$. In this way we can, roughly speaking, "kill" the negative part of the function remaining only with scaled copies of u_σ^* on some balls contained in B. Then it is easy to modify such functions continuously reconducing all of them to the function u_σ^* defined as in Section 1.5 on the whole ball. This homotopy extends φ to a hemisphere in dimension $N+2$. In any case, even with the present approach, we gain a further critical level by adding to the levels $c_1, c_2, \dots, c_{k+1}$ detected on $\mathcal{W}$ the level, which we have denoted by c_0, obtained as the infimum of I on $\mathcal{V}$, so that the difference apparently relies in a shift of the index which runs from 0 to $N + 1$ rather than from 1 to $N + 2$. However, the further information that, for $i \geqslant 1$, c_i is a critical level corresponding to a PS sequence in $\mathcal{W}$ will allow the use of Lemma 5.1 and motivates the choice of working on the double natural constraint. Nevertheless, approaching the problem on $\mathcal{V}$ would only give the further difficulty of avoiding the level $\frac{1}{N} S^{N/2}$ in Lemma 5.1 and would finally lead to prove the same result as Theorem 5.1 in even dimension but to find one solution less in odd dimension. On the other hand, Theorem 5.2 can be proved also in this other way.

We shall now observe that if two different levels c_i coincide, then (P) must necessarily have infinitely many solutions, even if the $[\text{P.S.}]_{c_i}$ condition fails. Note that this lack of compactness make us unable even to say that such a level must be critical, however Lemma 3.2 permits to conclude that the number of solutions at a possibly lower level is necessarily infinite.

LEMMA 5.3. *If there exist* $i, j \in \{1, \dots, N+1\}$, $i \neq j$, *such that* $c_i = c_j$, *then* (P) *has uncountably many solutions*.

PROOF. We can reduce ourselves to the case in which $j = i + 1$ with $i \in \{1, \dots, N\}$. We shall prove that, in this hypothesis, we can find a solution to (P) which is orthogonal to every given function $v \in H_0^1(\Omega)$. Let $(A_n)_{n\in\mathbb{N}} \in \Gamma_{i+1}^{\mathbb{N}}$ be a minimizing sequence, i.e., $\limsup_n(\sup_{A_n} I_\lambda) = c_{i+1}$. Let us fix $v \in H_0^1(\Omega)$ and consider for every $n \in \mathbb{N}$

$$A'_n = A_n \cap v^{\perp}.$$

The sequence $(A'_n)_{n\in\mathbb{N}}$ is, by Lemma 4.3, a sequence in Γ_i and, being

$$\limsup_n\Big(\sup_{A'_n} I_\lambda\Big) \leqslant \limsup_n\Big(\sup_{A_n} I_\lambda\Big) = c_{i+1} = c_i,$$

it is a minimizing sequence. Let $(u_n)_{n\in\mathbb{N}}$ be a constrained PS sequence at level c_i close to the sequence $(A'_n)_{n\in\mathbb{N}}$ (i.e., $\lim_n d(u_n, A_n) = 0$), then by Lemma 5.1 its weak limit $\bar{u}$ is a nontrivial solution to (CP) which is orthogonal to v. If (CP) has a only countably many solutions, the set of the functions which are orthogonal to some solution to (CP) is a first category set. So we can find a function $v \in H_0^1(\Omega)$ whose scalar product with every nontrivial solution is not null and we find a contradiction. □

THEOREM 5.1. *Let* $N \geqslant 4$, *then* (CP) *has at least* $\frac{N}{2} + 1$ *distinct* (*pairs of*) *solutions* $\forall\lambda \in]0, \lambda_1[$.

PROOF. Taking into account Lemma 5.3, we can suppose that $\forall i, j \in \{1, \dots, N+1\}$, $c_i \neq c_j$ so that we have $N + 1$ distinct levels c_i for $i = 1, \dots, k+1$. Adding the ground state level c_0, obtained as minimal level on $\mathcal{V}$, we have $N + 2$ distinct levels

$$0 < c_0 < c_1 < \cdots < c_{N+1} < \frac{2}{N} S^{N/2}.$$

By Lemma 3.2, we know that for every $i \in \{1, \dots, N+1\}$ we have the following alternative: or c_i or $c'_i = c_i - \frac{1}{N} S^{N/2}$ is a nonzero critical level, so we can deduce that at most two different values c_i can determine the same solution, so we get at least $\frac{N+2}{2} = \frac{N}{2} + 1$ solutions to (CP). More precisely, when N is even we have at least $\frac{N}{2} + 1$ pairs of solutions to (CP) while if N is odd we get at least $\frac{N+1}{2} + 1$ pairs of solutions to (CP). □

THEOREM 5.2. *Let* $N \geqslant 4$, *then there exists a positive number* $\bar{\lambda} \in]0, \lambda_1[$ *such that problem* (CP) *has at least* $N + 1$ *distinct* (*pairs of*) *solutions for every* $\lambda \in]0, \bar{\lambda}[$.

PROOF. The proof is obvious taking into account that there exists $\bar{\lambda} \in]0, \lambda_1[$ such that for all $\lambda \in]0, \bar{\lambda}[$ $2c_0 > \frac{1}{N} S^{N/2}$. This last property implies that $\forall i \in \{2, \dots, N+1\}$, $c_i - \frac{1}{N} S^{N/2} < c_1$, so the only critical value which can be given by two different min–max approaches of the type considered above is c_0. □

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